PICONE-TYPE INEQUALITIES FOR NONLINEAR ELLIPTIC EQUATIONS WITH FIRST-ORDER TERMS AND THEIR APPLICATIONS

JAROSLAV JAROŠ, KUSANO TAKAŜI, AND NORIO YOSHIDA

Received 24 February 2004; Revised 13 April 2004; Accepted 13 April 2004

Picone-type inequalities are established for nonlinear elliptic equations which are generalizations of nonself-adjoint linear elliptic equations, and Sturmian comparison theorems are derived as applications. Oscillation results are also obtained for forced superlinear elliptic equations and superlinear-sublinear elliptic equations.

1. Introduction

Beginning with the work of Picone [11], Picone identity has been investigated by many authors. In particular, we refer the reader to Allegretto [2], Kreith [8], Protter [12], Swanson [13] and the references cited therein for Picone identities and comparison theorems for nonself-adjoint linear elliptic equations.

Recently there has been an increasing interest in studying the forced oscillations of differential equations. We mention the papers [3–7, 10] dealing with forced oscillations of differential equations of self-adjoint type.

In Jaroš et al. [6], they have established Picone-type inequalities which connect the self-adjoint linear elliptic operator

$$p[u] \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + c(x)u \tag{1.1}$$

with the nonlinear elliptic operator

$$\begin{aligned} P[v] &\equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + C(x)|v|^{\beta-1}v, \\ \tilde{P}[v] &\equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + C(x)|v|^{\beta-1}v + D(x)|v|^{\gamma-1}v, \end{aligned} \tag{1.2}$$

Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2006, Article ID 52378, Pages 1–17
DOI 10.1155/JIA/2006/52378

where β and γ are positive constants with $\beta > 1$ and $0 < \gamma < 1$. They have derived Sturmian comparison theorems and oscillation theorems for the forced elliptic equation

$$P[v] = f(x) \tag{1.3}$$

as well as the superlinear-sublinear elliptic equation

$$\tilde{P}[v] = 0. \tag{1.4}$$

The objective of this paper is to extend the results obtained in [6] to the nonlinear elliptic equations with first-order terms

$$L[v] = f(x), \tag{1.5}$$

$$\tilde{L}[v] = 0, \tag{1.6}$$

where

$$\begin{aligned} L[v] &\equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + 2\sum_{i=1}^{n} B_i(x)\frac{\partial v}{\partial x_i} + C(x)|v|^{\beta-1}v, \\ \tilde{L}[v] &\equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + 2\sum_{i=1}^{n} B_i(x)\frac{\partial v}{\partial x_i} + C(x)|v|^{\beta-1}v + D(x)|v|^{\gamma-1}v. \end{aligned} \tag{1.7}$$

We note that if there exists a C^1-function $F(x)$ such that

$$\nabla F(x) = 2B(x)\left(A_{ij}(x)\right)^{-1}, \tag{1.8}$$

where $B(x) = (B_1(x), B_2(x), \dots, B_n(x))$, then (1.5) can be written in the form

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(e^{F(x)}A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + e^{F(x)}C(x)|v|^{\beta-1}v = e^{F(x)}f(x), \tag{1.9}$$

which was studied in [6].

In Section 2 we establish Picone-type inequalities for (1.5), and in Section 3 we obtain oscillation theorems for (1.5) in an unbounded domain $\Omega \subset \mathbb{R}^n$. Sections 4 and 5 concern Sturmian comparison theorems and oscillation theorems for (1.6), respectively.

2. Sturmian comparison theorems for (1.5)

Let G be a bounded domain in $\mathbb{R}^n$ with piecewise smooth boundary ∂G. It is assumed that

(A_1) $A_{ij}(x) \in C(\overline{G};\mathbb{R})$, $B_i(x) \in C(\overline{G};\mathbb{R})$, $C(x) \in C(\overline{G};[0,\infty))$ and $f(x) \in C(\overline{G};\mathbb{R})$;

(A_2) the matrix $(A_{ij}(x))$ is symmetric and positive definite in G;

(A_3) $\beta > 1$.

The domain $\mathcal{D}_L(G)$ of L is defined to be the set of all functions v of class $C^1(\overline{G};\mathbb{R})$ with the property that $A_{ij}(x)(\partial v/\partial x_j) \in C^1(G;\mathbb{R}) \cap C(\overline{G};\mathbb{R})$ $(i,j = 1,2,\dots,n)$.

Theorem 2.1. *If $v \in \mathcal{D}_L(G)$, $v \neq 0$ in G and $v \cdot f(x) \leq 0$ in G, then the following inequality holds for any $u \in C^1(G;\mathbb{R})$:*

$$\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)-\sum_{k=1}^{n}B_k(x)A^{ki}(x)u\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)-\sum_{k=1}^{n}B_k(x)A^{kj}(x)u\right)\\
&\quad+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u^2}{v}A_{ij}(x)\frac{\partial v}{\partial x_j}\right)\\
&\quad\leq \sum_{i,j=1}^{n} A_{ij}(x)\left(\frac{\partial u}{\partial x_i}-\sum_{k=1}^{n}B_k(x)A^{ki}(x)u\right)\left(\frac{\partial u}{\partial x_j}-\sum_{k=1}^{n}B_k(x)A^{kj}(x)u\right)\\
&\quad-\beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\,|\,f(x)\,|^{(\beta-1)/\beta}u^2+\frac{u^2}{v}\{L[v]-f(x)\},
\end{aligned} \tag{2.1}$$

where $(A^{ij}(x)) = (A_{ij}(x))^{-1}$.

Proof. The following Picone-type inequality was established by Jaroš et al. [6]:

$$\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)\right)+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u^2}{v}A_{ij}(x)\frac{\partial v}{\partial x_j}\right)\\
&\quad\leq \sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-\beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\,|\,f(x)\,|^{(\beta-1)/\beta}u^2\\
&\quad+\frac{u^2}{v}\left\{\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right)+C(x)|v|^{\beta-1}v-f(x)\right\}.
\end{aligned} \tag{2.2}$$

Since

$$-2u\sum_{i=1}^{n}B_i(x)v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) = -2u\sum_{i=1}^{n}B_i(x)\frac{\partial u}{\partial x_i}+2\frac{u^2}{v}\sum_{i=1}^{n}B_i(x)\frac{\partial v}{\partial x_i}, \tag{2.3}$$

combining (2.2) with (2.3) yields

$$\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)\right)-2u\sum_{i=1}^{n}B_i(x)v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\\
&\quad+B(x)\left(A_{ij}(x)\right)^{-1}B(x)^T u^2+\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u^2}{v}A_{ij}(x)\frac{\partial v}{\partial x_j}\right)\\
&\quad\leq \sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}-2u\sum_{i=1}^{n}B_i(x)\frac{\partial u}{\partial x_i}+B(x)\left(A_{ij}(x)\right)^{-1}B(x)^T u^2\\
&\quad-\beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\,|\,f(x)\,|^{(\beta-1)/\beta}u^2\\
&\quad+\frac{u^2}{v}\left\{\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right)+2\sum_{i=1}^{n}B_i(x)\frac{\partial v}{\partial x_i}+C(x)|v|^{\beta-1}v-f(x)\right\},
\end{aligned} \tag{2.4}$$

where $B(x) = (B_1(x),\dots,B_n(x))$ and the superscript T denotes the transpose. In view of the identities

$$\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)\right) - 2u\sum_{i=1}^{n} B_i(x)v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\\
&\quad + B(x)\left(A_{ij}(x)\right)^{-1}B(x)^T u^2\\
&\quad = \sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{ki}(x)u\right)\\
&\quad\quad \times\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{kj}(x)u\right),
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - 2u\sum_{i=1}^{n} B_i(x)\frac{\partial u}{\partial x_i} + B(x)\left(A_{ij}(x)\right)^{-1}B(x)^T u^2\\
&\quad = \sum_{i,j=1}^{n} A_{ij}(x)\left(\frac{\partial u}{\partial x_i} - \sum_{k=1}^{n} B_k(x)A^{ki}(x)u\right)\left(\frac{\partial u}{\partial x_j} - \sum_{k=1}^{n} B_k(x)A^{kj}(x)u\right),
\end{aligned} \tag{2.6}$$

we observe that (2.4) is equivalent to (2.1). □

We consider the comparison operator

$$\ell[u] \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + 2\sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u, \tag{2.7}$$

where the coefficients $a_{ij}(x)$, $b_i(x)$, $c(x)$ satisfy the following hypotheses:

(A_4) $a_{ij}(x), b_i(x), c(x) \in C(\overline{G};\mathbb{R})$;

(A_5) the matrix $(a_{ij}(x))$ is symmetric and positive definite in G.

The domain $\mathcal{D}_\ell(G)$ of ℓ is defined to be the set of all functions u of class $C^1(\overline{G};\mathbb{R})$ with the property that $a_{ij}(x)(\partial u/\partial x_j) \in C^1(G;\mathbb{R}) \cap C(\overline{G};\mathbb{R})$ $(i,j = 1,2,\dots,n)$.

Theorem 2.2. *Assume that $u \in \mathcal{D}_\ell(G)$, $v \in \mathcal{D}_L(G)$, $v \neq 0$ in G and $v \cdot f(x) \leq 0$ in G. Then we have the following Picone-type inequality*

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(ua_{ij}(x)\frac{\partial u}{\partial x_j} - \frac{u^2}{v}A_{ij}(x)\frac{\partial v}{\partial x_j}\right)\\
&\quad \geq \sum_{i,j=1}^{n}\left(a_{ij}(x) - A_{ij}(x)\right)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - 2u\sum_{i=1}^{n}\left(b_i(x) - B_i(x)\right)\frac{\partial u}{\partial x_i}\\
&\quad\quad + \left(\beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\left|f(x)\right|^{(\beta-1)/\beta} - c(x) - B(x)\left(A^{ij}(x)\right)B(x)^T\right)u^2\\
&\quad\quad + \sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{ki}(x)u\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{kj}(x)u\right)\\
&\quad\quad + \frac{u}{v}\left\{v\ell[u] - u\left(L[v] - f(x)\right)\right\}.
\end{aligned} \tag{2.8}$$

Proof. To prove the theorem it suffices to combine the inequalities (2.4) and (2.5) with the identity

$$u\ell[u] = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(ua_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + 2u\sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u^2. \tag{2.9}$$

□

Now we consider the first-order partial differential system

$$\nabla w - P(x)w = 0, \tag{2.10}$$

where $P(x) = (P_1(x), P_2(x), \dots, P_n(x))$ is a continuous vector function, and define the sequence of functions $\{q_k(x)\}_{k=1}^{n}$ by

$$\begin{gathered} q_1(x) = \int P_1(x)\,dx_1, \\ q_k(x) = q_{k-1}(x) + \int \left(P_k(x) - \frac{\partial}{\partial x_k} q_{k-1}(x)\right) dx_k \quad (k = 2, 3, \dots, n). \end{gathered} \tag{2.11}$$

Lemma 2.3. *The system (2.10) has a C^1-solution if and only if*

$$\frac{\partial}{\partial x_{k-1}}\left(P_k(x) - \frac{\partial}{\partial x_k} q_{k-1}(x)\right) = 0 \quad (k = 2, 3, \dots, n). \tag{2.12}$$

Then any C^1-solution w of (2.10) can be written in the form

$$w = C_n \exp q_n(x) \tag{2.13}$$

for some constant C_n.

Proof. Suppose that (2.10) has a C^1-solution w. Then we obtain

$$\frac{\partial w}{\partial x_1} - P_1(x)w = 0, \tag{2.14}$$

and hence

$$w = C_1(x_2, \dots, x_n) \exp \int P_1(x)\,dx_1 = C_1(x_2, \dots, x_n) \exp q_1(x) \tag{2.15}$$

for some function $C_1(x_2, \dots, x_n)$. From

$$\frac{\partial w}{\partial x_2} - P_2(x)w = 0 \tag{2.16}$$

we see that $C_1(x_2, \dots, x_n)$ must satisfy

$$\frac{\partial C_1}{\partial x_2} - \left(P_2(x) - \frac{\partial}{\partial x_2}\int P_1(x)\,dx_1\right) C_1 = 0. \tag{2.17}$$

Hence, it is necessary that

$$\frac{\partial}{\partial x_1}\left(P_2(x) - \frac{\partial}{\partial x_2}\int P_1(x)dx_1\right) = 0, \tag{2.18}$$

and we have

$$C_1 = C_2(x_3,\dots,x_n)\exp\int\left(P_2(x) - \frac{\partial}{\partial x_2}\int P_1(x)dx_1\right)dx_2 \tag{2.19}$$

for some function $C_2(x_3,\dots,x_n)$, and therefore

$$\begin{aligned} w &= C_2(x_3,\dots,x_n)\exp\left(\int P_1(x)dx_1 + \int\left(P_2(x) - \frac{\partial}{\partial x_2}\int P_1(x)dx_1\right)dx_2\right) \\ &= C_2(x_3,\dots,x_n)\exp q_2(x). \end{aligned} \tag{2.20}$$

Repeating this procedure, we observe that (2.12) is necessary and the solution w has the form (2.13). From the above consideration it is obvious that the condition (2.12) is sufficient for (2.10) to have a C^1-solution. □

Theorem 2.4. *If there exists a nontrivial function $u \in C^1(\overline{G};\mathbb{R})$ such that $u = 0$ on ∂G and*

$$\begin{aligned} M[u] \equiv \int_G \Bigg[&\sum_{i,j=1}^n A_{ij}(x)\left(\frac{\partial u}{\partial x_i} - \sum_{k=1}^n B_k(x)A^{ki}(x)u\right)\left(\frac{\partial u}{\partial x_j} - \sum_{k=1}^n B_k(x)A^{kj}(x)u\right) \\ &- \beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\,|f(x)|^{(\beta-1)/\beta}u^2\Bigg]dx \le 0, \end{aligned} \tag{2.21}$$

then every solution $v \in \mathfrak{D}_L(G)$ of (1.5) satisfying $v \cdot f(x) \le 0$ in G vanishes at some point of $\overline{G}$. Furthermore, if $\partial G \in C^1$, then either every solution $v \in \mathfrak{D}_L(G)$ of (1.5) satisfying $v \cdot f(x) \le 0$ in G has a zero in G or else $u = C_0 v \exp q(x)$ for some nonzero constant C_0 and some continuous function $q(x)$.

Proof

The first statement. Suppose to the contrary that there exists a solution $v \in \mathfrak{D}_L(G)$ of (1.5) satisfying $v \cdot f(x) \le 0$ in G and $v \ne 0$ on $\overline{G}$. We find that the inequality (2.1) of Theorem 2.1 holds. Integrating (2.1) over G and then using the divergence theorem yield

$$\begin{aligned} M[u] \ge \int_G \sum_{i,j=1}^n A_{ij}(x)&\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) - \sum_{k=1}^n B_k(x)A^{ki}(x)u\right) \\ &\times\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right) - \sum_{k=1}^n B_k(x)A^{kj}(x)u\right)dx. \end{aligned} \tag{2.22}$$

If

$$v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) - \sum_{k=1}^n B_k(x)A^{ki}(x)u \equiv 0 \quad \text{in } G\ (i = 1,2,\dots,n), \tag{2.23}$$

then it follows from Lemma 2.3 that

$$\frac{u}{v} = C_0 \exp q(x) \tag{2.24}$$

in G, by continuity on $\overline{G}$, where C_0 is some constant and $q(x)$ is some continuous function. Since $u = 0$ on ∂G, we see that $C_0 = 0$, which contradicts the fact that u is nontrivial. Therefore, we observe that

$$\nabla\left(\frac{u}{v}\right) - \left(\sum_{k=1}^{n} B_k(x)A^{ki}(x)\right)\left(\frac{u}{v}\right) \not\equiv 0 \quad \text{in } G. \tag{2.25}$$

Hence, we conclude that the right-hand side of (2.22) is positive, and hence $M[u] > 0$. This contradicts the hypothesis (2.21).

The second statement. Next we consider the case where $\partial G \in C^1$. Let $v \in \mathcal{D}_L(G)$ be a solution of (1.5) such that $v \cdot f(x) \leq 0$ in G and $v \neq 0$ in G. Since $\partial G \in C^1$, $u \in C^1(\overline{G};\mathbb{R})$ and $u = 0$ on ∂G, we see that u belongs to the Sobolev space $\overset{\circ}{H}_1(G)$ which is the closure in the norm

$$\|u\| = \|u\|_1 = \left(\int_G \sum_{|\alpha|\leq 1} |D^\alpha u|^2 \, dx\right)^{1/2} \tag{2.26}$$

of the class $C_0^\infty(G)$ of infinitely differentiable functions with compact support in G (see, e.g., Agmon [1, page 131]). Let $\{u_k\}$ be a sequence of functions in $C_0^\infty(G)$ converging to u in the norm (2.26). Then, the inequality (2.1) with $u = u_k$ holds. In view of the fact that (2.22) with $u = u_k$ holds, we find that $M[u_k] \geq 0$. Since

$$\begin{aligned} M[u] = \int_G \Bigg[&\sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - 2u\sum_{i=1}^{n} B_i(x)\frac{\partial u}{\partial x_i} \\ &+ \left(B(x)\left(A_{ij}(x)\right)^{-1}B(x)^T - \beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}|f(x)|^{(\beta-1)/\beta}\right)u^2\Bigg]dx \end{aligned} \tag{2.27}$$

and $A_{ij}(x)$, $B_i(x)$, $B(x)(A_{ij}(x))^{-1}B(x)^T - \beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}|f(x)|^{(\beta-1)/\beta}$ are uniformly bounded in G, there is a constant $K > 0$ such that

$$\begin{aligned} |M[u_k] - M[u]| \leq{}& K\int_G \left|\sum_{i,j=1}^{n}\left(\frac{\partial u_k}{\partial x_i}\frac{\partial(u_k-u)}{\partial x_j} + \frac{\partial(u_k-u)}{\partial x_i}\frac{\partial u}{\partial x_j}\right)\right| dx \\ &+ K\int_G \left|\sum_{i=1}^{n}\left(u_k\frac{\partial(u_k-u)}{\partial x_i} + (u_k-u)\frac{\partial u}{\partial x_i}\right)\right| dx \\ &+ K\int_G |u_k(u_k-u) + (u_k-u)u| \, dx. \end{aligned} \tag{2.28}$$

Application of Schwarz inequality yields

$$|M[u_k] - M[u]| \leq K(n^2+n+1)(\|u_k\| + \|u\|)\|u_k - u\|. \tag{2.29}$$

Since $\lim_{k\to\infty}|u_k - u| = 0$, we see that $\lim_{k\to\infty} M[u_k] = M[u] \geq 0$, and therefore $M[u] = 0$ in view of (2.21). Let B denote an arbitrary ball with $\overline{B} \subset G$ and define

$$J_B[u] \equiv \int_B \sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{ki}(x)u\right) \times \left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{kj}(x)u\right)dx \tag{2.30}$$

for $u \in C^1(G;\mathbb{R})$. We easily see that

$$0 \leq J_B[u_k] \leq M[u_k] \tag{2.31}$$

and that

$$\left|J_B[u_k] - J_B[u]\right| \leq K_1\left(\|w_k\|_B + \|w\|_B\right)\|w_k - w\|_B \tag{2.32}$$

holds, where K_1 is a positive constant, $w_k = u_k/v$, $w = u/v$ and the subscript B indicates the integrals involved in the norm (2.26) are taken over B. As $v \neq 0$ on $\overline{B}$, we observe that $\lim_{k\to\infty}\|w_k - w\|_B = 0$ when $\lim_{k\to\infty}\|u_k - u\| = 0$, and hence $\lim_{k\to\infty} J_B[u_k] = J_B[u]$. Since $\lim_{k\to\infty} M[u_k] = M[u] = 0$, we obtain $\lim_{k\to\infty} J_B[u_k] = J_B[u] = 0$. It follows from Lemma 2.3 that $u/v = C_0 \exp q(x)$ in B, by arbitrariness of B in G, and hence by continuity on $\overline{G}$ for nonzero constant C_0 and some continuous function $q(x)$. This completes the proof of the second statement. □

Corollary 2.5. *Assume that $f(x) \geq 0$ (or $f(x) \leq 0$) in G. If there exists a nontrivial function $u \in C^1(\overline{G};\mathbb{R})$ such that $u = 0$ on ∂G and $M[u] \leq 0$, then (1.5) has no negative (or positive) solution on $\overline{G}$.*

Proof. Let (1.5) have a solution v which is negative (or positive) on $\overline{G}$. Then, it is obvious that $v \cdot f(x) \leq 0$ in G, and hence Theorem 2.4 implies that v must vanish at some point of $\overline{G}$. This is a contradiction and the proof is complete. □

Theorem 2.6. *If there exists a nontrivial solution $u \in \mathfrak{D}_\ell(G)$ of $\ell[u] = 0$ in G such that $u = 0$ on ∂G and*

$$V[u] \equiv \int_G \Bigg[\sum_{i,j=1}^{n}\left(a_{ij}(x) - A_{ij}(x)\right)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - 2u\sum_{i=1}^{n}\left(b_i(x) - B_i(x)\right)\frac{\partial u}{\partial x_i} + \left(\beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\left|f(x)\right|^{(\beta-1)/\beta} - c(x) - B(x)\left(A^{ij}(x)\right)B(x)^T\right)u^2\Bigg]dx \geq 0, \tag{2.33}$$

then every solution $v \in \mathfrak{D}_L(G)$ of (1.5) satisfying $v \cdot f(x) \leq 0$ in G vanishes at some point of $\overline{G}$. Furthermore, if $\partial G \in C^1$, then either every solution $v \in \mathfrak{D}_L(G)$ of (1.5) satisfying $v \cdot f(x) \leq 0$ in G has a zero in G or else $u = C_0 v \exp q(x)$ for some nonzero constant C_0 and some continuous function $q(x)$.

Proof. It suffices to start the inequality (2.8) instead of (2.1) and use the same arguments as in the proof of Theorem 2.4. □

Corollary 2.7. *Assume that $f(x) \geq 0$ (or $f(x) \leq 0$) in G. If there exists a nontrivial solution $u \in \mathcal{D}_\ell(G)$ of $\ell[u] = 0$ in G such that $u = 0$ on ∂G and $V[u] \geq 0$, then (1.5) has no negative (or positive) solution on $\overline{G}$.*

Proof. It is easily verified that

$$V[u] = -\int_G u\ell[u]\,dx - M[u] \tag{2.34}$$

for any $u \in C^1(\overline{G};\mathbb{R})$ satisfying $u = 0$ on ∂G. Hence, we conclude that

$$V[u] = -M[u] \tag{2.35}$$

for the solution u of $\ell[u] = 0$ such that $u = 0$ on ∂G. The conclusion follows from Corollary 2.5. □

Remark 2.8. If $(a_{ij}(x) - A_{ij}(x))$ is positive definite in G and

$$\begin{aligned} &\beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\,|\,f(x)\,|^{(\beta-1)/\beta} \\ &\quad \geq c(x) + B(x)\big(A^{ij}(x)\big)B(x)^T \\ &\qquad + \big(b(x)-B(x)\big)\big(a_{ij}(x)-A_{ij}(x)\big)^{-1}\big(b(x)-B(x)\big)^T, \end{aligned} \tag{2.36}$$

then $V[u] \geq 0$ for any $u \in C^1(\overline{G};\mathbb{R})$, where

$$b(x) - B(x) = \big(b_1(x)-B_1(x), b_2(x)-B_2(x), \dots, b_n(x)-B_n(x)\big). \tag{2.37}$$

In the case where $b_i(x) = B_i(x)$ $(i = 1,2,\dots,n)$, we see that $V[u] \geq 0$ for any $u \in C^1(\overline{G};\mathbb{R})$ if $(a_{ij}(x) - A_{ij}(x))$ is positive semidefinite in G and

$$\beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\,|\,f(x)\,|^{(\beta-1)/\beta} \geq c(x) + B(x)\big(A^{ij}(x)\big)B(x)^T. \tag{2.38}$$

Theorem 2.9. *Suppose that G is divided into two subdomains G_1 and G_2 by an $(n-1)$-dimensional piecewise smooth hypersurface in such a way that*

$$f(x) \geq 0 \quad \text{in } G_1, \qquad f(x) \leq 0 \quad \text{in } G_2. \tag{2.39}$$

If there exist nontrivial functions $u_p \in C^1(\overline{G_p};\mathbb{R})$ $(p = 1,2)$ such that $u_p = 0$ on ∂G_p and

$$\begin{aligned} M_p[u_p] \equiv \int_{G_p} \Bigg[&\sum_{i,j=1}^{n} A_{ij}(x)\left(\frac{\partial u_p}{\partial x_i} - \sum_{k=1}^{n} B_k(x)A^{ki}(x)u_p\right)\left(\frac{\partial u_p}{\partial x_j} - \sum_{k=1}^{n} B_k(x)A^{kj}(x)u_p\right) \\ &- \beta(\beta-1)^{(1-\beta)/\beta}C(x)^{1/\beta}\,|\,f(x)\,|^{(\beta-1)/\beta}u_p^2\Bigg]dx \leq 0, \end{aligned} \tag{2.40}$$

then every solution $v \in \mathcal{D}_L(G)$ of (1.5) has a zero on $\overline{G}$.

Proof. Assume that (1.5) has a solution v which has no zero on $\overline{G}$. Then, either $v < 0$ on $\overline{G}$ or $v > 0$ on $\overline{G}$. If $v < 0$ on $\overline{G}$, then $v < 0$ on $\overline{G_1}$, and therefore $v \cdot f(x) \leq 0$ in G_1. It follows from Corollary 2.5 that (1.5) has no negative solution on $\overline{G_1}$. This is a contradiction. The case where $v > 0$ on $\overline{G}$ can be treated similarly, and we are also led to a contradiction. The proof is complete. □

THEOREM 2.10. *Suppose that G is divided into two adjacent subdomains G_1 and G_2 as mentioned in Theorem 2.9. If there exist nontrivial solutions $u_p \in \mathfrak{D}_\ell(G_p)$ $(p = 1,2)$ of $\ell[u_p] = 0$ in G_p such that $u_p = 0$ on ∂G_p and*

$$V_p[u_p] \equiv \int_{G_p} \Bigg[\sum_{i,j=1}^{n} (a_{ij}(x) - A_{ij}(x)) \frac{\partial u_p}{\partial x_i} \frac{\partial u_p}{\partial x_j} - 2u_p \sum_{i=1}^{n} (b_i(x) - B_i(x)) \frac{\partial u_p}{\partial x_i} + \Big(\beta(\beta-1)^{(1-\beta)/\beta} C(x)^{1/\beta} \left| f(x) \right|^{(\beta-1)/\beta} - c(x) - B(x)(A^{ij}(x))B(x)^T \Big) u_p^2 \Bigg] dx \geq 0, \tag{2.41}$$

then every solution $v \in \mathfrak{D}_L(G)$ of (1.5) has a zero on $\overline{G}$.

Proof. By using the same arguments as in the proof of Theorem 2.9, we conclude that the conclusion follows from Corollary 2.7. □

3. Oscillation theorems for (1.5)

In this section we derive an oscillation criterion for (1.5) in an unbounded domain $\Omega \subset \mathbb{R}^n$. Assume that

(H_1) $A_{ij}(x), A_i(x), C(x), f(x) \in C(\Omega; \mathbb{R})$;

(H_2) the matrix $(A_{ij}(x))$ is symmetric and positive definite in Ω.

The domain $\mathfrak{D}_L(\Omega)$ of L is defined to be the set of all functions v of class $C^1(\Omega; \mathbb{R})$ with the property that $A_{ij}(x)(\partial v/\partial x_j) \in C^1(\Omega; \mathbb{R})$ $(i, j = 1, 2, \ldots, n)$.

Definition 3.1. A function $v : \Omega \to \mathbb{R}$ is said to be *oscillatory* in Ω if v has a zero in Ω_r for any $r > 0$, where

$$\Omega_r = \Omega \cap \{x \in \mathbb{R};\ |x| > r\}. \tag{3.1}$$

THEOREM 3.2. *Assume that for any $r > 0$ there is a bounded domain G in Ω_r with piecewise smooth boundary, which can be divided into two subdomains G_1 and G_2 by an $(n-1)$-dimensional hypersurface in such a way that $f(x) \geq 0$ in G_1 and $f(x) \leq 0$ in G_2. Furthermore, assume that $C(x) \geq 0$ in G and there exist nontrivial functions $u_p \in C^1(\overline{G_p}; \mathbb{R})$ $(p = 1,2)$ such that $u_p = 0$ on ∂G and $M_p[u_p] \leq 0$, where M_p are given by (2.40). Then every solution $v \in \mathfrak{D}_L(\Omega)$ of (1.5) is oscillatory in Ω.*

Proof. We need only to apply Theorem 2.9 to make sure that every solution v has a zero in any domain as mentioned in the hypotheses of Theorem 3.2. □

Example 3.3. We consider the forced superlinear elliptic equation

$$\Delta v + 2\frac{\partial v}{\partial x_1} + 2\frac{\partial v}{\partial x_2} + K(\sin(x_1 - \pi)\sin x_2)|v|^{\beta-1}v = \cos x_1 \sin x_2, \quad (x_1, x_2) \in \Omega, \quad (3.2)$$

where $K > 0$ is a constant, Δ is the two-dimensional Laplacian, and Ω is an unbounded domain in $\mathbb{R}^2$ containing a horizontal strip such that

$$[\pi, \infty) \times [0, \pi] \subset \Omega. \quad (3.3)$$

Let m be any fixed natural number, and consider the square

$$G = ((2m-1)\pi, 2m\pi) \times (0, \pi), \quad (3.4)$$

which is divided into two subdomains

$$\begin{aligned} G_1 &= ((2m-1)\pi, (2m-(1/2))\pi) \times (0, \pi), \\ G_1 &= ((2m-(1/2))\pi, 2m\pi) \times (0, \pi) \end{aligned} \quad (3.5)$$

by the vertical line $x_1 = (2m - (1/2))\pi$. It is easy to see that $C(x) = K\sin(x_1 - \pi)\sin x_2 \geq 0$ in G, $f(x) = \cos x_1 \sin x_2 \leq 0$ in G_1 and $f(x) \geq 0$ in G_2. Letting $u_p = \sin 2x_1 \sin x_2$ ($p = 1, 2$), we observe that $u_p = 0$ on ∂G_p. An easy calculation shows that

$$\begin{aligned} M_p[u_p] = \int_{G_p} \Bigg[\sum_{i=1}^{2} \left(\frac{\partial u_p}{\partial x_i} - u_p \right)^2 &- \beta(\beta-1)^{(1-\beta)/\beta} (K(\sin(x_1-\pi)\sin x_2))^{1/\beta} \\ &\times |\cos x_1 \sin x_2|^{(\beta-1)\beta} u_p^2 \Bigg] dx_1\, dx_2 \\ = \frac{7}{8}\pi^2 - \frac{8}{3} K^{1/\beta} \beta(\beta-1)^{(1-\beta)/\beta} B\left(\frac{3}{2} + \frac{1}{2\beta}, 2 - \frac{1}{2\beta}\right)&, \end{aligned} \quad (3.6)$$

where $B(s, t)$ denotes the beta function. Hence, we find that $M_p[u_p] \leq 0$ ($p = 1, 2$) if $K > 0$ is chosen so large that

$$K \geq \left[\frac{21}{64}\pi^2 \cdot \left(\beta(\beta-1)^{(1-\beta)/\beta} B\left(\frac{3}{2} + \frac{1}{2\beta}, 2 - \frac{1}{2\beta}\right)\right)^{-1} \right]^{\beta}. \quad (3.7)$$

It follows from Theorem 3.2 that every solution $v \in C^2(\Omega; \mathbb{R})$ of (3.2) is oscillatory in Ω for all sufficiently large $K > 0$.

4. Sturmian comparison theorems for (1.6)

We deal with the elliptic equation (1.6) and establish Picone-type inequalities for (1.6). Sturmian comparison theorems for (1.6) are derived by using the Picone-type inequalities.

We assume that the coefficients $A_{ij}(x)$, $B_i(x)$, $C(x)$, $D(x)$ and the constants β, γ appearing in (1.6) satisfy the following:

($\tilde{\text{A}}_1$) $A_{ij}(x) \in C(\overline{G};\mathbb{R})$, $B_i(x) \in C(\overline{G};\mathbb{R})$, $C(x) \in C(\overline{G};[0,\infty))$ and $D(x) \in C(\overline{G};[0,\infty))$;

($\tilde{\text{A}}_2$) the matrix $(A_{ij}(x))$ is symmetric and positive definite in G;

($\tilde{\text{A}}_3$) $\beta > 1$ and $0 < \gamma < 1$.

The domain $\mathcal{D}_{\tilde{L}}(G)$ of $\tilde{L}$ is defined to be the same as that of L, that is, $\mathcal{D}_{\tilde{L}}(G) = \mathcal{D}_L(G)$.

Theorem 4.1. *If $v \in \mathcal{D}_{\tilde{L}}(G)$ and $v \neq 0$ in G, then the following inequality holds for any $u \in C^1(G;\mathbb{R})$:*

$$
\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{ki}(x)u\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right) - \sum_{k=1}^{n} B_k(x)A^{kj}(x)u\right)\\
&\qquad + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u^2}{v}A_{ij}(x)\frac{\partial v}{\partial x_j}\right)\\
&\qquad \le \sum_{i,j=1}^{n} A_{ij}(x)\left(\frac{\partial u}{\partial x_i} - \sum_{k=1}^{n} B_k(x)A^{ki}(x)u\right)\left(\frac{\partial u}{\partial x_j} - \sum_{k=1}^{n} B_k(x)A^{kj}(x)u\right)\\
&\qquad - \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)}D(x)^{(\beta-1)/(\beta-\gamma)}u^2 + \frac{u^2}{v}\tilde{L}[v].
\end{aligned}
\tag{4.1}
$$

Proof. Starting with the following inequality

$$
\begin{aligned}
&\sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right)\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right)\right) + \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{u^2}{v}A_{ij}(x)\frac{\partial v}{\partial x_j}\right)\\
&\qquad \le \sum_{i,j=1}^{n} A_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)}\\
&\qquad \times D(x)^{(\beta-1)/(\beta-\gamma)}u^2\\
&\qquad + \frac{u^2}{v}\left\{\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial v}{\partial x_j}\right) + C(x)|v|^{\beta-1}v + D(x)|v|^{\gamma-1}v\right\},
\end{aligned}
\tag{4.2}
$$

which was established by Jaroš et al. [6, Theorem 7], and proceeding as in the proof of Theorem 2.1, we find that the inequality (4.1) holds. □

Theorem 4.2. *Assume that $u \in \mathfrak{D}_\ell(G)$, $v \in \mathfrak{D}_{\tilde{L}}(G)$ and $v \neq 0$ in G. Then we have the following Picone-type inequality:*

$$\begin{aligned}
&\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(u a_{ij}(x)\frac{\partial u}{\partial x_j} - \frac{u^2}{v}A_{ij}(x)\frac{\partial v}{\partial x_j}\right)\\
&\quad \geq \sum_{i,j=1}^{n}\left(a_{ij}(x)-A_{ij}(x)\right)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - 2u\sum_{i=1}^{n}\left(b_i(x)-B_i(x)\right)\frac{\partial u}{\partial x_i}\\
&\qquad + \left(\frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)}D(x)^{(\beta-1)/(\beta-\gamma)}\right.\\
&\qquad\quad \left. - c(x) - B(x)\left(A^{ij}(x)\right)B(x)^T\right)u^2\\
&\qquad + \sum_{i,j=1}^{n} A_{ij}(x)\left(v\frac{\partial}{\partial x_i}\left(\frac{u}{v}\right) - \sum_{k=1}^{n}B_k(x)A^{ki}(x)u\right)\left(v\frac{\partial}{\partial x_j}\left(\frac{u}{v}\right) - \sum_{k=1}^{n}B_k(x)A^{kj}(x)u\right)\\
&\qquad + \frac{u}{v}\left(v\ell[u] - u\tilde{L}[v]\right).
\end{aligned} \tag{4.3}$$

Proof. Arguing as in the proof of Theorem 2.2, we observe that the conclusion follows from (4.1). □

Theorem 4.3. *If there exists a nontrivial function $u \in C^1(\overline{G};\mathbb{R})$ such that $u = 0$ on ∂G and*

$$\begin{aligned}
\tilde{M}[u] \equiv \int_G \Bigg[&\sum_{i,j=1}^{n} A_{ij}(x)\left(\frac{\partial u}{\partial x_i} - \sum_{k=1}^{n}B_k(x)A^{ki}(x)u\right)\left(\frac{\partial u}{\partial x_j} - \sum_{k=1}^{n}B_k(x)A^{kj}(x)u\right)\\
&- \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)}D(x)^{(\beta-1)/(\beta-\gamma)}u^2\Bigg]dx \leq 0,
\end{aligned} \tag{4.4}$$

then every solution $v \in \mathfrak{D}_{\tilde{L}}(G)$ of (1.6) vanishes at some point of $\overline{G}$. Furthermore, if $\partial G \in C^1$, then either every solution $v \in \mathfrak{D}_{\tilde{L}}(G)$ of (1.6) has a zero in G or else $u = C_0 v \exp q(x)$ for some nonzero constant C_0 and some continuous function $q(x)$.

Proof. Suppose that there is a solution v of (1.6) such that $v \neq 0$ on $\overline{G}$. Then, the inequality (4.1) of Theorem 4.1 holds for the nontrivial function u. Integrating (4.1) over G and proceeding as in the proof of Theorem 2.4 yield the conclusion $\tilde{M}[u] > 0$, which contradicts the hypothesis (4.4). This completes the proof of the first statement. Next we consider the case where $\partial G \in C^1$. Let v be a solution of (1.6) satisfying $v \neq 0$ in G. Using the same arguments as in the proof of Theorem 2.4, we see that $\tilde{M}[u] = 0$, which implies that $u = C_0 v \exp q(x)$ for some nonzero constant C_0 and some continuous function $q(x)$. This completes the proof of the second statement. □

Theorem 4.4. *If there exists a nontrivial solution $u \in \mathcal{D}_\ell(G)$ of $\ell[u] = 0$ in G such that $u = 0$ on ∂G and*

$$\begin{aligned}\tilde{V}[u] \equiv \int_G \Bigg[& \sum_{i,j=1}^{n} \big(a_{ij}(x) - A_{ij}(x)\big) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} - 2u \sum_{i=1}^{n} \big(b_i(x) - B_i(x)\big) \frac{\partial u}{\partial x_i} \\ & + \bigg(\frac{\beta - \gamma}{1 - \gamma} \Big(\frac{\beta - 1}{1 - \gamma} \Big)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} \\ & - c(x) - B(x)\big(A^{ij}(x)\big)B(x)^T \bigg) u^2 \Bigg] dx \geq 0,\end{aligned} \tag{4.5}$$

then every solution $v \in \mathcal{D}_{\tilde{L}}(G)$ of (1.6) vanishes at some point of $\overline{G}$. Furthermore, if $\partial G \in C^1$, then either every solution $v \in \mathcal{D}_{\tilde{L}}(G)$ of (1.6) has a zero in G or else $u = C_0 v \exp q(x)$ for some nonzero constant C_0 and some continuous function $q(x)$.

Proof. The proof follows by using the same arguments as in Theorem 2.6. □

Remark 4.5. In the case where $b_i(x) = 0$ $(i = 1,2,\dots,n)$ and $B_i(x) \in C^1(\overline{G};\mathbb{R})$ $(i = 1,2,\dots, n)$, it can be shown that $\tilde{V}[u] \geq 0$ for any $u \in C^1(\overline{G};\mathbb{R})$ if $(a_{ij}(x) - A_{ij}(x))$ is positive semidefinite in G and

$$\begin{aligned}& \frac{\beta - \gamma}{1 - \gamma} \Big(\frac{\beta - 1}{1 - \gamma} \Big)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} \\ & \qquad \geq c(x) + \nabla \cdot B(x) + B(x)\big(A^{ij}(x)\big)B(x)^T \quad \text{in } G.\end{aligned} \tag{4.6}$$

5. Oscillation theorems for (1.6)

Now we establish oscillation criteria for (1.6) in an unbounded domain $\Omega \subset \mathbb{R}^n$. It is assumed that

($\tilde{\mathrm{H}}_1$) $A_{ij}(x) \in C(\Omega;\mathbb{R})$ and the matrix $(A_{ij}(x))$ is symmetric and positive definite in Ω; and the same is true of $a_{ij}(x)$;

($\tilde{\mathrm{H}}_2$) $B_i(x) \in C^1(\Omega;\mathbb{R})$, $C(x) \in C(\Omega;[0,\infty))$, $D(x) \in C(\Omega;[0,\infty))$ and $b_i(x)$, $c(x) \in C(\Omega;\mathbb{R})$;

($\tilde{\mathrm{H}}_3$) $\beta > 1$ and $0 < \gamma < 1$.

The domain $\mathcal{D}_{\tilde{L}}(\Omega)$ of $\tilde{L}$ is defined to be the same as that of L, that is, $\mathcal{D}_{\tilde{L}}(\Omega) = \mathcal{D}_L(\Omega)$. The domain $\mathcal{D}_\ell(\Omega)$ of ℓ is defined similarly.

Definition 5.1. A bounded domain G with $\overline{G} \subset \Omega$ is said to be a *nodal domain* for $\ell[u] = 0$ if there is a nontrivial function $u \in \mathcal{D}_\ell(G)$ such that $\ell[u] = 0$ in G and $u = 0$ on ∂G. The equation $\ell[u] = 0$ is called *nodally oscillatory* in Ω if it has a nodal domain contained in Ω_r for any $r > 0$.

Theorem 5.2. *Let $b_i(x) = 0$ $(i = 1,2,\dots,n)$, and assume that*

$$\big(a_{ij}(x) - A_{ij}(x)\big) \text{ is positive semidefinite in } \Omega, \tag{5.1}$$

$$\begin{aligned} c(x) \le{}& \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} \\ & - \nabla\cdot B(x) - B(x)\big(A^{ij}(x)\big)B(x)^T \quad \text{in } \Omega. \end{aligned} \tag{5.2}$$

Every solution $v \in \mathcal{D}_{\tilde{L}}(\Omega)$ of (1.6) is oscillatory in Ω if $\ell[u] = 0$ is nodally oscillatory in Ω.

Proof. Since $\ell[u] = 0$ is nodally oscillatory in Ω, there exists a nodal domain $G \subset \Omega_r$ for any $r > 0$, and therefore there is a nontrivial solution u of $\ell[u] = 0$ in G such that $u = 0$ on ∂G. It follows from the hypotheses (5.1) and (5.2) that $\tilde{V}[u] \ge 0$. Theorem 4.4 implies that every solution $v \in \mathcal{D}_{\tilde{L}}(\Omega)$ of (1.6) must vanish at some point of $\overline{G}$, that is, v has a zero in Ω_r for any $r > 0$. This implies that v is oscillatory in Ω. □

The following corollary is an immediate consequence of Theorem 5.2.

Corollary 5.3. *If the elliptic equation*

$$\Delta u + \left(\frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} - \nabla\cdot B(x) - \big|B(x)\big|^2\right) u = 0 \tag{5.3}$$

is nodally oscillatory in Ω, then every solution $v \in C^2(\Omega;\mathbb{R})$ of

$$\Delta v + 2\sum_{i=1}^{n} B_i(x)\frac{\partial v}{\partial x_i} + C(x)|v|^{\beta-1}v + D(x)|v|^{\gamma-1}v = 0 \tag{5.4}$$

is oscillatory in Ω.

Various nodal oscillation criteria for

$$\Delta u + d(x)u = 0, \quad x \in \mathbb{R}^n \tag{5.5}$$

have been obtained by Kreith and Travis [9]. They have shown that (5.5) is nodally oscillatory in $\mathbb{R}^n$ if

$$\begin{aligned} \int_{\mathbb{R}^2} d(x)\,dx = \infty \quad (n = 2), \\ \int^{\infty} S[d(x)](r)\,dr = \infty \quad (n \ge 3), \end{aligned} \tag{5.6}$$

where $S[d(x)](r)$ denotes the spherical mean of $d(x)$ over the sphere $\{x \in \mathbb{R}^n;\ |x| = r\}$.

Corollary 5.4. *Let $\Omega = \mathbb{R}^n$ and assume that*

$$\begin{aligned} \int_{\mathbb{R}^2} \Psi(x)\,dx = \infty \quad (n = 2), \\ \int^{\infty} S[\Psi(x)](r)\,dr = \infty \quad (n \ge 3), \end{aligned} \tag{5.7}$$

where

$$\Psi(x) = \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} C(x)^{(1-\gamma)/(\beta-\gamma)} D(x)^{(\beta-1)/(\beta-\gamma)} - \nabla\cdot B(x) - \left|B(x)\right|^2. \tag{5.8}$$

Then every solution $v \in C^2(\mathbb{R}^n;\mathbb{R})$ of (5.4) is oscillatory in $\mathbb{R}^n$.

Proof. The conclusion follows by combining the oscillation results due to Kreith and Travis [9] with Corollary 5.3. □

Corollary 5.5. *Let $\Omega = \mathbb{R}^n$ and assume that there are positive constants k_0, k_i $(i = 1,2,\dots, n)$ such that*

$$C(x) \geq k_0, \quad D(x) \geq k_0, \quad B_i(x) = k_i \quad (i = 1,2,\dots,n). \tag{5.9}$$

If

$$\frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} k_0 > k_1^2 + \cdots + k_n^2, \tag{5.10}$$

then every solution $v \in C^2(\mathbb{R}^n;\mathbb{R})$ of (5.4) is oscillatory in $\mathbb{R}^n$.

Proof. Since

$$\Psi(x) \geq \frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} k_0 - \left(k_1^2 + \cdots + k_n^2\right) > 0, \tag{5.11}$$

we find that the hypotheses of Corollary 5.4 are satisfied, and consequently the conclusion follows from Corollary 5.4. □

Example 5.6. We consider the elliptic equation

$$\Delta u + 4\frac{\partial v}{\partial x_1} + 2\frac{\partial v}{\partial x_2} + 4|v|^2 v + 5|v|^{-1/2} v = 0 \quad \text{in } \mathbb{R}^2. \tag{5.12}$$

Here $n = 2$, $k_1 = 2$, $k_2 = 1$, $k_0 = 4$, $\beta = 3$, and $\gamma = 1/2$. It is easily seen that

$$\frac{\beta-\gamma}{1-\gamma}\left(\frac{\beta-1}{1-\gamma}\right)^{(1-\beta)/(\beta-\gamma)} k_0 = 5\cdot 2^{2/5}, \qquad k_1^2 + k_2^2 = 5. \tag{5.13}$$

From Corollary 5.5 it follows that every solution $v \in C^2(\mathbb{R}^2;\mathbb{R})$ of (5.12) is oscillatory in $\mathbb{R}^2$.

References

[1] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand Mathematical Studies, no. 2, D. Van Nostrand, New Jersey, 1965.

[2] W. Allegretto, *A comparison theorem for nonlinear operators*, Annali della Scuola Normale Superiore di Pisa. Seris III **25** (1971), 41–46.

[3] M. A. El-Sayed, *An oscillation criterion for a forced second order linear differential equation*, Proceedings of the American Mathematical Society **118** (1993), no. 3, 813–817.

[4] J. Jaroš and T. Kusano, *Second-order semilinear differential equations with external forcing terms*, Sūrikaisekikenkyūsho Kōkyūroku (1997), no. 984, 191–197 (Japanese).

[5] J. Jaroš, T. Kusano, and N. Yoshida, *Forced superlinear oscillations via Picone's identity*, Acta Mathematica Universitatis Comenianae. New Series **69** (2000), no. 1, 107–113.

[6] ______, *Picone-type inequalities for nonlinear elliptic equations and their applications*, Journal of Inequalities and Applications **6** (2001), no. 4, 387–404.

[7] ______, *Generalized Picone's formula and forced oscillations in quasilinear differential equations of the second order*, Universitatis Masarykianae Brunensis. Facultas Scientiarum Naturalium. Archivum Mathematicum **38** (2002), no. 1, 53–59.

[8] K. Kreith, *A comparison theorem for general elliptic equations with mixed boundary conditions*, Journal of Differential Equations **8** (1970), 537–541.

[9] K. Kreith and C. C. Travis, *Oscillation criteria for selfadjoint elliptic equations*, Pacific Journal of Mathematics **41** (1972), 743–753.

[10] A. H. Nasr, *Sufficient conditions for the oscillation of forced super-linear second order differential equations with oscillatory potential*, Proceedings of the American Mathematical Society **126** (1998), no. 1, 123–125.

[11] M. Picone, *Sui valori eccezionali di un parametro da cui dipende un'equazione differenziale lineare ordinaria del second'ordine*, Annali della Scuola Normale Superiore di Pisa **11** (1909), 1–141 (Italian).

[12] M. H. Protter, *A comparison theorem for elliptic equations*, Proceedings of the American Mathematical Society **10** (1959), 296–299.

[13] C. A. Swanson, *A comparison theorem for elliptic differential equations*, Proceedings of the American Mathematical Society **17** (1966), 611–616.

Jaroslav Jaroš: Department of Mathematical Analysis, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovak Republic
E-mail address: jaros@fmph.uniba.sk

Kusano Takaŝi: Department of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka 814-0180, Japan
E-mail address: tkusano@cis.fukuok-au.ac.jp

Norio Yoshida: Department of Mathematics, Faculty of Science, University of Toyama, Toyama 930-8555, Japan
E-mail address: nori@sci.toyama-u.ac.jp