ON SOME TURÁN-TYPE INEQUALITIES

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We prove Turán-type inequalities for some special functions by using a generalization of the Schwarz inequality.

1. Introduction

The importance, in many fields of mathematics, of the inequalities of the type

$$f_n(x)f_{n+2}(x) - f_{n+1}^2(x) \le 0, \tag{1.1}$$

where $n = 0,1,2,\dots,$ is well known. They are named, by Karlin and Szegö, Turán-type inequalities because the first of this type of inequalities was proved by Turán [12]. More precisely, by using the classical recurrence relation [10, page 81]

$$\begin{gathered}(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad n = 0,1,\dots \\ P_{-1}(x) = 0, \qquad P_0(x) = 1\end{gathered} \tag{1.2}$$

and the differential relation [10, page 83]

$$(1-x^2)P_n'(x) = nP_{n-1}(x) - nxP_n(x), \tag{1.3}$$

he proved the following inequality:

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} \le 0, \quad -1 \le x \le 1, \tag{1.4}$$

where $P_n(x)$ is the Legendre polynomial of degree n. In (1.4) equality occurs only if $x = \pm 1$. This classical result has been extended in several directions: ultraspherical polynomials, Laguerre and Hermite polynomials, Bessel functions of first kind, modified Bessel functions, and so forth.

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For example, Lorch [8] established Turán-type inequalities for the positive zeros $c_{\nu k}$, $k = 1,2,\ldots$, of the general Bessel function

$$C_\nu(x) = J_\nu(x)\cos\alpha - Y_\nu(x)\sin\alpha, \quad 0 \le \alpha < \pi, \tag{1.5}$$

where $J_\nu(x)$ and $Y_\nu(x)$ denote the Bessel functions of the first and the second kind, respectively, while the corresponding results for the positive zeros $c'_{\nu k}$, $\nu \ge 0$, $k = 1,2,\ldots$, of the derivative $C'_\nu(x) = (d/dx)C_\nu(x)$ and for the zeros of ultraspherical, Laguerre, and Hermite polynomials have been established in [2, 3, 6], respectively.

Recently, in [7], we have proved Turán-type inequalities for some special functions, as well as the polygamma and the Riemann zeta functions, by using the following generalization of the Schwarz inequality:

$$\int_a^b g(t)[f(t)]^m dt \cdot \int_a^b g(t)[f(t)]^n dt \ge \left[\int_a^b g(t)[f(t)]^{(m+n)/2} dt\right]^2, \tag{1.6}$$

where f and g are two nonnegative functions of a real variable and m and n belong to a set S of real numbers, such that the integrals in (1.6) exist.

As mentioned in [7] this approach represents an alternative method with respect to the classical ones used by the above-cited authors and based, prevalently, on the Sturm theory.

In this paper, we continue, in this direction, to investigate about Turán-type inequalities satisfied by some special functions. In the next section, we will give three results. In the first one, we will use the well-known psi function defined as follows:

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \quad x > 0, \tag{1.7}$$

with the usual notation for the gamma function.

In the second one, we will use the so-called Riemann ξ-function which can be defined (see [11, page 16], cf. [9, page 285]) by

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s), \tag{1.8}$$

where ζ is the Riemann ζ-function. This function has the following representation (see [5]):

$$\xi\left(s+\frac{1}{2}\right) = \sum_{k=0}^{\infty} b_k s^{2k}, \tag{1.9}$$

where the coefficients b_k are given by the formula

$$b_k = 8\frac{2^{2k}}{(2k)!}\int_0^\infty t^{2k}\Phi(t)dt, \quad k = 0,1,\ldots, \tag{1.10}$$

$$\Phi(t) = \sum_{n=1}^{\infty} \left(2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}\right)e^{-\pi n^2 e^{4t}}. \tag{1.11}$$

In [1] the following Turán-type inequalities were proved:

$$b_k^2 - \frac{k+1}{k} b_{k+1} b_{k-1} \geq 0, \quad k = 0, 1, \ldots, \tag{1.12}$$

which are very important in the theory of the Riemann ξ-function (see [5]).

In the third one, we will use the modified Bessel functions of the third kind $K_\nu(x)$, $x > 0$, defined as follows:

$$\begin{aligned} K_\nu(x) &= \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}, \quad \nu \neq 0, \pm 1, \pm 2, \ldots, \\ K_n(x) &= \lim_{\nu \to n} K_\nu(x), \quad n = 0, \pm 1, \pm 2, \ldots, \end{aligned} \tag{1.13}$$

where

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{\nu+2k}}{k!\Gamma(\nu+k+1)} \tag{1.14}$$

are the modified Bessel functions of the first kind.

2. The results

Theorem 2.1. *For $n = 1, 2, \ldots$, denote by $h_n = \sum_{k=1}^{n}(1/k)$ the partial sum of the harmonic series. Let*

$$a_n = h_n - \log n, \tag{2.1}$$

then

$$(a_n - \gamma)(a_{n+2} - \gamma) \geq (a_{n+1} - \gamma)^2, \tag{2.2}$$

where γ is the Euler-Mascheroni constant defined by

$$\gamma = -\psi(1) = 0,5772156649\ldots. \tag{2.3}$$

Proof. For the *psi* function, we use the following expression:

$$\psi(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \gamma, \quad n = 1, 2, \ldots, \tag{2.4}$$

and the following integral representation:

$$\psi(z+1) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{e^{-t} - 1} \right) dt, \quad \operatorname{Re} z > 0. \tag{2.5}$$

By putting $z = n$ in (2.5), for $n = 1, 2, \ldots$, we obtain from (2.4) and (2.5),

$$\sum_{k=1}^{n} \frac{1}{k} - \gamma = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-nt}}{e^{-t} - 1} \right) dt = \int_0^\infty \frac{e^{-t} - e^{-nt}}{t} dt + \int_0^\infty e^{-nt} \frac{e^t - 1 - t}{t(e^t - 1)} dt. \tag{2.6}$$

Since

$$\int_0^\infty \frac{e^{-t}-e^{-nt}}{t}\,dt=\log n, \tag{2.7}$$

we have

$$\sum_{k=1}^{n}\frac{1}{k}-\log n-\gamma=\int_0^\infty \frac{e^t-1-t}{t(e^t-1)}e^{-nt}\,dt. \tag{2.8}$$

By (1.6) with $g(t)=(e^t-1-t)/t(e^t-1)$, $f(t)=e^{-t}$ and $a=0$, $b=+\infty$, we get

$$\int_0^\infty \frac{e^t-1-t}{t(e^t-1)}e^{-nt}\,dt\cdot\int_0^\infty \frac{e^t-1-t}{t(e^t-1)}e^{-(n+2)t}\,dt\geq\left[\int_0^\infty \frac{e^t-1-t}{t(e^t-1)}e^{-(n+1)t}\,dt\right]^2 \tag{2.9}$$

that is the inequality (2.2). □

Theorem 2.2. *For $k=1,2,\ldots$, let b_k $(k=1,2,\ldots)$ be the coefficients in (1.9), then*

$$b_k^2-\frac{(2k+1)(k+1)}{k(2k-1)}b_{k+1}b_{k-1}\leq 0,\quad k=1,2,\ldots. \tag{2.10}$$

Proof. By (1.6) and (1.10), with $g(t)=8\Phi(t)$, $f(t)=(2t)^2$ and $a=0$, $b=+\infty$, we get

$$\int_0^\infty 8\Phi(t)(2t)^{2k+2}\,dt\cdot\int_0^\infty 8\Phi(t)(2t)^{2k-2}\,dt\geq\left[\int_0^\infty 8\Phi(t)(2t)^{2k}\,dt\right]^2. \tag{2.11}$$

Dividing (2.11) by $(2k)!$ this inequality becomes

$$\frac{(2k+2)!}{(2k)!}b_{k+1}\frac{(2k-2)!}{(2k)!}b_{k-1}\leq b_k^2,\quad k=1,2,\ldots, \tag{2.12}$$

from which, since $((2k+2)!/(2k)!)((2k-2)!/(2k)!)=((2k+1)(k+1))/k(2k-1)$, we obtain the conclusion of Theorem 2.2. □

Remark 2.3. It is important to note that inequalities (1.12) and (2.10) together give

$$\frac{k+1}{k}b_{k+1}b_{k-1}\leq b_k^2\leq\frac{k+1}{k}\frac{2k+1}{2k-1}b_{k+1}b_{k-1},\quad k=1,2,\ldots. \tag{2.13}$$

Theorem 2.4. *Let $K_\nu(x)$, $x>0$, be the modified Bessel function of the third kind. Then, for $\nu>-1/2$ and $\mu>-1/2$,*

$$K_\nu(x)\cdot K_\mu(x)\geq K^2_{(\nu+\mu)/2}(x). \tag{2.14}$$

Proof. By (1.6) with $g(t)=e^{-\beta/t-\gamma t}$, $f(t)=t^{-1}$ and $a=0$, $b=+\infty$, we get

$$\int_0^\infty t^{m-1}e^{-\beta/t-\gamma t}\,dt\cdot\int_0^\infty t^{n-1}e^{-\beta/t-\gamma t}\,dt\geq\left[\int_0^\infty t^{(m+n)/2-1}e^{-\beta/t-\gamma t}\,dt\right]^2. \tag{2.15}$$

Using the following formula (see [4, Integral 3.471(9)]):

$$\int_0^\infty t^{\nu-1}e^{-\beta/t-\gamma t}dt = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2} K_\nu\left(2\sqrt{\beta\gamma}\right), \quad \nu > -\frac{1}{2}, \tag{2.16}$$

from (2.15) we have

$$K_\nu\left(2\sqrt{\beta\gamma}\right) \cdot K_\mu\left(2\sqrt{\beta\gamma}\right) \geq K^2_{(\nu+\mu)/2}\left(2\sqrt{\beta\gamma}\right) \tag{2.17}$$

which, putting $x = 2\sqrt{\beta\gamma}$, is equivalent to the conclusion of Theorem 2.4.

In the particular case $\mu = \nu + 2$, we find

$$K_\nu(x) \cdot K_{\nu+2}(x) \geq K^2_{\nu+1}(x), \quad \nu > -\frac{1}{2}. \tag{2.18}$$

□

Concluding Remark 2.5. By means of (1.6) Turán-type inequalities for many complicated integrals as well as, for example, $s_n = \int_0^\pi (\log \sin x)^n dx$ $(n = 0, 1, \ldots)$ for which we have

$$s_n(x)s_{n+2}(x) \geq s^2_{n+1}(x), \tag{2.19}$$

can be obtained.

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