THE JAMES CONSTANT OF NORMALIZED NORMS ON $\mathbb{R}^2$

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We introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms. We also give a criterion for deciding whether a given norm in this class is uniformly nonsquare. Moreover, an estimate for the James constant is presented and the exact value of some certain norms is computed. This gives a partial answer to the question raised by Kato et al.

1. Introduction and preliminaries

A norm $\|\cdot\|$ on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$) is said to be *absolute* if $\|(z,w)\| = \|(|z|,|w|)\|$ for all $z,w \in \mathbb{C}$ (resp., $\mathbb{R}$), and *normalized* if $\|(1,0)\| = \|(0,1)\| = 1$. The ℓ_p-norms $\|\cdot\|_p$ are such examples:

$$\|(z,w)\|_p = \begin{cases} \left(|z|^p + |w|^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{|z|,|w|\} & \text{if } p = \infty. \end{cases} \tag{1.1}$$

Let AN_2 be the family of all absolute normalized norms on $\mathbb{C}^2$ (resp., $\mathbb{R}^2$), and Ψ_2 the family of all continuous convex functions ψ on $[0,1]$ such that $\psi(0) = \psi(1) = 1$ and $\max\{1-t,\ t\} \le \psi(t) \le 1$ $(0 \le t \le 1)$. According to Bonsall and Duncan [1], AN_2 and Ψ_2 are in a one-to-one correspondence under the equation

$$\psi(t) = \|(1-t,t)\| \quad (0 \le t \le 1). \tag{1.2}$$

Indeed, for all $\psi \in \Psi_2$, let

$$\|(z,w)\|_\psi = \begin{cases} (|z|+|w|)\psi\left(\dfrac{|w|}{|z|+|w|}\right) & \text{if } (z,w) \ne (0,0), \\ 0 & \text{if } (z,w) = (0,0). \end{cases} \tag{1.3}$$

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Then $\|\cdot\|_\psi \in AN_2$, and $\|\cdot\|_\psi$ satisfies (1.2). From this result, we can consider many non-ℓ_p-type norms easily. Now let

$$\psi_p(t) = \begin{cases} \left((1-t)^p + t^p\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \max\{1-t,\ t\} & \text{if } p = \infty. \end{cases} \tag{1.4}$$

Then $\psi_p(t) \in \Psi_2$ and, as is easily seen, the ℓ_p-norm $\|\cdot\|_p$ is associated with ψ_p.

If X is a Banach space, then X is *uniformly nonsquare* if there exists $\delta \in (0,1)$ such that for any $x, y \in S_X$,

$$\text{either } \|x+y\| \le 2(1-\delta) \qquad \text{or} \qquad \|x-y\| \le 2(1-\delta), \tag{1.5}$$

where $S_X = \{x \in X : \|x\| = 1\}$. The *James constant* $J(X)$ is defined by

$$J(X) = \sup\{\min\{\|x+y\|,\ \|x-y\|\} : x,\ y \in S_X\}. \tag{1.6}$$

The *modulus of convexity of* X, $\delta_X : [0,2] \to [0,1]$ is defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x,\ y \in S_X,\ \|x-y\| \ge \varepsilon\right\}. \tag{1.7}$$

The preceding parameters have been recently studied by several authors (cf. [4–6, 8, 9]). We collect together some known results.

Proposition 1.1. *Let X be a nontrivial Banach space, then*

(i) $\sqrt{2} \le J(X) \le 2$ *(Gao and Lau [5]),*
(ii) *if X is a Hilbert space, then $J(X) = \sqrt{2}$; the converse is not true (Gao and Lau [5]),*
(iii) *X is uniformly nonsquare if and only if $J(X) < 2$ (Gao and Lau [5]),*
(iv) $2J(X) - 2 \le J(X^*) \le J(X)/2 + 1$, $J(X^{**}) = J(X)$, *and there exists a Banach space X such that $J(X^*) \ne J(X)$ (Kato et al. [8]),*
(v) *if $2 \le p \le \infty$, then $\delta_{\ell_p}(\varepsilon) = 1 - (1 - (\varepsilon/2)^p)^{1/p}$ (Hanner [6]),*
(vi) $J(X) = \sup\{\varepsilon \in (0,2) : \delta_X(\varepsilon) \le 1 - \varepsilon/2\}$ *(Gao and Lau [5]).*

The paper is organized as follows. In Section 2 we introduce a new class of normalized norms on $\mathbb{R}^2$. This class properly contains all absolute normalized norms of Bonsall and Duncan [1]. The so-called generalized Day-James space, ℓ_ψ-ℓ_φ, where $\psi, \varphi \in \Psi_2$, is introduced and studied. More precisely, we prove that $(\ell_\psi\text{-}\ell_\varphi)^* = \ell_{\psi^*}\text{-}\ell_{\varphi^*}$ where ψ^* and φ^* are the dual functions of ψ and φ, respectively. In Section 3, the upper bound of the James constant of the generalized Day-James space is given. Furthermore, we compute $J(\ell_\psi\text{-}\ell_\infty)$ and deduce that every generalized Day-James space except ℓ_1-ℓ_1 and ℓ_∞-ℓ_∞ is uniformly nonsquare. This result strengthens Corollary 3 of Saito et al. [10].

2. Generalized Day-James spaces

In this section, we introduce a new class of normalized norms on $\mathbb{R}^2$ which properly contains all absolute normalized norms of Bonsall and Duncan [1]. Moreover, we introduce a two-dimensional normed space which is a generalization of Day-James ℓ_p-ℓ_q spaces.

Lemma 2.1. *Let $\psi \in \Psi_2$ and let $\|\cdot\|_{\psi,\psi_\infty}$ be a function on $\mathbb{R}^2$ defined by, for all $(z,w) \in \mathbb{R}^2$,*

$$\begin{aligned}||(z,w)||_{\psi,\psi_\infty} &:= \max\{||(z^+,w^+)||_\psi, ||(z^-,w^-)||_\psi\},\\ &= \begin{cases}||(z,w)||_\psi & \text{if } zw \geq 0,\\ ||(z,w)||_\infty & \text{if } zw \leq 0,\end{cases}\end{aligned} \tag{2.1}$$

where x^+ and x^- are positive and negative parts of $x \in \mathbb{R}$, that is, $x^+ = \max\{x,0\}$ and $x^- = \max\{-x,0\}$. Then $\|\cdot\|_{\psi,\psi_\infty}$ is a norm on $\mathbb{R}^2$.

For convenience, we put $\mathcal{B}_{\psi_1,\psi_2} := \{(z,w) \in \mathbb{R}^2 : \|(z,w)\|_{\psi_1,\psi_2} \leq 1\}$.

Theorem 2.2. *Let $\psi,\varphi \in \Psi_2$ and*

$$||(z,w)||_{\psi,\varphi} := \begin{cases}||(z,w)||_\psi & \text{if } zw \geq 0,\\ ||(z,w)||_\varphi & \text{if } zw \leq 0\end{cases} \tag{2.2}$$

for all $(z,w) \in \mathbb{R}^2$. Then $\|\cdot\|_{\psi,\varphi}$ is a norm on $\mathbb{R}^2$. Denote by N_2 the family of all such preceding norms.

Proof. Let $\psi,\varphi \in \Psi_2$, we only show $\|\cdot\|_{\psi,\varphi}$ satisfies the triangle inequality. To this end, it suffices to prove that $\mathcal{B}_{\psi,\varphi}$ is convex. By Lemma 2.1, we have that $\mathcal{B}_{\psi,\psi_\infty}$ and $\mathcal{B}_{\varphi,\psi_\infty}$ are closed unit balls of $\|\cdot\|_{\psi,\psi_\infty}$ and $\|\cdot\|_{\varphi,\psi_\infty}$, respectively, and so $\mathcal{B}_{\psi,\psi_\infty}$ and $\mathcal{B}_{\varphi,\psi_\infty}$ are convex sets. We define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T\big((z,w)\big) = (-z,w) \quad \forall (z,w) \in \mathbb{R}^2. \tag{2.3}$$

Then T is a linear operator and $T(\mathcal{B}_{\varphi,\psi_\infty}) = \mathcal{B}_{\psi_\infty,\varphi}$, which implies that $\mathcal{B}_{\psi_\infty,\varphi}$ is convex and so $\mathcal{B}_{\psi,\varphi} = \mathcal{B}_{\psi_\infty,\varphi} \cap \mathcal{B}_{\psi,\psi_\infty}$ is convex. □

Taking $\psi = \psi_p$ and $\varphi = \psi_q$ $(1 \leq p,q \leq \infty)$ in Theorem 2.2, we obtain the following.

Corollary 2.3 (Day-James ℓ_p-ℓ_q spaces). *For $1 \leq p$, $q \leq \infty$, denote by ℓ_p-ℓ_q the Day-James space, that is, $\mathbb{R}^2$ with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,*

$$||(z,w)||_{p,q} = \begin{cases}||(z,w)||_p & \text{if } zw \geq 0,\\ ||(z,w)||_q & \text{if } zw \leq 0.\end{cases} \tag{2.4}$$

James [7] considered the ℓ_p-$\ell_{p'}$ space as an example of a Banach space which is isometric to its dual but which is not given by a Hilbert norm when $p \neq 2$. Day [2] considered even more general spaces, namely, if $(X,\|\cdot\|)$ is a two-dimensional Banach space and $(X^*,\|\cdot\|^*)$ its dual, then the X-X^* space is the space X with the norm defined by, for all $(z,w) \in \mathbb{R}^2$,

$$||(z,w)||_{X,X^*} = \begin{cases}||(z,w)|| & \text{if } zw \geq 0,\\ ||(z,w)||^* & \text{if } zw \leq 0.\end{cases} \tag{2.5}$$

For $\psi,\varphi \in \Psi_2$, denote by ℓ_ψ-ℓ_φ the *generalized Day-James space*, that is, $\mathbb{R}^2$ with the norm $\|\cdot\|_{\psi,\varphi}$ defined by (2.2). For ψ_p defined by (1.4), we write ℓ_ψ-ℓ_p for ℓ_ψ-ℓ_{ψ_p}. For example, if $1 \le p,q \le \infty$, ℓ_p-ℓ_q means ℓ_{ψ_p}-ℓ_{ψ_q}.

It is worthwhile to mention that there is a normalized norm which is not absolute.

Proposition 2.4. *There is $\psi \in \Psi_2$ such that ℓ_ψ-ℓ_∞ is not isometrically isomorphic to ℓ_φ-ℓ_φ for all $\varphi \in \Psi_2$.*

Proof. Let

$$\psi(t) := \begin{cases} 1-t & \text{if } 0 \le t \le \dfrac{1}{8}, \\ \dfrac{11-4t}{12} & \text{if } \dfrac{1}{8} \le t \le \dfrac{1}{2}, \\ \dfrac{1+t}{2} & \text{if } \dfrac{1}{2} \le t \le 1. \end{cases} \tag{2.6}$$

We observe that the sphere of ℓ_ψ-ℓ_∞ is the octagon whose right half consists of 4 segments of different lengths. Suppose that there are $\varphi \in \Psi_2$ and an isometric isomorphism from ℓ_ψ-ℓ_∞ onto ℓ_φ-ℓ_φ. Since the image of each segment in ℓ_ψ-ℓ_∞ is again a segment of the same length in ℓ_φ-ℓ_φ, the sphere of ℓ_φ-ℓ_φ must be the octagon whose each corresponding side has the same length (measured by $\|\cdot\|_\varphi$). We show that this cannot happen. Consider $(1,0) \in S_{\ell_\varphi\text{-}\ell_\varphi}$. If $(1,0)$ is an extreme point of $B_{\ell_\varphi\text{-}\ell_\varphi}$, then $S_{\ell_\varphi\text{-}\ell_\varphi}$ contains 4 segments of same lengths since $\|\cdot\|_\varphi$ is absolute. On the other hand, if $(1,0)$ is an not extreme point of $B_{\ell_\varphi\text{-}\ell_\varphi}$, again $S_{\ell_\varphi\text{-}\ell_\varphi}$ contains 4 segments of same lengths. □

Next, we prove that the dual of a generalized Day-James space is again a generalized Day-James space. Recall that, for $\psi \in \Psi_2$, the *dual function* ψ^* of ψ is defined by

$$\psi^*(s) = \max_{0 \le t \le 1} \frac{(1-s)(1-t)+st}{\psi(t)} \tag{2.7}$$

for all $s \in [0,1]$. It was proved that $\psi^* \in \Psi_2$ and $(\ell_\psi\text{-}\ell_\psi)^* = \ell_{\psi^*}\text{-}\ell_{\psi^*}$ (see [3, Proposition 1 and Theorem 2]). We generalize this result to our spaces as follows.

Theorem 2.5. *For $\psi,\varphi \in \Psi_2$, there is an isometric isomorphism that identifies $(\ell_\psi\text{-}\ell_\varphi)^*$ with $\ell_{\psi^*}\text{-}\ell_{\varphi^*}$ such that if $f \in (\ell_\psi\text{-}\ell_\varphi)^*$ is identified with the element $(z,w) \in \ell_{\psi^*}\text{-}\ell_{\varphi^*}$, then*

$$f(u,v) = zu + wv \tag{2.8}$$

for all $(u,v) \in \mathbb{R}^2$.

Proof. We can prove analogous to [3, Theorem 2]. □

3. The James constant and uniform nonsquareness

The next lemmas are crucial for proving the main theorems.

Lemma 3.1. *Let $\psi,\varphi \in \Psi_2$. Then*

(i) $\|\cdot\|_\infty \le \|\cdot\|_{\psi,\varphi} \le \|\cdot\|_1$,

(ii) $(1/M_{\psi,\varphi})\|\cdot\|_\psi \le \|\cdot\|_{\psi,\varphi} \le M_{\varphi,\psi}\|\cdot\|_\psi$,
(iii) $(1/M_{\varphi,\psi})\|\cdot\|_\varphi \le \|\cdot\|_{\psi,\varphi} \le M_{\psi,\varphi}\|\cdot\|_\varphi$,
where $M_{\varphi,\psi} = \max_{0\le t\le 1}\varphi(t)/\psi(t)$ *and* $M_{\psi,\varphi} = \max_{0\le t\le 1}\psi(t)/\varphi(t)$.

Lemma 3.2. *Let* $\psi,\varphi \in \Psi_2$ *and let* Q_i $(i = 1,\dots,4)$ *denote the ith quadrant in* $\mathbb{R}^2$. *Suppose that* $x, y \in S_{\ell_\psi\text{-}\ell_\varphi}$, *then the following statements are true.*
(i) *If* $x, y \in Q_1$, *then* $x + y \in Q_1$ *and* $x - y \in Q_2 \cup Q_4$.
(ii) *If* $x, y \in Q_2$, *then* $x + y \in Q_2$ *and* $x - y \in Q_1 \cup Q_3$.
(iii) *If* $\psi(t) \le \varphi(t)$ *for all* $t \in [0,1]$ *and* $x - y \in Q_2^\circ \cup Q_4^\circ$, *where* Q_2° *and* Q_4° *are the interiors of* Q_2 *and* Q_4, *respectively, then* $x + y \in Q_1 \cup Q_3$.

We will estimate the James constant of ℓ_ψ-ℓ_φ.

Theorem 3.3. *Let* $\psi,\varphi\in\Psi_2$ *with* $\psi(t)\le\varphi(t)$ *for all* $t\in[0,1]$, *let* $M_{\varphi,\psi} = \max_{0\le t\le 1}\varphi(t)/\psi(t)$, *and let* $\delta_\psi(\cdot)$ *be the modulus of convexity of* ℓ_ψ-ℓ_ψ. *Then for* $\varepsilon \in [0,2]$,

$$\delta_{\psi,\varphi}(\varepsilon) \ge \min\left\{1 - M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)),\ \delta_\psi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right\}, \tag{3.1}$$

where $\delta_{\psi,\varphi}(\cdot)$ *is the modulus of convexity of* ℓ_ψ-ℓ_φ. *Consequently,*

$$J(\ell_\psi\text{-}\ell_\varphi) \le \sup\left\{\varepsilon \in (0,2) : \varepsilon \le 2M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)) \text{ or } \varepsilon \le 2\left(1-\delta_\psi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right)\right\}. \tag{3.2}$$

Proof. By Lemma 3.1(ii), we have

$$\|\cdot\|_\psi \le \|\cdot\|_{\psi,\varphi} \le M_{\varphi,\psi}\|\cdot\|_\psi. \tag{3.3}$$

We now evaluate the modulus of convexity $\delta_{\psi,\varphi}$ for ℓ_ψ-ℓ_φ. We consider two cases.

Case 1. Take $\|x\|_{\psi,\varphi} = \|y\|_{\psi,\varphi} = 1$ with $\|x-y\|_{\psi,\varphi} \ge \varepsilon$, where $x - y \in Q_1 \cup Q_3$. Thus $\|x\|_\psi \le 1$, $\|y\|_\psi \le 1$, and $\|x-y\|_\psi \ge \varepsilon$, which implies that

$$\frac{1}{2}\|x+y\|_\psi \le 1 - \delta_\psi(\varepsilon). \tag{3.4}$$

This in turn implies

$$\frac{1}{2}\|x+y\|_{\psi,\varphi} \le \frac{1}{2}M_{\varphi,\psi}\|x+y\|_\psi \le M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)), \tag{3.5}$$

thus

$$1 - \frac{1}{2}\|x+y\|_{\psi,\varphi} \ge 1 - M_{\varphi,\psi}(1-\delta_\psi(\varepsilon)). \tag{3.6}$$

Case 2. Now take x, y as above, but with $x - y \in Q_2^\circ \cup Q_4^\circ$. By Lemma 3.2(iii), $x + y \in Q_1 \cup Q_3$. Since $\|x-y\|_{\psi,\varphi} \ge \varepsilon$,

$$\|x-y\|_\psi \ge \frac{\|x-y\|_{\psi,\varphi}}{M_{\varphi,\psi}} \ge \frac{\varepsilon}{M_{\varphi,\psi}}. \tag{3.7}$$

Then

$$\frac{1}{2}\|x+y\|_{\psi,\varphi} = \frac{1}{2}\|x+y\|_{\psi} \leq 1-\delta_{\psi}\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right), \tag{3.8}$$

and so

$$1-\frac{1}{2}\|x+y\|_{\psi,\varphi} \geq \delta_{\psi}\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right). \tag{3.9}$$

Hence we obtain (3.1). By Proposition 1.1(vi), (3.2) follows. □

The following corollary shows that we can have equality in (3.2).

Corollary 3.4 [4, 8]. *If $1 \leq q \leq p < \infty$ and $p \geq 2$, then*

$$J(\ell_p\text{-}\ell_q) \leq 2\left(\frac{2^{p/q}}{2^{p/q}+2}\right)^{1/p}. \tag{3.10}$$

In particular, if $p = 2$ and $q = 1$, then $J(\ell_2\text{-}\ell_1) = \sqrt{8/3}$.

Proof. It follows that since

$$M_{\psi_q,\psi_p} = 2^{1/q-1/p}, \qquad \delta_{\ell_p\text{-}\ell_p}(\varepsilon) = 1-\left(1-\left(\frac{\varepsilon}{2}\right)^{p}\right)^{1/p}. \tag{3.11}$$

Moreover, if $p = 2$ and $q = 1$, then $J(\ell_2\text{-}\ell_1) \leq \sqrt{8/3}$. Now we put

$$x_0 = \left(\frac{2+\sqrt{2}}{2\sqrt{3}}, \frac{2-\sqrt{2}}{2\sqrt{3}}\right), \qquad y_0 = \left(\frac{2-\sqrt{2}}{2\sqrt{3}}, \frac{2+\sqrt{2}}{2\sqrt{3}}\right). \tag{3.12}$$

Then

$$||x_0||_{2,1} = ||y_0||_{2,1} = 1, \qquad ||x_0 \pm y_0||_{2,1} = \sqrt{\frac{8}{3}}. \tag{3.13}$$

□

Theorem 3.5. *Let $\psi,\varphi \in \Psi_2$ with $\psi(t) \leq \varphi(t)$ for all $t \in [0,1]$, let $M_{\varphi,\psi} = \max_{0\leq t\leq 1}\varphi(t)/\psi(t)$, and let $\delta_{\varphi}(\cdot)$ be the modulus of convexity of $\ell_{\varphi}\text{-}\ell_{\varphi}$. Then for $\varepsilon \in [0,2]$,*

$$\delta_{\psi,\varphi}(\varepsilon) \geq 1-M_{\varphi,\psi}\left(1-\delta_{\varphi}\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right), \tag{3.14}$$

where $\delta_{\psi,\varphi}(\cdot)$ is the modulus of convexity of $\ell_{\psi}\text{-}\ell_{\varphi}$. Consequently,

$$J(\ell_{\psi}\text{-}\ell_{\varphi}) \leq \sup\left\{\varepsilon \in (0,2) : \varepsilon \leq 2M_{\varphi,\psi}\left(1-\delta_{\varphi}\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right)\right\}. \tag{3.15}$$

Proof. By Lemma 3.1(iii), we have

$$\frac{1}{M_{\varphi,\psi}}\|\cdot\|_{\varphi} \leq \|\cdot\|_{\psi,\varphi} \leq \|\cdot\|_{\varphi}. \tag{3.16}$$

We now evaluate the modulus of convexity $\delta_{\psi,\varphi}$ for ℓ_ψ-ℓ_φ. Let

$$\|x\|_{\psi,\varphi} = \|y\|_{\psi,\varphi} = 1 \quad \text{with } \|x-y\|_{\psi,\varphi} \geq \varepsilon. \tag{3.17}$$

Then

$$\begin{gathered} \frac{1}{M_{\varphi,\psi}}\|x\|_\varphi \leq 1, \qquad \frac{1}{M_{\varphi,\psi}}\|y\|_\varphi \leq 1, \\ \frac{1}{M_{\varphi,\psi}}\|x-y\|_\varphi \geq \frac{1}{M_{\varphi,\psi}}\|x-y\|_{\psi,\varphi} \geq \frac{\varepsilon}{M_{\varphi,\psi}}, \end{gathered} \tag{3.18}$$

which implies that

$$\frac{1}{2M_{\varphi,\psi}}\|x+y\|_\varphi \leq 1 - \delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right). \tag{3.19}$$

This in turn implies that

$$\frac{1}{2M_{\varphi,\psi}}\|x+y\|_{\psi,\varphi} \leq \frac{1}{2M_{\varphi,\psi}}\|x+y\|_\varphi \leq 1 - \delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right), \tag{3.20}$$

thus

$$1 - \frac{1}{2}\|x+y\|_{\psi,\varphi} \geq 1 - M_{\varphi,\psi}\left(1 - \delta_\varphi\left(\frac{\varepsilon}{M_{\varphi,\psi}}\right)\right). \tag{3.21}$$

Hence we obtain (3.14). By Proposition 1.1(vi), (3.15) follows. □

Corollary 3.6. *If $2 \leq q \leq p < \infty$, then*

$$J(\ell_p\text{-}\ell_q) \leq 2^{1-1/p}. \tag{3.22}$$

It is easy to see that the estimate (3.22) is better than one obtained in [4, Example 2.4(3)].

For some generalized Day-James spaces, [8, Corollary 4] of Kato et al. gives only rough result for the estimate of the James constant, that is, for $\psi \in \Psi_2$,

$$\frac{2}{M} \leq J(\ell_\psi\text{-}\ell_\infty) \leq 2M, \tag{3.23}$$

where $M = \max_{0\leq t\leq 1} \psi_\infty(t)/\psi(t)$.

However, the following theorem gives the exact value of the James constant of these spaces.

Theorem 3.7. *Let $\psi \in \Psi_2$. Then*

$$J(\ell_\psi\text{-}\ell_\infty) = 1 + \frac{1/2}{\psi(1/2)}. \tag{3.24}$$

Proof. For our convenience, we write $\|\cdot\|$ instead of $\|\cdot\|_{\psi,\psi_\infty}$. Let $x,y\in S_{\ell_\psi\text{-}\ell_\infty}$. We prove that

$$\text{either } \|x+y\| \le 1+\frac{1/2}{\psi(1/2)} \quad \text{or} \quad \|x-y\| \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.25}$$

Let us consider the following cases.

Case 1. $x,y\in Q_1$. Let $x=(a,b)$ and $y=(c,d)$ where $a,b,c,d\in[0,1]$. By Lemma 3.2(i), we have $x-y\in Q_2\cup Q_4$. Then

$$\|x-y\| = \max\{|a-c|,\ |b-d|\} \le 1 \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.26}$$

Case 2. $x,y\in Q_2$. If x,y lies in the same segment, then $\|x-y\|\le 1$. We now suppose that $x=(-1,a)$ and $y=(-c,1)$ where $a,c\in[0,1]$.

Subcase 2.1. $a\le(1/2)/\psi(1/2)$ and $c\le(1/2)/\psi(1/2)$. Then

$$\|x+y\| = \big\|(-1-c,1+a)\big\|_\infty = \max\{1+c,\ 1+a\} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.27}$$

Subcase 2.2. $a\ge(1/2)/\psi(1/2)$ or $c\ge(1/2)/\psi(1/2)$. Put $z=(-1,1)$, then

$$\|x-y\| \le \|x-z\|+\|z-y\| = 1-a+1-c \le 1+1-\frac{1/2}{\psi(1/2)} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.28}$$

From now on, we may assume without loss of generality that there is $\beta\in[1/2,1]$ such that $\psi(\beta)\le\psi(t)$ for all $t\in[0,1]$. Indeed, $J(\ell_\psi\text{-}\ell_\infty)=J(\ell_{\tilde\psi}\text{-}\ell_\infty)$ where $\tilde\psi(t)=\psi(1-t)$ for all $t\in[0,1]$.

Case 3. $x\in Q_1$ and $y\in Q_2$. Let $x=(a,b)$, $y=(-c,1)$ where $a,b,c\in[0,1]$. We consider three subcases.

Subcase 3.1. $a\le(1/2)/\psi(1/2)$ or $c\le(1/2)/\psi(1/2)$. Then

$$\|x-y\| = \big\|(a+c,b-1)\big\|_\infty = \max\{a+c,\ 1-b\} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.29}$$

Subcase 3.2. $(1/2)/\psi(1/2)\le a\le c$. Then $b\le(1/2)/\psi(1/2)$ and

$$\|x+y\| = \big\|(a-c,b+1)\big\|_\infty = \max\{c-a,\ 1+b\} \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.30}$$

Subcase 3.3. $(1/2)/\psi(1/2)<c\le a$. We write $a=(1-t_0)/\psi(t_0)$, $b=t_0/\psi(t_0)$ where $t_0=b/(a+b)$ and $0\le t_0\le 1/2$. By the convexity of ψ and $\psi(t)\ge\psi(\beta)$ for all $0\le t\le 1$, we

have $\psi(t_0) \geq \psi(1/2)$ and so $1/\psi(t_0) \leq 1/\psi(1/2)$. By Lemma 3.1(i),

$$\begin{aligned}\|x+y\| &= \|(a,b)+(-c,1)\| \leq \|(a-c,b+1)\|_1 \\ &= a-c+b+1 = \frac{1}{\psi(t_0)}+1-c \\ &\leq \frac{1}{\psi(1/2)}+1-\frac{1/2}{\psi(1/2)} = 1+\frac{1/2}{\psi(1/2)}.\end{aligned} \tag{3.31}$$

Case 4. $x \in Q_1$ and $y \in Q_2$. Let $x=(a,b)$, $y=(-1,c)$ where $a,b,c \in [0,1]$. We consider three subcases.

Subcase 4.1. $b \leq (1/2)/\psi(1/2)$ or $c \leq (1/2)/\psi(1/2)$. Then

$$\|x+y\| = \|(a-1,b+c)\|_\infty = \max\{1-a,\ b+c\} \leq 1+\frac{1/2}{\psi(1/2)}. \tag{3.32}$$

Subcase 4.2. $(1/2)/\psi(1/2) < b \leq c$. Then $a \leq (1/2)/\psi(1/2)$ and

$$\|x-y\| = \|(1+a,b-c)\|_\infty = \max\{1+a,c-b\} \leq 1+\frac{1/2}{\psi(1/2)}. \tag{3.33}$$

Subcase 4.3. $(1/2)/\psi(1/2) < c \leq b$. We write $a=(1-t_0)/\psi(t_0)$, $b=t_0/\psi(t_0)$, where $t_0 = b/(a+b)$ and $1/2 \leq t_0 \leq 1$. We choose $\alpha = b/(a+2b-1)$, then

$$\frac{1}{2} \leq \alpha \leq 1, \qquad a = \frac{1-2\alpha}{\alpha}b+1. \tag{3.34}$$

Since $b-c \leq 1+a$ and $b \leq 1$,

$$\frac{b-c}{1+a+b-c} \leq \frac{1}{2} \leq t_0 \leq \alpha. \tag{3.35}$$

Let

$$\psi_\alpha(t) = \begin{cases} \dfrac{\alpha-1}{\alpha}t+1 & \text{if } 0 \leq t \leq \alpha, \\ t & \text{if } \alpha \leq t \leq 1. \end{cases} \tag{3.36}$$

We see that $\psi_\alpha(t_0) = \psi(t_0)$. By the convexity of ψ, we have

$$\psi(t) \leq \psi_\alpha(t) \quad \forall t \leq t_0. \tag{3.37}$$

Therefore,

$$\begin{aligned}
\|x-y\| &= \|(a+1,b-c)\|_{\psi} = (1+a+b-c)\psi\left(\frac{b-c}{1+a+b-c}\right)\\
&\le (1+a+b-c)\psi_{\alpha}\left(\frac{b-c}{1+a+b-c}\right) = \frac{\alpha-1}{\alpha}(b-c)+1+a+b-c\\
&= 1+a+\frac{2\alpha-1}{\alpha}b-\frac{2\alpha-1}{\alpha}c = 1+1-\frac{2\alpha-1}{\alpha}c\\
&< 1+1-\frac{2\alpha-1}{\alpha}\frac{1/2}{\psi(1/2)} = 1+\frac{1/2}{\psi(1/2)}+1-\frac{3\alpha-1}{2\alpha}\frac{1}{\psi(1/2)}\\
&= 1+\frac{1/2}{\psi(1/2)}+1-\frac{\psi_{\alpha}(1/2)}{\psi(1/2)} \le 1+\frac{1/2}{\psi(1/2)}.
\end{aligned} \tag{3.38}$$

Finally, we conclude that

$$J(\ell_{\psi}\text{-}\ell_{\infty}) \le 1+\frac{1/2}{\psi(1/2)}. \tag{3.39}$$

Now, we put $x_0 = ((1/2)/\psi(1/2),(1/2)/\psi(1/2))$ and $y_0 = (-1,1)$, then

$$\|x_0\| = \|y_0\| = 1, \qquad \|x_0 \pm y_0\| = 1+\frac{1/2}{\psi(1/2)}. \tag{3.40}$$

Thus,

$$J(\ell_{\psi}\text{-}\ell_{\infty}) \ge \min\{\|x_0-y_0\|, \|x_0+y_0\|\} = 1+\frac{1/2}{\psi(1/2)}. \tag{3.41}$$

This together with (3.39) completes the proof. □

Corollary 3.8 [4, Example 2.4(2)]. *Let $1 \le p \le \infty$, then*

$$J(\ell_p\text{-}\ell_{\infty}) = 1+\left(\frac{1}{2}\right)^{1/p}. \tag{3.42}$$

Indeed, $\psi_p(1/2) = 2^{1/p-1}$.

We now obtain the bounds for $J(\ell_{\psi}\text{-}\ell_1)$.

Corollary 3.9. *Let $\psi \in \Psi_2$. Then*

$$2\min_{0\le t\le 1}\psi(t) \le J(\ell_{\psi}\text{-}\ell_1) \le \frac{3}{2}+\frac{1}{2}\min_{0\le t\le 1}\psi(t). \tag{3.43}$$

Proof. Note that $\psi^*(1/2) = \max_{0\le t\le 1}(1/2)/\psi(t) = 1/2\min_{0\le t\le 1}\psi(t)$. By Theorem 3.7, we have $J(\ell_{\psi^*}\text{-}\ell_{\infty}) = 1+\min_{0\le t\le 1}\psi(t)$. Applying Proposition 1.1(iv), the assertion is obtained. □

We now improve the upper bound for $J(\ell_p\text{-}\ell_1)$ (see also Corollary 3.4).

Corollary 3.10. *Let* $1 \le p < \infty$. *Then*

$$J(\ell_p\text{-}\ell_1) \le \frac{3}{2} + \left(\frac{1}{2}\right)^{2-1/p}. \tag{3.44}$$

In particular, if $p \ge 2$, *then*

$$J(\ell_p\text{-}\ell_1) \le \min\left\{\frac{4}{(2^p+2)^{1/p}}, \frac{3}{2} + \left(\frac{1}{2}\right)^{2-1/p}\right\}. \tag{3.45}$$

The following corollary follows by Theorem 3.7 and Corollary 3.9.

Corollary 3.11. *Let* $\psi \in \Psi_2$. *Then*

(i) ℓ_ψ-ℓ_∞ *is uniformly nonsquare if and only if* $\psi \ne \psi_\infty$,
(ii) ℓ_ψ-ℓ_1 *is uniformly nonsquare if and only if* $\psi \ne \psi_1$.

We can say more about the uniform nonsquareness of ℓ_ψ-ℓ_φ.

Theorem 3.12. *Let* $\psi, \varphi \in \Psi_2$. *Then all* ℓ_ψ-ℓ_φ *except* ℓ_1-ℓ_1 *and* ℓ_∞-ℓ_∞ *are uniformly nonsquare.*

Proof. If $\psi = \varphi$, we are done by [10, Corollary 3]. Assume that $\psi \ne \varphi$. We prove that ℓ_ψ-ℓ_φ is uniformly nonsquare. Suppose not, that is, there are $x, y \in S_{\ell_\psi\text{-}\ell_\varphi}$ such that $\|x \pm y\|_{\psi,\varphi} = 2$. We consider three cases.

Case 1. $x, y \in Q_1$. Then

$$\begin{aligned} \|x\|_{\psi,1} &= \|x\|_\psi = \|x\|_{\psi,\varphi} = 1, \\ \|y\|_{\psi,1} &= \|y\|_\psi = \|y\|_{\psi,\varphi} = 1. \end{aligned} \tag{3.46}$$

It follows by Lemma 3.2(i) that $x + y \in Q_1$ and $x - y \in Q_2 \cup Q_4$. Therefore

$$\begin{aligned} \|x+y\|_{\psi,1} &= \|x+y\|_{\psi,\varphi} = 2, \\ 2 = \|x-y\|_{\psi,\varphi} &\le \|x-y\|_1 = \|x-y\|_{\psi,1} \le 2. \end{aligned} \tag{3.47}$$

Hence $\|x \pm y\|_{\psi,1} = 2$ and this implies that ℓ_ψ-ℓ_1 is not uniformly nonsquare. By Corollary 3.11(ii), we have $\psi = \psi_1$. Again, since ℓ_ψ-$\ell_\varphi = \ell_1$-ℓ_φ is not uniformly nonsquare, $\varphi = \psi_1 = \psi$; a contradiction.

Case 2. $x, y \in Q_2$. It is similar to Case 1, so we omit the proof.

Case 3. $x := (a, b) \in Q_1$ and $y := (-c, d) \in Q_2$ where $a, b, c, d \in [0, 1]$. Since $\|x + y\|_{\psi,\varphi} = 2$, the line segment joining x and y must lie in the sphere. In particular, there is $\alpha \in [0, 1]$ such that

$$(0, 1) = \alpha x + (1 - \alpha) y. \tag{3.48}$$

It follows that $b = 1$ since $b, d \le 1$. Similarly consider x and $-y$ instead of x and y, we can also conclude that $a = 1$. Hence $\|(1,1)\|_\psi = \|(1,1)\|_{\psi,\varphi} = 1$, that is, $\psi(1/2) = 1/2$. Then $\psi = \psi_\infty$ and so ℓ_ψ-$\ell_\varphi = \ell_\infty$-$\ell_\varphi$ is not uniformly nonsquare. By Corollary 3.11(i), we have $\varphi = \psi_\infty = \psi$; a contradiction. □

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