EXPLICIT BOUNDS OF COMPLEX EXPONENTIAL FRAMES

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We discuss the stability of complex exponential frames $\{e^{i\lambda_n x}\}$ in $L^2(-\gamma,\gamma)$, $\gamma > 0$. Specifically, we improve the 1/4-theorem and obtain explicit upper and lower bounds for some complex exponential frames perturbed along the real and imaginary axes, respectively. Two examples are given to show that the bounds are best possible. In addition, the growth of the entire functions of exponential type γ ($\gamma > \pi$) on the integer sequence is estimated.

1. Introduction

Complex exponentials capable of function reconstruction can be derived from various sources and they may serve as a Riesz basis, or provide series representations such as the Fourier series. As natural generations of Riesz bases by allowing redundancies, frames provide another powerful reconstruction approach. Suppose $\{\lambda_n\}$, $n \in \mathbb{Z}$, is a sequence of distinct complex numbers. We say that the set of exponential functions $\{e^{i\lambda_n t}\}$ is a *frame* over an interval $(-\gamma,\gamma)$ if there exist positive constants A and B, which depend exclusively on γ and the set of functions $\{e^{i\lambda_n t}\}$, such that

$$A \leq \frac{\sum_n \left| \int_{-\gamma}^{\gamma} g(t) e^{i\lambda_n t} dt \right|^2}{\int_{-\gamma}^{\gamma} |g(t)|^2 dt} \leq B \tag{1.1}$$

for every function $g(t) \in L^2(-\gamma,\gamma)$, where $n \in \mathbb{Z}$. In this case, $\{\lambda_n\}$ is called a *frame sequence* and A and B are called the *bounds* of the frame. If $A = B$, the frame is called *tight* and if $A = B = 1$, it is called a *Parseval* frame.

The Paley-Wiener space P is the Hilbert space of all entire functions of exponential type at most π that are square integrable on the real axis. The inner product on P is given by $(f,g) = \int_{-\infty}^{\infty} f(x)\bar{g}(x)dx$ for $f,g \in P$. From Paley-Wiener theorem, P is isometrically isomorphic to $L^2[-\pi,\pi]$, that is, for each $f \in P$, there is a function $\phi \in L^2[-\pi,\pi]$ such

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that $f(z) = (1/\sqrt{2\pi})\int_{-\pi}^{\pi}\phi(t)e^{izt}dt$ and $\|f\| = \int_{-\pi}^{\pi}|\phi|^2 d\phi$. Consequently, the frame condition (1.1) is equivalent to

$$\widetilde{A}\|f\| \le \sum_n |f(\lambda_n)|^2 \le \widetilde{B}\|f\| \tag{1.2}$$

for any function $f \in P$, where $\widetilde{A} = A/2\pi$ and $\widetilde{B} = B/2\pi$.

An optimal estimation of the bounds of a frame is important in many frame applications, and they often play a decisive role in speeding the convergence of reconstruction algorithms. For example, when $|\lambda_n - n| \le \delta < L$, good estimates for the lower and upper bounds of an exponential frame can be obtained in terms of L (see Theorem 1.1 below).

It was shown by Paley and Wiener that $e^{i\lambda_n t}$ is a Riesz basis if $L = 1/\pi^2$. This was later shown to hold for $L = \ln 2/\pi$ by Duffin and Eachus [5, page 43] and then for $L = 1/4$ by Kadec (see [11, page 38]). For exponential frames, a similar result independently obtained by Balan [1] and Christensen [4] can be stated as follows.

Theorem 1.1. *Suppose* $\{e^{i\lambda_n t}\}$ *is a frame for* $L^2(-\gamma,\gamma)$ *with bounds A, B, where* $\{\lambda_n\}$ *are real. Set*

$$L(\gamma) = \frac{\pi}{4\gamma} - \frac{1}{\gamma}\arcsin\left\{\frac{1}{\sqrt{2}}\left(1 - \sqrt{\frac{A}{B}}\right)\right\}. \tag{1.3}$$

If the real sequence $\{\mu_n\}$ *satisfies* $|\mu_n - \lambda_n| \le \delta < L(\gamma)$, *then* $\{e^{i\mu_n t}\}$ *is a frame for* $L^2(-\gamma,\gamma)$ *with bounds*

$$A\left\{1 - \sqrt{\frac{A}{B}}(1 - \cos\gamma\delta + \sin\gamma\delta)\right\}^2, \qquad B(2 - \cos\gamma\delta + \sin\gamma\delta)^2. \tag{1.4}$$

Since $L(\gamma) > L_0(\gamma) = (1/\gamma)\ln(1 + \sqrt{A/B})$, Theorem 1.1 is an improvement of the earlier result of Duffin and Schaeffer [6] where the variation of the sequence $\{\lambda_n\}$ was shown to be bounded by $L_0(\gamma)$. It also extends Kadec's 1/4-theorem from Riesz bases to frames. This result has been employed in the construction of the solution space of some Sturm-Liouville equations [7].

It follows from a result of Verblunsky (see [10] and [3]) that after rescaling, the imaginary parts of the characteristic roots of the delay-differential equation $y'(t) = ay(t-1)$ tend to 1/4. So if the value of the above $L(\gamma)$ could be enlarged, more characteristic roots would satisfy the condition on the frame sequence in Theorem 1.1, which could give a better approximation to the solution of the delay-differential equation in a finite-dimensional Hilbert space. Interested readers may refer to [2] for details. Motivated by this consideration, we will improve Theorem 1.1 and evaluate the bounds of complex exponential frames perturbed along the real and imaginary axes, respectively.

2. Explicit bounds

Theorem 2.1. *Suppose* $\{\lambda_n\}$ *is a frame sequence of real numbers for* $L^2(-\pi,\pi)$ *with bounds A, B. Let* $\{\rho_n\}$ *be a real sequence satisfying* $0 < \theta \le |\rho_n - \lambda_n| \le \delta$, *and let* $\sigma \ge 0$ *satisfy*

Table 2.1

A/B	θ	L_0	L	$\widetilde{L}$
0.76	0.20	0.1995	0.2211	0.2234
0.15	0.10	0.1042	0.1074	0.1099

$(1+\sigma)(\sin\pi\theta/\pi\theta) < 1$. *Then* $\{e^{i\rho_n t}\}$ *is a frame over* $L^2(-\pi,\pi)$ *with bounds*

$$A\left\{1-\sqrt{\frac{A}{B}}(1-\cos\pi\delta+\sin\pi\delta)-\frac{\sigma}{1+\sigma}\left(1-\sqrt{\frac{A}{B}}\right)\right\}^2, \qquad B\left\{1+(1-\cos\pi\delta+\sin\pi\delta)\frac{1+\sigma}{1-\sigma}\right\}^2 \tag{2.1}$$

provided that δ *satisfies*

$$\delta < \widetilde{L} = \frac{1}{4}-\frac{1}{\pi}\arcsin\left\{\frac{1}{(1+\sigma)\sqrt{2}}\left(1-\sqrt{\frac{A}{B}}\right)\right\}. \tag{2.2}$$

Theorem 2.1 shows that $L(\gamma)$ obtained in Theorem 1.1 is not optimal if $A \neq B$. Table 2.1 shows the numeric differences among L_0, L, and $\widetilde{L}$, defined in [6], Theorems 1.1 and 2.1, respectively.

Before proving Theorem 2.1, we first introduce a perturbation theorem given in [4] for general frames.

Theorem 2.2. *Let* $\{f_i\}_{i=1}^{\infty}$ *be a frame for a Hilbert space* H *with bounds* A, B. *Let* $\{g_i\}_{i=1}^{\infty}$ *be a sequence in* H. *Assume there exist nonnegative constants* μ_1, μ_2, *and* μ *such that* $\max(\mu_1+\mu/\sqrt{A},\mu_2) < 1$, *and*

$$\left\|\sum_{i=1}^{n} c_i(f_i-g_i)\right\| \le \mu_1\left\|\sum_{i=1}^{n} c_i f_i\right\| + \mu_2\left\|\sum_{i=1}^{n} c_i g_i\right\| + \mu\left(\sum_{i=1}^{n}|c_i|^2\right)^{1/2} \tag{2.3}$$

for all $c_1,c_2,\dots,c_n$. *Then* $\{g_i\}_{i=1}^{\infty}$ *is a frame with bounds*

$$A\left(1-\frac{\mu_1+\mu_2+\mu/\sqrt{A}}{1+\mu_2}\right)^2, \qquad B\left(1+\frac{\mu_1+\mu_2+\mu/\sqrt{B}}{1-\mu_2}\right)^2. \tag{2.4}$$

Proof of Theorem 2.1. Let $n \in \mathbb{N}$ and $c_k \in \mathbb{C}$, $k = 1,2,\dots,n$, be arbitrary. Set $\delta_k = \rho_k - \lambda_k$, and set

$$U = \left\|\sum_{k=1}^{n} c_k\left(e^{i\rho_k x}-e^{i\lambda_k x}\right)\right\|. \tag{2.5}$$

The conditions on δ and σ imply that $\sigma \in [0,1)$. Consequently,

$$U \le \left\|\sum_{k=1}^{n} c_k e^{i\lambda_k x}\left(1-(1+\sigma)e^{i\delta_k x}\right)\right\| + \sigma\left\|\sum_{k=1}^{n} c_k e^{i\rho_k x}\right\|. \tag{2.6}$$

Expanding $1-(1+\sigma)e^{i\delta_k x}$ in the system $\{1, \cos nx, \sin(n-1/2)x\}_{n=1}^{\infty}$, we obtain

$$
\begin{aligned}
1-(1+\sigma)e^{i\delta_k x} = {} & \left(1-(1+\sigma)\frac{\sin\pi\delta_k}{\pi\delta_k}\right)+(1+\sigma)\sum_{\tau=1}^{\infty}\frac{(-1)^{\tau}2\delta_k\sin\pi\delta_k}{\pi(\tau^2-\delta_k^2)}\cos(\tau x) \\
& +(1+\sigma)i\sum_{\tau=1}^{\infty}\frac{(-1)^{\tau}2\delta_k\cos\pi\delta_k}{\pi\left((\tau-1/2)^2-\delta_k^2\right)}\sin\left\{\left(\tau-\frac{1}{2}\right)x\right\}.
\end{aligned}
\tag{2.7}
$$

Since $\|\cos(\tau x)\phi(x)\| \le \|\phi\|$ and $\|\sin\{(\tau-1/2)x\}\phi(x)\| \le \|\phi\|$, it follows that

$$
\begin{aligned}
U \le {} & \left\|\sum_{k=1}^{n}\left\{1-(1+\sigma)\frac{\sin\pi\delta_k}{\pi\delta_k}\right\}c_k e^{i\lambda_k x}\right\|+(1+\sigma)\sum_{\tau=1}^{\infty}\left\|\sum_{k=1}^{n}\frac{2\delta_k\sin\pi\delta_k}{\pi(\tau^2-\delta_k^2)}c_k e^{i\lambda_k x}\right\| \\
& +(1+\sigma)\sum_{\tau=1}^{\infty}\left\|\sum_{k=1}^{n}\frac{2\delta_k\cos\pi\delta_k}{\pi((\tau-1/2)^2-\delta_k^2)}c_k e^{i\lambda_k x}\right\|+\sigma\left\|\sum_{k=1}^{n}c_k e^{i\rho_k x}\right\|.
\end{aligned}
\tag{2.8}
$$

Since σ satisfies $1+\sigma<\pi\theta/\sin\pi\theta$, then we have

$$
\begin{gathered}
\left|1-(1+\sigma)\frac{\sin\pi\delta_k}{\pi\delta_k}\right| \le 1-(1+\sigma)\frac{\sin\pi\delta}{\pi\delta}, \\
\left|\frac{2\delta_k\sin\pi\delta_k}{\pi(\tau^2-\delta_k^2)}\right| \le \frac{2\delta\sin\pi\delta}{\pi(\tau^2-\delta^2)}, \\
\left|\frac{2\delta_k\cos\pi\delta_k}{\pi((\tau-1/2)^2-\delta_k^2)}\right| \le \frac{2\delta\cos\pi\delta}{\pi((\tau-1/2)^2-\delta^2)}.
\end{gathered}
\tag{2.9}
$$

Considering that

$$
\left\|\sum_{k=1}^{n}a_k c_k e^{i\lambda_k x}\right\| \le \sqrt{B}\left(\sum_{k=1}^{n}\left\|a_k c_k\right\|^2\right)^{1/2} \le \sqrt{B}\sup|a_k|\left(\sum_{k=1}^{n}\left\|c_k\right\|^2\right)^{1/2},
\tag{2.10}
$$

we obtain

$$
\begin{aligned}
U \le {} & \sqrt{B}\left\{1-(1+\sigma)\frac{\sin\pi\delta}{\pi\delta}+(1+\sigma)\sum_{\tau=1}^{\infty}\frac{2\delta\sin\pi\delta}{\pi(\tau^2-\delta^2)}\right. \\
& \left.+(1+\sigma)\sum_{\tau=1}^{\infty}\frac{2\delta\cos\pi\delta}{\pi((\tau-1/2)^2-\delta^2)}\right\}\left(\sum_{k=1}^{n}\left\|c_k\right\|^2\right)^{1/2}+\sigma\left\|\sum_{k=1}^{n}c_k e^{i\rho_k x}\right\| \\
= {} & \sqrt{B}\left\{1-(1+\sigma)\frac{\sin\pi\delta}{\pi\delta}+(1+\sigma)\sin\pi\delta\left(\frac{1}{\pi\delta}-\cot\pi\delta\right)\right. \\
& \left.+(1+\sigma)\cos\pi\delta\tan\pi\delta\right\}\left(\sum_{k=1}^{n}\left\|c_k\right\|^2\right)^{1/2}+\sigma\left\|\sum_{k=1}^{n}c_k e^{i\rho_k x}\right\| \\
= {} & \sqrt{B}\{1+(1+\sigma)(\sin\pi\delta-\cos\pi\delta)\}\left(\sum_{k=1}^{n}\left\|c_k\right\|^2\right)^{1/2}+\sigma\left\|\sum_{k=1}^{n}c_k e^{i\rho_k x}\right\|,
\end{aligned}
\tag{2.11}
$$

which implies that $1+(1+\sigma)(\sin\pi\delta-\cos\pi\delta)>0$. Now assuming $\mu_1=0$, $\mu_2=\sigma$, and $\mu=\sqrt{B}\{1+(1+\sigma)(\sin\pi\delta-\cos\pi\delta)\}$ in Theorem 2.2, we see that for $\{e^{i\rho_k x}\}$ to be a frame over $L^2(-\pi,\pi)$, we only require $\mu<\sqrt{A}$. This means that

$$\sin\pi\delta-\cos\pi\delta<\frac{1}{1+\sigma}\left\{\sqrt{\frac{A}{B}}-1\right\}. \tag{2.12}$$

Thus $\delta<\widetilde{L}=1/4-1/\pi\arcsin\{(1/(1+\sigma)\sqrt{2})(1-\sqrt{A/B})\}$ and the bounds of the frame now follow directly from Theorem 2.2. This completes the proof. □

The sequence considered in Theorem 2.1 is perturbed along the real axis. Perturbation results along the imaginary axis were established by Duffin and Schaeffer [6]. Here we first explicitly specify their upper and lower bounds, and then illustrate their accuracy.

Theorem 2.3. *Let $\lambda_n=\alpha_n+i\beta_n$ be a complex sequence with α_n, β_n real, $|\beta_n|<\beta$. If $\{e^{i\alpha_n t}\}$ is a frame over an interval $(-\gamma,\gamma)$ with bounds A and B, and $f(z)$ is an entire function of exponential type γ with $0<\gamma\le\pi$, $f\in L^2(-\infty,\infty)$, then*

$$Ae^{-2\gamma\beta}\le\frac{\sum_{n=-\infty}^{\infty}|f(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx}\le B\left\{e^{-\beta\gamma}+\sqrt{\frac{B}{A}}\left(1-e^{-\beta\gamma}\right)\right\}^2e^{2\gamma\beta}. \tag{2.13}$$

Before giving the proof of this theorem, we state two lemmas directly cited from [6].

Lemma 2.4. *If $f(z)$ is an entire function of exponential type γ and $f\in L^2(-\infty,\infty)$, then*

$$\int_{-\infty}^{\infty}|f^{(k)}(x)|^2\le\gamma^{2k}\int_{-\infty}^{\infty}|f(x)|^2dx. \tag{2.14}$$

If we choose $\rho=(\gamma/M)^{1/2}$, then Lemma 2.5 in [6] can be expressed as follows.

Lemma 2.5. *Let $\{e^{i\sigma_n t}\}$ be a frame over the interval $(-\gamma,\gamma)$, $0\le\gamma\le\pi$, with bounds A and B. If $\{\mu_n\}$ is a sequence satisfying $|\mu_n-\sigma_n|\le M$ for some constant M, then for any function f in the Paley-Wiener space,*

$$\frac{\sum_{n\in\mathbb{N}}|f(\mu_n)|^2}{\sum_{n\in\mathbb{N}}|f(\sigma_n)|^2}\le\left\{1+\sqrt{\frac{B}{A}}\left(e^{\gamma M}-1\right)\right\}^2. \tag{2.15}$$

Lemma 2.6. *Let $\{e^{i\lambda_n t}\}$ be a frame over the interval $(-\gamma,\gamma)$ with bounds A and B. Then for any given $\epsilon>0$, there exists $\delta>0$ such that when $|\mu_n-\lambda_n|<\delta$ for all $n\in\mathbb{N}$,*

$$(1-\epsilon)A<\frac{\sum_n|f(\mu_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx}<(1+\epsilon)B \tag{2.16}$$

for all entire functions $f(z)$ of exponential type γ with $f\in L^2(-\infty,\infty)$.

Proof of Lemma 2.6. Given $\epsilon_1 > 0$, suppose $|\mu_n - \lambda_n| < \delta$ where $\delta > 0$ satisfies that $|(B/A)(e^{\gamma\delta} - 1)^2| < \epsilon_1$, and choose $\rho = \{\gamma/\delta\}^{1/2}$. Then with the Taylor's series expansion of f at $z = \lambda_n$, we have

$$\begin{aligned}|f(\mu_n) - f(\lambda_n)|^2 &\le \left\{\sum_{k=1}^{\infty} \frac{|f^{(k)}(\lambda_n)|^2}{k!}\right\}\left\{\sum \frac{|\mu_n - \lambda_n|^{2k}}{k!}\right\} \\ &\le \left\{\sum_{k=1}^{\infty} \frac{|f^{(k)}(\lambda_n)|^2}{\rho^{2k}k!}\right\}\left\{\sum_{k=1}^{\infty} \frac{(\rho\delta)^{2k}}{k!}\right\}.\end{aligned} \tag{2.17}$$

Since $f^{(k)}(z)$ is an entire function of type γ, and since $\{e^{i\lambda_n t}\}$ is a frame over the interval $(-\gamma, \gamma)$, we can combine the property of the upper bound B of the frame with Lemma 2.4 to generate the following inequalities:

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |f(\mu_n) - f(\lambda_n)|^2 &\le \{e^{\gamma\delta} - 1\}\left\{\sum_{k=1}^{\infty} \frac{1}{\rho^{2k}k!} \sum_{n=-\infty}^{\infty} |f^{(k)}(\lambda_n)|^2\right\} \\ &\le \{e^{\gamma\delta} - 1\} \sum_{k=1}^{\infty} \frac{B}{\rho^{2k}k!} \int_{-\infty}^{\infty} |f^{(k)}(x)|^2 dx \\ &= B(e^{\gamma\delta} - 1)(e^{\gamma^2/\rho^2} - 1) \int_{-\infty}^{\infty} |f(x)|^2 dx \\ &\le \frac{B}{A}(e^{\gamma\delta} - 1)^2 \sum_{n\in\mathbb{N}} |f(\lambda_n)|^2 < \epsilon_1 \sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2.\end{aligned} \tag{2.18}$$

By Minkowski's inequality, it follows that

$$\left(\sum_{n\in\mathbb{N}} |f(\mu_n)|^2\right)^{1/2} \le (1 + \epsilon_1^{1/2})\left(\sum_{n\in\mathbb{N}} |f(\lambda_n)|^2\right)^{1/2}. \tag{2.19}$$

Thus

$$\frac{\sum_{n\in\mathbb{N}} |f(\mu_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} = \frac{\sum_{n\in\mathbb{N}} |f(\mu_n)|^2}{\sum_{n\in\mathbb{N}} |f(\lambda_n)|^2} \frac{\sum_{n\in\mathbb{N}} |f(\lambda_n)|^2}{\int_{-\infty}^{\infty} |f(x)|^2 dx} \le \left(1 + \epsilon_1^{1/2}\right)^2 B. \tag{2.20}$$

On the other hand,

$$\begin{aligned}\left(\sum_{n\in\mathbb{N}} |f(\lambda_n)|^2\right)^{1/2} &\le \left(\sum_{n\in\mathbb{N}} |f(\lambda_n) - f(\mu_n)|^2\right)^{1/2} + \left(\sum_{n\in\mathbb{N}} |f(\mu_n)|^2\right)^{1/2} \\ &\le \epsilon_1^{1/2}\left(\sum_{n} |f(\lambda_n)|^2\right)^{1/2} + \left(\sum_{n\in\mathbb{N}} |f(\mu_n)|^2\right)^{1/2}.\end{aligned} \tag{2.21}$$

It follows that

$$\left(1 - \epsilon_1^{1/2}\right)^2 \left(\sum_{n\in\mathbb{N}} |f(\lambda_n)|^2\right) \le \sum_{n\in\mathbb{N}} |f(\mu_n)|^2. \tag{2.22}$$

Therefore,

$$\frac{\sum_{n\in\mathbb{N}}|f(\mu_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx} = \frac{\sum_{n\in\mathbb{N}}|f(\mu_n)|^2}{\sum_{n\in\mathbb{N}}|f(\lambda_n)|^2}\frac{\sum_{n\in\mathbb{N}}|f(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx} \geq \left(1-\epsilon_1^{1/2}\right)^2 A. \tag{2.23}$$

It is obvious that ϵ_1 can be chosen such that both $(1-\epsilon_1^{1/2})^2 > 1-\epsilon$ and $(1+\epsilon_1^{1/2})^2 < 1+\epsilon$ hold for any given $\epsilon > 0$. Thus the proof of the lemma is completed. □

Proof of Theorem 2.3. The second inequality can be obtained directly from the frame's definition and Lemma 2.5 if σ_n and μ_n in Lemma 2.5 are replaced by α_n and λ_n, respectively. For the first one, assuming $f(z)$ is in the Paley-Wiener space, then as in [6] we construct a new function f_1 and a new sequence $\lambda_n^{(1)} = \alpha_n + i\beta_n^{(1)}$ with $|\beta_n^{(1)}| \leq \beta/2$, such that

$$e^{-\beta\gamma}\frac{\sum_n|f_1(\lambda_n^{(1)})|^2}{\int_{-\infty}^{\infty}|f_1(x)|^2dx} \leq \frac{\sum_n|f(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx}. \tag{2.24}$$

Next for any given $\epsilon > 0$, there is $\delta > 0$ defined in Lemma 2.6 such that $|\lambda_n^{(K_0)} - \alpha_n| = |\beta_n^{(K_0)}| \leq |\beta/2^{K_0}| < \delta$ for sufficiently large K_0. Then Lemma 2.6 guarantees that

$$\frac{\sum_{n\in\mathbb{N}}|f_{K_0}(\lambda_n^{(K_0)})|^2}{\int_{-\infty}^{\infty}|f_{K_0}(x)|^2dx} \geq (1-\epsilon)A. \tag{2.25}$$

Repeating the procedure for (2.24) K_0 times, we obtain that

$$\begin{aligned}\frac{\sum_{n\in\mathbb{N}}|f(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx} &\geq e^{-\gamma(\beta+\beta/2+\cdots+\beta/2^{K_0-1})}\frac{\sum_{n\in\mathbb{N}}\left|f_{K_0}(\lambda_n^{(K_0)})\right|^2}{\int_{-\infty}^{\infty}|f_{K_0}(x)|^2dx} \\ &\geq (1-\epsilon)Ae^{-2\beta\gamma}\end{aligned} \tag{2.26}$$

for an arbitrary $\epsilon > 0$, which completes the proof. □

Corollary 2.7. *Under the assumption of Theorem 2.3, if $\gamma = \pi$ and $|\lambda_n - n| < L$ for some constant L, then*

$$e^{-2\beta\pi} \leq \frac{\sum_{n=-\infty}^{\infty}|f(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f(x)|^2dx} \leq e^{2L\pi} \tag{2.27}$$

for all entire functions of exponential type π belonging to $L^2(-\infty,\infty)$.

Proof. Actually, it suffices to prove the second inequality. In Lemma 2.5, if we set $\gamma = \pi$, $\sigma_n = n$, and $\mu_n = \lambda_n$, then from Parseval's identity (Theorem 4.1), we know that $A = B = 1$. The conclusion of Lemma 2.5 immediately yields that $\sum_{n=-\infty}^{\infty}|f(\lambda_n)|^2/\int_{-\infty}^{\infty}|f(x)|^2dx \leq e^{2L\pi}$, which completes the proof. □

Corollary 2.8. *Suppose $\{\lambda_n = n + i\beta_n\}$ is a sequence satisfying $|\beta_n| < \beta$, then $\{e^{i\lambda_n t}\}$ is a frame over $(-\pi,\pi)$ with lower bound $e^{-2\pi\beta}$ and upper bound $e^{2\pi\beta}$, respectively.*

Remark 2.9. In Corollary 2.8, the upper and lower bounds cannot be replaced by $c_1 e^{2\gamma\beta}$ $(c_1 < 1)$ and $c_2 e^{-2\gamma\beta}$ $(c_2 > 1)$, respectively. It is obvious that $c_1 e^{2\gamma\beta} \to c_1 < 1$ and $c_2 e^{-2\gamma\beta} \to c_2 > 1$ as $\beta \to 0$. But when $\beta \to 0$, $\lambda_n \to n$, Theorem 2.3 implies that the upper and lower bounds B_β and A_β satisfy $B_\beta \to 1$ and $A_\beta \to 1$. It forces that $c_1 = c_2 = 1$.

Remark 2.10. Two examples given in the next section show that the two exponents $-2\gamma\beta$ and $2\gamma\beta$ in Theorem 2.3 are best possible.

3. Two examples

Let $y = \cosh a(\pi - x)$, $0 \le x \le 2\pi$, then its Fourier expansion is

$$y = \frac{2}{\pi}\sinh a\pi \left\{ \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a}{a^2+n^2}\cos nx \right\}. \tag{3.1}$$

It follows that $\sum_{n=1}^{\infty}(a/(a^2+n^2))\cos nx = (\pi/2)(\cosh a(\pi-x)/\sinh a\pi) - 1/2a$. Since $\cos nx$ is even, we may extend n to the negative infinity, and obtain that $\sum_{n=-\infty}^{\infty}(\cos nx/a^2 + n^2) = (\pi/a)(\cosh a(\pi - x)/\sinh a\pi)$. Now set $a = \beta$ with $x = 0$ and $x = 2\gamma \le 2\pi$, respectively, then we have

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{\beta^2+n^2} &= \frac{\pi}{\beta}\frac{e^{\pi\beta}+e^{-\pi\beta}}{e^{\pi\beta}-e^{-\pi\beta}}, \\ \sum_{n=-\infty}^{\infty} \frac{\cos 2\gamma n}{\beta^2+n^2} &= \frac{\pi}{\beta}\frac{e^{\beta(\pi-2\gamma)}+e^{-\beta(\pi-2\gamma)}}{e^{\pi\beta}-e^{-\pi\beta}}. \end{aligned} \tag{3.2}$$

With the identities (3.2), we are going to evaluate the following two examples.

Example 3.1. Suppose $g_1(t) = e^{it}$ and $f_1(z) = (1/2\pi)^{1/2}\int_{-\gamma}^{\gamma} g_1(t)e^{izt}dt$. Then from the function f_1, it can be demonstrated that the exponent of the upper bound in Theorem 2.3 cannot be reduced.

In fact, f_1 is an entire function of exponent type γ, and can be represented as $f_1(z) = (1/2\pi)^{1/2}(e^{\gamma(1+z)i} - e^{-\gamma(1+z)i}/(1+z)i)$. Substituting z with $\lambda_n = n + i\beta$, we get that

$$\begin{aligned} \sum_{n=-\infty}^{\infty} |f_1(\lambda_n)|^2 &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{|1+\lambda_n|^2}|e^{\gamma(1+\lambda_n)i} - e^{-\gamma(1+\lambda_n)i}|^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{(1+n)^2+\beta^2}|e^{-\gamma\beta+(1+n)\gamma i} - e^{\gamma\beta-(1+n)\gamma i}|^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\frac{1}{n^2+\beta^2}(e^{2\gamma\beta}+e^{-2\gamma\beta} - 2\cos(2\gamma n)) \\ &= \frac{1}{2\pi}\left\{ (e^{2\gamma\beta}+e^{-2\gamma\beta})\sum_{n=-\infty}^{\infty}\frac{1}{n^2+\beta^2} - 2\sum_{n=-\infty}^{\infty}\frac{\cos 2\gamma n}{n^2+\beta^2} \right\}. \end{aligned} \tag{3.3}$$

From the identities of (3.2), we obtain that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} |f_1(\lambda_n)|^2 &= \frac{1}{2\pi}\left\{\frac{\pi}{\beta}\frac{(e^{2\gamma\beta}+e^{-2\gamma\beta})(e^{\pi\beta}+e^{-\pi\beta})}{e^{\pi\beta}-e^{-\pi\beta}} - \frac{2\pi}{\beta}\frac{e^{(\pi-2\gamma)\beta}+e^{-(\pi-2\gamma)\beta}}{e^{\pi\beta}-e^{-\pi\beta}}\right\} \\ &= \frac{e^{2\gamma\beta}-e^{-2\gamma\beta}}{2\beta}.\end{aligned} \tag{3.4}$$

By Plancherel's theorem [11, page 85], we have $\int_{-\infty}^{\infty}|f_1(x)|^2dx = \int_{-\gamma}^{\gamma}|g_1(t)|^2dt = 2\gamma$. It follows that

$$\frac{\sum_{-\infty}^{\infty}|f_1(\lambda_n)|^2}{\int_{-\infty}^{\infty}|f_1(x)|^2dx} = \frac{e^{2\gamma\beta}-e^{-2\gamma\beta}}{4\gamma\beta} = B_\beta. \tag{3.5}$$

It implies that the γ in the upper bound of Theorem 2.3 cannot be replaced by $\gamma-\epsilon$ for any $\epsilon > 0$. Otherwise, there is a contradiction for any sufficiently large β.

Example 3.2. Suppose $g_2(t) = e^{s+it}$ $(s > 0)$ and $f_2(z) = (1/2\pi)^{1/2}\int_{-\gamma}^{\gamma} g_2(t)e^{izt}dt$. Then f_2 can assume the lower bound of Theorem 2.3 for $\gamma = 1$.

Actually, since $f_2(z) = (1/2\pi)^{1/2}(e^{\gamma(s+(1+z)i)} - e^{-\gamma(s+(1+z)i)}/s+(1+z)i)$, by substitution of z with $\lambda_n = n+i\beta$, we obtain that

$$\sum_{-\infty}^{\infty}|f_2(\lambda_n)|^2 = \left(\frac{1}{2\pi}\right)^{1/2}\frac{|e^{-\gamma(\beta-s)+(1+n)\gamma i} - e^{\gamma(\beta-s)-(1+n)\gamma i}|^2}{|-(\beta-s)+(1+n)i|^2}. \tag{3.6}$$

Since (3.6) is similar to that in Example 3.1 except for that β is replaced by $\beta - s$, so we obtain that

$$\sum_{-\infty}^{\infty}|f_2(\lambda_n)|^2 = \frac{e^{2\gamma(\beta-s)}-e^{-2\gamma(\beta-s)}}{2(\beta-s)} \longrightarrow 2\gamma \tag{3.7}$$

as $s \to \beta$. On the other hand, since

$$\int_{-\infty}^{\infty}|f_2(x)|^2dx = \int_{-\gamma}^{\gamma}|g_2(t)|^2dt = 2\gamma e^{2s}, \tag{3.8}$$

it follows that $\sum_{-\infty}^{\infty}|f_2(\lambda_n)|^2/\int_{-\infty}^{\infty}|f_2(x)|^2dx \to e^{-2\beta}$. Thus the lower bound can be achieved when $\gamma = 1$.

4. Entire functions on integer sequence

Suppose f is in the Paley-Wiener space and is written as $f(z) = (1/\sqrt{2\pi})\int_{-\pi}^{\pi} g(t)e^{izt}dt$ with $g \in L^2(-\pi,\pi)$. Then, from Plancherel's theorem, we have

$$\int_{-\infty}^{\infty}|f(x)|^2dx = \int_{-\pi}^{\pi}|g(t)|^2dt. \tag{4.1}$$

Consequently, Parseval's identity can be expressed as follows [11, page 90].

Theorem 4.1. *Assume that $f(z)$ is an entire function of exponential type at most π, and is square integrable on the real axis, then*

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{4.2}$$

From Theorem 4.1 and (1.2), we know that $\{e^{int}\}$ is a tight frame in $L^2(-\pi,\pi)$ with the bound $A = 2\pi$, but this is not true in $L^2(-\gamma,\gamma)$ if $\gamma > \pi$ (see [6]). It therefore will be interesting to find all the tight frames or Parseval frames in $L^2[-\gamma,\gamma]$. We next consider the space $P(\gamma)$ consisting of all the entire functions of exponential type at most γ, $\gamma > \pi$. The space $P(\gamma)$ is then isomorphic to $L^2[-\gamma,\gamma]$. For the functions in $P(\gamma)$, Pólya and Plancherel [8, 9] proved the following theorem.

Theorem 4.2. *If f is an entire function of exponential type γ, then for any real increasing sequence $\{\lambda_n\}$ such that $\lambda_{n+1} - \lambda_n \geq \delta$ for some $\delta > 0$,*

$$\sum_{n=-\infty}^{\infty} |f(\lambda_n)|^2 \leq \frac{4(e^{\gamma\delta} - 1)}{\pi\gamma\delta^2} \int_{-\infty}^{\infty} |f(x)|^2 dx, \tag{4.3}$$

and in particular

$$\sum_{n=-\infty}^{\infty} |f(n)|^2 \leq \frac{4(e^{\gamma} - 1)}{\pi\gamma} \int_{-\infty}^{\infty} |f(x)|^2 dx. \tag{4.4}$$

While the coefficient in (4.4) depends on the exponential type γ, there are some entire functions in $P(\gamma)$, but out of the Paley-Wiener space P, which still have nice properties.

Theorem 4.3. *For any entire function $f(z)$ of exponential type $\gamma > 0$ satisfying*

$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty, \tag{4.5}$$

there exists a constant c such that the function $g(z) = f(z+c)$ satisfies that

$$\sum_{n=-\infty}^{\infty} |g(n)|^2 \leq \frac{4}{\pi} \int_{-\infty}^{\infty} |g(x)|^2 dx < \infty. \tag{4.6}$$

Proof. Let $f(z)$ be an entire function of exponential type $\gamma > 0$. Since $|f|^2$ is subharmonic, then for $\delta > 0$ and $w \in \mathbb{R}$, we have

$$|f(w)|^2 \leq \frac{1}{\pi\delta^2} \iint_{|z-w|<\delta} |f(z)|^2 dx\,dy \tag{4.7}$$

$$\leq \frac{1}{\pi\delta^2} \int_{-\delta}^{\delta} \int_{w-\delta}^{w+\delta} |f(x+iy)|^2 dx\,dy. \tag{4.8}$$

Suppose k is a positive integer. Let $\delta = 1/2^k$ and $w = n + 2j/2^k$ for $j = 1,\dots,2^{k-1}$. Then it follows that

$$\left| f\left(n + \frac{2j}{2^k}\right) \right|^2 \leq \frac{1}{\pi\delta^2} \int_{-\delta}^{\delta} \int_{(n+2j/2^k)-\delta}^{(n+2j/2^k)+\delta} |f(x+iy)|^2 dx\,dy \tag{4.9}$$

for $j = 1,\dots,2^{k-1}$. Set $f_j(z) = f(z+2j/2^k)$. Then f_j is an entire function of exponential type γ, and we consequently have that

$$\begin{aligned}\sum_{j=1}^{2^{k-1}}\sum_{n=-\infty}^{\infty}|f_j(n)|^2 &\le \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\int_{-\infty}^{\infty}|f(x+iy)|^2dx\,dy\\ &\le \frac{1}{\pi\delta^2}\int_{-\delta}^{\delta}\left(e^{2\gamma|y|}\int_{-\infty}^{\infty}|f(x)|^2dx\right)dy\\ &= \frac{e^{2\gamma\delta}-1}{\pi\gamma\delta^2}\int_{-\infty}^{\infty}|f(x)|^2dx.\end{aligned}\tag{4.10}$$

Choose f_{j_0} from $\{f_j\}$ such that $\sum_{n=-\infty}^{\infty}|f_{j_0}(n)|^2 \le \sum_{n=-\infty}^{\infty}|f_j(n)|^2$ for $j = 1,\dots,2^{k-1}$, then

$$2^{k-1}\sum_{n=-\infty}^{\infty}|f_{j_0}(n)|^2 \le \frac{2^{2k}}{\pi\gamma}\left(e^{\gamma/2^{k-1}}-1\right)\int_{-\infty}^{\infty}|f(x)|^2dx.\tag{4.11}$$

Consequently,

$$\sum_{n=-\infty}^{\infty}|f_{j_0}(n)|^2 \le \frac{4}{\pi}\frac{e^{\gamma/2^{k-1}}-1}{\gamma/2^{k-1}}\int_{-\infty}^{\infty}|f(x)|^2dx.\tag{4.12}$$

Since $(e^{\gamma/2^{k-1}}-1)/\gamma/2^{k-1} \to 1$ as $k\to\infty$, we have obtained that

$$\sum_{n=-\infty}^{\infty}|f_{j_0}(n)|^2 \le \frac{4}{\pi}\int_{-\infty}^{\infty}|f(x)|^2dx = \frac{4}{\pi}\int_{-\infty}^{\infty}|f_{j_0}(x)|^2dx,\tag{4.13}$$

which completes the proof of Theorem 4.3. □

Note. It will be interesting to know if the constant $4/\pi$ in Theorem 4.3 could be replaced by one as in Parseval's identity.

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