AN UPPER BOUND FOR THE ℓ_p NORM OF A GCD-RELATED MATRIX

PENTTI HAUKKANEN

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We find an upper bound for the ℓ_p norm of the $n \times n$ matrix whose ij entry is $(i,j)^s/[i,j]^r$, where (i,j) and $[i,j]$ are the greatest common divisor and the least common multiple of i and j and where r and s are real numbers. In fact, we show that if $r > 1/p$ and $s < r - 1/p$, then $\|((i,j)^s/[i,j]^r)_{n\times n}\|_p < \zeta(rp)^{2/p}\zeta(rp-sp)^{1/p}/\zeta(2rp)^{1/p}$ for all positive integers n, where ζ is the Riemann zeta function.

1. Introduction

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers, and let f be an arithmetical function. Let $(S)_f$ denote the $n \times n$ matrix having f evaluated at the greatest common divisor (x_i, x_j) of x_i and x_j as its ij entry, that is, $(S)_f = (f((x_i, x_j)))$. Analogously, let $[S]_f$ denote the $n \times n$ matrix having f evaluated at the least common multiple $[x_i, x_j]$ of x_i and x_j as its ij entry, that is, $[S]_f = (f([x_i, x_j]))$. The matrices $(S)_f$ and $[S]_f$ are referred to as the GCD and LCM matrix on S associated with f respectively. Smith [12] calculated $\det(S)_f$ when S is a factor-closed set and $\det[S]_f$ in a more special case. Since Smith a large number of results on GCD and LCM matrices have been presented in the literature. For general accounts see, for example, [3, 5–8].

Norms of GCD matrices have not been studied much in the literature. Some results are obtained in [1, 4], see also the references of [4] and [10, Chapter 3].

Let $p \in \mathbf{Z}^+$. The ℓ_p norm of an $n \times n$ matrix M is defined as

$$\|M\|_p = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} |m_{ij}|^p\right)^{1/p}. \tag{1.1}$$

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Let $r, s \in \mathbf{R}$. It is known [1, Theorem 3] that if $r > 1/p$, then

$$\lim_{n\to\infty} \left\| \left(\frac{1}{[i,j]^r} \right)_{n\times n} \right\|_p = \frac{\zeta(pr)^{3/p}}{\zeta(2pr)^{1/p}}. \tag{1.2}$$

We here generalize this result by showing that if $r > 1/p$ and $s < r - 1/p$, then

$$\lim_{n\to\infty} \left\| \left(\frac{(i,j)^s}{[i,j]^r} \right)_{n\times n} \right\|_p = \frac{\zeta(pr)^{2/p}\zeta(pr-ps)^{1/p}}{\zeta(2pr)^{1/p}}, \tag{1.3}$$

see Theorem 3.1. This result also sharpens the rough estimation

$$\left\| \left(\frac{(i,j)^s}{[i,j]^r} \right)_{n\times n} \right\|_p = O(1) \tag{1.4}$$

given in [4, Theorem 3.1(3)].

2. Preliminaries

In this section we review the basic results on arithmetical functions needed in this paper. For more comprehensive treatments on arithmetical functions we refer to [2, 9–11].

The Dirichlet convolution $f * g$ of two arithmetical functions f and g is defined as

$$(f*g)(n) = \sum_{d\mid n} f(d)g(n/d). \tag{2.1}$$

Let N^u, $u \in \mathbf{R}$, denote the arithmetical function defined as $N^u(n) = n^u$ for all $n \in \mathbf{Z}^+$, and let E denote the arithmetical function defined as $E(n) = 1$ for all $n \in \mathbf{Z}^+$. The Jordan totient function $J_k(n)$, $k \in \mathbf{Z}^+$, is defined as the number of k-tuples $a_1, a_2, \ldots, a_k \pmod n$ such that the greatest common divisor of $a_1, a_2, \ldots, a_k$ and n is 1. By convention, $J_k(1) = 1$. The Möbius function μ is the inverse of E under the Dirichlet convolution. It is well known that $J_k = N^k * \mu$. This suggests we define

$$J_u(n) = (N^u * \mu)(n) = \sum_{d\mid n} d^u \mu(n/d) \tag{2.2}$$

for all $u \in \mathbf{R}$. Since μ is the inverse of E under the Dirichlet convolution, we have

$$n^u = \sum_{d\mid n} J_u(d). \tag{2.3}$$

An arithmetical function f is said to be multiplicative if $f(1) = 1$ and

$$f(mn) = f(m)f(n) \tag{2.4}$$

whenever $(m,n) = 1$, and an arithmetical function f is said to be completely multiplicative if $f(1) = 1$ and (2.4) holds for all m and n. For example, the function N^u is completely multiplicative. Each completely multiplicative function f distributes over the Dirichlet convolution, that is,

$$f(g*h) = (fg)*(fh) \tag{2.5}$$

for all arithmetical functions g and h. The inverse f^{-1} of a completely multiplicative function f under the Dirichlet convolution is given as

$$f^{-1} = \mu f. \tag{2.6}$$

The Dirichlet series of an arithmetical function f is defined as

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}, \tag{2.7}$$

where we assume (for brevity) that s is a real number. The Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{2.8}$$

where $s > 1$. If the series $\sum_{n=1}^{\infty} f(n)/n^s$ and $\sum_{n=1}^{\infty} g(n)/n^s$ converge absolutely for $s > s_0$, then

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \sum_{n=1}^{\infty} \frac{g(n)}{n^s} = \sum_{n=1}^{\infty} \frac{(f*g)(n)}{n^s} \tag{2.9}$$

and this last series converges absolutely for $s > s_0$. Further, if the inverse f^{-1} of f under the Dirichlet convolution exists, then

$$\sum_{n=1}^{\infty} \frac{f^{-1}(n)}{n^s} = \left(\sum_{n=1}^{\infty} \frac{f(n)}{n^s} \right)^{-1} \tag{2.10}$$

and this series also converges absolutely for $s > s_0$.

3. Results

THEOREM 3.1. *Let $r > 1/p$ and $s < r - 1/p$. Then*

$$\lim_{n\to\infty} \left\| \left(\frac{(i,j)^s}{[i,j]^r} \right)_{n\times n} \right\|_p = \frac{\zeta(rp)^{2/p}\zeta(rp-sp)^{1/p}}{\zeta(2rp)^{1/p}}. \tag{3.1}$$

Proof. Denote

$$s_n = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{(i,j)^{sp}}{[i,j]^{rp}}. \tag{3.2}$$

Since $(i,j)[i,j]=ij$, we have for all p, r, s

$$s_n=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(i,j)^{(r+s)p}}{i^{rp}j^{rp}}. \tag{3.3}$$

It is clear that

$$s_n<\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(i,j)^{(r+s)p}}{i^{rp}j^{rp}}. \tag{3.4}$$

Making the change of variables $\lambda=(i,j)$, $i=u\lambda$ and $j=v\lambda$, we see that

$$\begin{aligned} s_n&<\sum_{u=1}^{\infty}\sum_{v=1}^{\infty}\sum_{\lambda=1}^{\infty}\frac{\lambda^{(s-r)p}}{u^{rp}v^{rp}}\\ &=\left(\sum_{u=1}^{\infty}\frac{1}{u^{rp}}\right)\left(\sum_{v=1}^{\infty}\frac{1}{v^{rp}}\right)\left(\sum_{\lambda=1}^{\infty}\frac{1}{\lambda^{(r-s)p}}\right)\\ &=\zeta(rp)^2\zeta(rp-sp). \end{aligned} \tag{3.5}$$

Note that all these series have only positive terms and rp, $rp-sp>1$. Thus, $\{s_n\}$ is increasing and bounded above, and so $\lim_{n\to\infty}s_n=S$ exists. We deduce that the double series $\sum\sum((i,j)^{sp}/[i,j]^{rp})$ converges absolutely, with sum S.

We calculate the number S as follows. We have

$$S=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(i,j)^{sp}}{[i,j]^{rp}}=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(i,j)^{(r+s)p}}{i^{rp}j^{rp}}. \tag{3.6}$$

From (2.3) we obtain

$$\begin{aligned} S&=\sum_{i=1}^{\infty}\frac{1}{i^{rp}}\sum_{j=1}^{\infty}\frac{1}{j^{rp}}\sum_{d\mid(i,j)}J_{(r+s)p}(d)\\ &=\sum_{i=1}^{\infty}\frac{1}{i^{rp}}\sum_{d\mid i}J_{(r+s)p}(d)\sum_{\substack{1\le j<\infty\\ j\equiv 0\ (\mathrm{mod}\ d)}}\frac{1}{j^{rp}}. \end{aligned} \tag{3.7}$$

Since $rp>1$, we can write

$$S=\zeta(rp)\sum_{i=1}^{\infty}\frac{1}{i^{rp}}\sum_{d\mid i}\frac{J_{(r+s)p}(d)}{d^{rp}}. \tag{3.8}$$

Since the function $1/d^{rp}$ (i.e., the function N^{-rp}) is completely multiplicative in d, on the basis of (2.2) and (2.5) we have

$$S=\zeta(rp)\sum_{i=1}^{\infty}\frac{1}{i^{rp}}\left(E*N^{sp}*\mu N^{-rp}\right)(i). \tag{3.9}$$

Since the function $1/i^{rp}$ (i.e., the function N^{-rp} again) is completely multiplicative in i, on the basis of (2.5) again we have

$$S = \zeta(rp)\sum_{i=1}^{\infty}\left(N^{-rp} * N^{-(rp-sp)} * \mu N^{-2rp}\right)(i). \tag{3.10}$$

Since $rp, rp - sp > 1$, we can apply (2.6)–(2.10) to obtain

$$S = \zeta(rp)\zeta(rp)\zeta(rp-sp)/\zeta(2rp). \tag{3.11}$$

This completes the proof of Theorem 3.1. □

Corollary 3.2. *Let $r > 1/p$ and $s < r - 1/p$. Then, for all $n \in \mathbf{Z}^+$,*

$$\left\|\left(\frac{(i,j)^s}{[i,j]^r}\right)_{n\times n}\right\|_p < \frac{\zeta(rp)^{2/p}\zeta(rp-sp)^{1/p}}{\zeta(2rp)^{1/p}}. \tag{3.12}$$

The spectral norm of an $n \times n$ matrix M is defined as

$$\|M\|_S = \max\left\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } M^*M\right\}. \tag{3.13}$$

Corollary 3.3. *Let $r > 1/2$ and $s < r - 1/2$. Then, for all $n \in \mathbf{Z}^+$,*

$$\left\|\left(\frac{(i,j)^s}{[i,j]^r}\right)_{n\times n}\right\|_S < \frac{\left(\zeta(2r)\zeta(2(r-s))\right)^{1/2}}{\zeta(4r)^{1/2}}. \tag{3.14}$$

Proof. It is known that $\|M\|_S \le \|M\|_2$. Thus Corollary 3.3 follows from Corollary 3.2. □

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Pentti Haukkanen: Department of Mathematics, Statistics and Philosophy, University of Tampere, 33014 Tampere, Finland
E-mail address: mapehau@uta.fi