A LINEARIZATION METHOD IN OSCILLATION THEORY OF HALF-LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

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Using inequalities for a certain function appearing in the half-linear version of Picone's identity, we show that oscillatory properties of the half-linear second-order differential equation $(r(t)\Phi(x'))' + c(t)\Phi(x) = 0$, $\Phi(x) = |x|^{p-2}x$, $p > 1$, can be investigated via oscillatory properties of a certain associated second-order *linear* differential equation. This linear equation plays the role of a Sturmian majorant, in a certain sense, if $p \geq 2$, and the role of a minorant if $p \in (1,2]$.

1. Introduction

In this paper, we deal with oscillatory properties of the half-linear second-order differential equation

$$\big(r(t)\Phi(x')\big)' + c(t)\Phi(x) = 0, \quad \Phi(x) = |x|^{p-2}x, \ p > 1. \tag{1.1}$$

In the recent years, considerable similarity between oscillatory properties of (1.1) and its special case, the linear Sturm-Liouville equation

$$\big(r(t)x'\big)' + c(t)x = 0, \tag{1.2}$$

has been found, see, for example, [1] and the references given therein. On the other hand, some natural differences were pointed out, mostly caused by the fact that the solution space of (1.1) has only one half of the properties which characterize linearity, namely homogeneity, but generally not additivity. This fact is also a motivation for the terminology *half-linear* equation.

One of the important differences between (1.1) and (1.2) is missing transformation theory for half-linear equations. More precisely, the transformation $x = h(t)y$, where h is a differentiable function such that rh' is also differentiable, applied to (1.2), gives the following (linear) identity:

$$h(t)\big[\big(r(t)x'\big)' + c(t)x\big] = \big(R(t)y'\big)' + C(t)y, \tag{1.3}$$

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where

$$R(t) = h^2(t)r(t), \qquad C(t) = h(t)\left[\left(r(t)h'(t)\right)' + c(t)h(t)\right]. \tag{1.4}$$

In particular, if x is a solution of (1.2), y is a solution of the equation

$$\left(R(t)y'\right)' + C(t)y = 0. \tag{1.5}$$

Identity (1.3) can be verified by a direct differentiation and one can easily find that the important role is played by the linearity of the differential operator $(r(t)x')'$. Since the operator $(r(t)\Phi(x'))'$ is no longer linear, the transformation (1.3) has no immediate half-linear extension.

The transformation formula (1.3) for linear equation (1.2) is used in many oscillation criteria for this equation, with the following idea. Equation (1.2) is transformed into an "easier" equation (1.5), oscillatory properties of the easier equation (1.5) are studied and then the obtained results are "translated" back to original equation (1.2). The missing half-linear extension of (1.3) excludes the possibility to study (1.1) using this method.

In this paper we try, in a certain sense, to eliminate this disadvantage of the qualitative theory of (1.1). We elaborate a method which enables to compare oscillatory properties of (1.1) with oscillatory properties of a certain associated *linear* equation of the form (1.2) and then to use the results of the deeply developed linear oscillation theory in the investigation of (1.1). If $p = 2$ in (1.1), this "certain associated linear equation" is just (1.5), so our method can be regarded as an extension of the linear transformation method to half-linear equations (1.1). The linearization method which we establish is based on the inequalities for the function P which appears in the half-linear Picone identity (see Lemma 2.1 below) and on inequalities for solutions of Riccati equations associated with (1.2).

The idea to investigate half-linear equations via associated linear equations has been introduced in the paper [5], where the case $r(t) \equiv 1$ in (1.1) considered, (1.1) is viewed as a perturbation of the half-linear Euler-type differential and then it is compared with some associated linear differential equation. Here we extend this idea to the general half-linear equation (1.1). This equation is viewed as a perturbation of a half-linear equation of the same form (not necessarily of Euler-type) and then compared with an associated linear equation. Similarly to results of [5], we find considerable difference between the case $p \geq 2$ and $1 < p \leq 2$ in (1.1).

2. Preliminaries

In this section, we recall some basic facts of the half-linear oscillation theory. Recall that (1.1) is said to be *disconjugate* in an interval $I \subset \mathbb{R}$ if every nontrivial solution of this equation has at most one zero point in I. Equation (1.1) is said to be *nonoscillatory* if there exists $T \in \mathbb{R}$ such that this equation is disconjugate on $[T, \infty)$, in the opposite case (1.1) is said to be *oscillatory*. This definition of oscillation and nonoscillation of (1.1) is the same as in the linear case since it is known, see, for example, [2] or [4], that the linear Sturmian theory extends verbatim to (1.1), in particular, all solutions of this equations are either oscillatory, that is, have infinitely many zeros tending to ∞, or are nonoscillatory, they have only a finite number of zeros on each interval of the form $[T,\infty)$.

If x is a solution of (1.1) such that $x(t) \neq 0$ in some interval $I \subset \mathbb{R}$, then the function $w = r\Phi(x')/\Phi(x)$ is a solution of the generalized Riccati equation

$$w' + c(t) + (p-1)r^{1-q}(t)|w|^q = 0, \tag{2.1}$$

where q is the conjugate number of p, that is, $1/p + 1/q = 1$.

An important role in our investigation is played by half-linear Picone's identity. We present this identity in a modified form here, the general formulation can be found in [7].

Lemma 2.1. *Let w be a solution of (2.1) which exists on some interval $I \subset \mathbb{R}$. Then for every function ξ which is differentiable in this interval*

$$r(t)|\xi'|^p - c(t)|\xi|^p = \left[w(t)|\xi|^p\right]' + pr^{1-q}(t)P\left(r^{q-1}\xi', w(t)\Phi(\xi)\right), \tag{2.2}$$

where

$$P(u,v) = \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \geq 0 \tag{2.3}$$

with equality if and only if $v = \Phi(u)$.

In the following lemma, the function $P(u,v)$ is compared with a certain associated quadratic function. The proof of inequalities given in this lemma can be found, for example, in [3].

Lemma 2.2. *We have the following inequalities for $u,v \in \mathbb{R}$, $u \neq 0$:*

$$P(u,v) \leq \frac{1}{2}|u|^{2-p}\left(v - \Phi(u)\right)^2, \quad p \geq 2, \tag{2.4}$$

$$P(u,v) \geq \frac{1}{2}|u|^{2-p}\left(v - \Phi(u)\right)^2, \quad p \in (1,2]. \tag{2.5}$$

We will also need the following inequalities for solutions of a pair of Riccati equations associated with (1.1), for the proof, which follows from a general statement for solutions of differential inequalities, we refer to [6].

Lemma 2.3. *Consider a pair of half-linear differential equations (1.1) and*

$$\left(R(t)\Phi(y')\right)' + C(t)\Phi(y) = 0, \tag{2.6}$$

where the functions R, C satisfy the same assumptions as r and c. Suppose that (2.6) is a Sturmian majorant of (1.1) in an interval $I \subset \mathbb{R}$, that is, $0 < R(t) \leq r(t)$, $c(t) \leq C(t)$ for $t \in I$, and suppose that Riccati equation associated with (2.6)

$$v' + C(t) + (p-1)R^{1-q}(t)|v|^q = 0, \tag{2.7}$$

has a solution v which exists on the whole interval I. If w is a solution of the Riccati equation (2.1) associated with (1.1), such that $w(t_0) = v(t_0)$ for some $t_0 \in I$, then w also exists on the whole interval I. Moreover, $w(t) \geq v(t)$ for $t \geq t_0$ and $w(t) \leq v(t)$ for $t \leq t_0$.

We finish this section with basic properties of "half-linear harmonic oscillator" equation as presented in [4]. Consider the equation

$$\left(\Phi(x')\right)' + (p-1)\Phi(x) = 0 \tag{2.8}$$

and let

$$\pi_p = \frac{2\pi}{p\sin(\pi/p)} \tag{2.9}$$

is the "half-linear π." Further, denote by $\sin_p t$ the solution of (2.8) given by the initial condition $x(0) = 0$, $x'(0) = 1$. This solution has many of the properties of the classical sine function. In particular, it is the odd $2\pi_p$ periodic function satisfying $\sin_p t > 0$ for $t \in (0, \pi_p)$, $\sin_p \pi_p = 0$, $(\sin_p t)' > 0$ for $t \in (0, \pi_p/2)$, and $(\sin_p t)' < 0$ for $t \in (\pi_p/2, \pi_p)$. We also denote $\cos_p t = (\sin_p t)'$.

3. Linearization method

Together with (1.1) we consider the equation of the same form

$$\left(r(t)\Phi(x')\right)' + \tilde{c}(t)\Phi(x) = 0, \tag{3.1}$$

this equation is supposed to be disconjugate in an interval $I \subset \mathbb{R}$ and let h be its solution such that $h(t) > 0$ for $t \in I$. Moreover, we will suppose that $h'(t) \neq 0$ in this interval. The Picone identity for (3.1) and the associated Riccati equation reads as follows:

$$r(t)|\xi'|^p - \tilde{c}(t)|\xi|^p = \left[w_h(t)|\xi|^p\right]' + pr^{1-q}(t)P\left(r^{q-1}\xi', w_h(t)\Phi(\xi)\right), \tag{3.2}$$

where

$$w_h(t) = \frac{r(t)\Phi(h')}{\Phi(h)}. \tag{3.3}$$

We denote

$$R(t) := r(t)h^2(t)\left|h'(t)\right|^{p-2}, \qquad C(t) := \left(c(t) - \tilde{c}(t)\right)h^p(t) \tag{3.4}$$

and consider the linear Sturm-Liouville equation

$$\left(R(t)y'\right)' + \frac{p}{2}C(t)y = 0. \tag{3.5}$$

The Riccati equation associated with this linear equation (related by the substitution $u = 2R(t)y'/py$) is

$$u' + C(t) + \frac{p}{2}\frac{u^2}{R(t)} = 0. \tag{3.6}$$

The first of our main results reads as follows.

Theorem 3.1. *Suppose that $p \geq 2$ and that linear equation (3.5) is disconjugate in an interval I. Then half-linear equation (1.1) is also disconjugate in this interval.*

Proof. Disconjugacy of (3.5) implies the existence of a solution u of (3.6) on interval I. Fix a $t_0 \in I$ and consider the solution of (2.1) satisfying the initial condition $w(t_0) = h^{-p}(t_0)u(t_0) + w_h(t_0)$, where w_h is given by (3.3). We will show that this solution exists on the whole interval I. This then implies the required result—disconjugacy of (1.1) on I.

Substituting $\xi = h$ in (3.2) and using the fact that $P(r^{q-1}h', w_h\Phi(h)) = 0$, we get

$$r(t)|h'|^p - \tilde{c}(t)h^p = \big[w_h(t)h^p\big]'. \tag{3.7}$$

Subtracting this equation from (2.2) with $\xi = h$ and using the identity

$$P\big(r(t)h', w(t)\Phi(h)\big) = h^p(t)P\big(\Phi_q(w_h), w\big), \quad \Phi_q(s) = |s|^{q-2}s, \tag{3.8}$$

we obtain

$$-\big(c(t) - \tilde{c}(t)\big)h^p(t) = \big[\big(w(t) - w_h(t)\big)h^p(t)\big]' + pr^{1-q}(t)h^p(t)P\big(\Phi_q(w_h), w\big). \tag{3.9}$$

Further, using inequality (2.4) of Lemma 2.2, we have

$$\begin{aligned} pr^{1-q}h^pP\big(\Phi_q(w_h), w\big) &\leq \frac{p}{2}r^{1-q}h^p\left|r^{q-1}\frac{h'}{h}\right|^{2-p}(w - w_h)^2 \\ &= \frac{p}{2}r^{1-q+(q-1)(2-p)}|h'|^{2-p}h^{p-2+p-2p}\big[h^p(w - w_h)\big]^2 \\ &= \frac{p}{2r|h'|^{p-2}h^2}\big[h^p(w - w_h)\big]^2. \end{aligned} \tag{3.10}$$

Hence, if we denote $v = h^p(w - w_h)$, using notation (3.4), we see that v satisfies the inequality

$$v' + C(t) + \frac{p}{2}\frac{v^2}{R(t)} \geq 0. \tag{3.11}$$

Denote $\tilde{C}(t) := -v' - (p/2)(v^2/R)$. Then $v' + \tilde{C}(t) + (p/2)(v^2/R) = 0$ and $C(t) - \tilde{C}(t) \geq 0$ for $t \in I$, that is, (3.5) is a majorant of the equation

$$\big(R(t)y'\big)' + \frac{p}{2}\tilde{C}(t)y = 0. \tag{3.12}$$

Since $v(t_0) = h^p(t_0)(w(t_0) - w_h(t_0)) = u(t_0)$, by Lemma 2.3 (which covers also the linear case treated here) the function v, which is a solution of the Riccati equation associated with (3.12) (again by the substitution $v = 2Ry'/py$)

$$v' + \tilde{C}(t) + \frac{p}{2}\frac{v^2}{R(t)} = 0, \tag{3.13}$$

exists on the whole interval I. This means that $w = h^{-p}v + w_h$ exists on this interval as well, and hence (1.1) is disconjugate on I. □

Theorem 3.2. *Suppose that* $1 < p \le 2$ *and that half-linear equation (1.1) is disconjugate in an interval* I. *Then linear equation (3.5) is also disconjugate in* I.

Proof. We proceed similarly as in the previous proof. Disconjugacy of (1.1) implies the existence of a solution w of the associated Riccati equation (2.1) which is defined in the whole interval I. The function $v = h^p(w - w_h)$ satisfies (3.9). Now, since $p \in (1,2]$, we have by Lemma 2.2

$$v' + C(t) + \frac{p}{2}\frac{v^2}{R(t)} \le 0. \tag{3.14}$$

We again denote $\widetilde{C} = -v' - (p/2)(v^2/R)$. Then $\widetilde{C}(t) \ge C(t)$, that is, (3.12) is a majorant of (3.5) and v satisfies (3.13). Hence the solution u of (3.6) satisfying the same initial condition as v at some $t_0 \in I$ exists on the whole interval I by Lemma 2.3 and this means that (3.5) is also disconjugate on I. □

As an immediate consequence of the previous two theorems we have the following statements.

Corollary 3.3. *The following statements hold.*

(i) *Let* $1 < p \le 2$. *If linear equation (3.5) is not disconjugate in an interval I, then (1.1) is also not disconjugate in this interval.*

(ii) *Let* $p \ge 2$. *If half-linear equation (1.1) is not disconjugate on I, then linear equation (3.5) is not disconjugate in I as well.*

Remark 3.4. (i) Roughly speaking, if $p \ge 2$, the linear equation (3.5) is a Sturmian majorant equation of (1.1), in a certain sense. For $p \in (1,2]$, the majorant equation to (3.5) is half-linear equation (1.1).

(ii) If $p = 2$, that is, (1.1) reduces to the linear equation (1.2), then (3.5) reduces to the equation

$$\left[r(t)h^2(t)y'\right]' + h^2(t)\left[c(t) - \tilde{c}(t)\right]y = 0 \tag{3.15}$$

and this is just the equation which results from (1.2) upon the transformation $x = h(t)y$, where h is a solution of (3.1). From this point of view, Theorems 3.1, 3.2 can be regarded as an extension of the linear transformation method to half-linear equations.

(iii) Elbert and Schneider [5] considered the equation

$$\left(\Phi(x')\right)' + c(t)\Phi(x) = 0 \tag{3.16}$$

as a perturbation of the Euler-type half-linear differential equation

$$\left(\Phi(x')\right)' + \frac{\gamma_p}{t^p}\Phi(x) = 0, \quad \gamma_p = \left(\frac{p-1}{p}\right)^p \tag{3.17}$$

and oscillatory properties of (3.16) are related to the oscillatory properties of the linear differential equation

$$(ty')' + \frac{1}{2}\left(\frac{p}{p-1}\right)^{p-1} t^{p-1}\left(c(t) - \gamma_p t^{-p}\right)y = 0. \tag{3.18}$$

More precisely, under the assumption that $\int^\infty (c(t) - \gamma_p t^{-p})t^{p-1}dt$ is convergent and

$$\int_t^\infty \left(c(s) - \gamma_p s^{-p}\right)s^{p-1}ds \geq 0 \quad \text{for large } t, \tag{3.19}$$

it is proved: (i) if $p > 2$, then nonoscillation of (3.18) implies nonoscillation of (3.16); (ii) if $p \in (1,2)$, then nonoscillation of (3.16) implies nonoscillation of (3.18). If we substitute $r \equiv 1$, $\tilde{c}(t) = \gamma_p t^{-p}$, $h(t) = t^{(p-1)/p}$ in Theorems 3.1, 3.2, oscillatory properties of (3.16) are related to oscillatory properties of the linear equation

$$(ty')' + \frac{p}{2}\left(\frac{p}{p-1}\right)^{p-2} t^{p-1}\left(c(t) - \gamma_p t^{-p}\right)y = 0. \tag{3.20}$$

The constant $(p/2)(p/(p-1))^{p-2}$ in (3.20) is worse than the corresponding constant in (3.18) (it is $(p-1)$-times bigger). The explanation of the fact that the constant in [5] is better is the following. In the proof of main results of that paper, a modified version of Lemma 2.2 has been used. In this modified version, the variables u, v are restricted to the region $0 \leq v \leq \Phi(u)$ and under this restriction the constant $p/2$ in (2.4), (2.5) can be replaced by a better constant $(q-1)/2$. Note that this restriction on u, v is enabled by additional assumption (3.19), see [5] for details. However, if no additional restriction on u, v is available, the constant $p/2$ in Lemma 2.2 is exact since (2.4), (2.5) reduce to equalities if $v = -\Phi(u)$.

4. Applications

Now we offer some applications of the linearization method established in the previous section. These applications are only a very limited sample of possibilities to use the results of linear oscillation theory when investigating (1.1), and are of rather straightforward character. More sophisticated applications, including looking for additional conditions under which linear equation (3.5) and half-linear equation (1.1) have the same oscillatory nature, *regardless* whether $1 < p \leq 2$ or $p \geq 2$, is a subject of the present investigation.

As stated in Corollary 3.3, in case $p \in (1,2]$, *any* conjugacy or oscillation criterion for (3.5) can be applied also to (1.1), and if $p \geq 2$, *any* disconjugacy and nonoscillation criterion for (3.5) can be used to study disconjugacy and nonoscillation of (1.1).

We start with applications which are related to the results presented in [10], where (3.16) is viewed as a perturbation of the one-term equation $(\Phi(x'))' = 0$, this approach corresponds to the special case $\tilde{c}(t) \equiv 0$ in the previous section. Recall that we suppose that (3.1) possesses a solution h such that $h(t) > 0$ and $h'(t) \neq 0$ in the interval where this equation and (1.1) are considered.

THEOREM 4.1. *Let $1 < p \le 2$ and $c(t) \ge \tilde{c}(t)$ for large t, say $t \in [t_0, \infty)$. Suppose that there exist t_1, t_2 such that $t_0 < t_1 < t_2$ and*

$$\left[\int_{t_0}^{t_1} r(t)\,|h'(t)|^{p-2} h^2(t)\,dt\right]^{-1} < \frac{p}{2}\int_{t_1}^{t_2} [c(t) - \tilde{c}(t)] h^p(t)\,dt. \tag{4.1}$$

Then the solution x of (1.1) given by the initial condition $x(t_0) = 0$, $x'(t_0) = 1$ has at least one zero in (t_0, ∞).

Proof. Consider (3.5). The transformation $X(s) = y(t)$, $s = \int_{t_0}^{t} R^{-1}(\tau)\,d\tau$, transforms this equation into the equation

$$\frac{d^2}{ds^2}X + \frac{p}{2}R(t(s))C(t(s))X = 0, \tag{4.2}$$

where R, C are given by (1.4), $t = t(s)$ being the inverse function of $s = s(t)$. Denote $s_1 = \int_{t_0}^{t_1} R^{-1}(\tau)\,d\tau$, $s_2 = \int_{t_0}^{t_2} R^{-1}(\tau)\,d\tau$. We have

$$\int_{s_1}^{s_2} C(t(s))R(t(s))\,ds = \int_{t_1}^{t_2} C(t)\,dt = \int_{t_1}^{t_2} [c(t) - \tilde{c}(t)] h^p(t)\,dt, \tag{4.3}$$

hence (4.1) can be written in the form

$$\frac{1}{s_1} < \frac{p}{2}\int_{s_1}^{s_2} C(t(s))R(t(s))\,ds. \tag{4.4}$$

Now consider the solution X of (4.2) given by the initial condition $X(0) = 0$, $X'(0) = 1$. Then (4.4) and [12, Theorem 1] imply that this solution has a zero in $(0, \infty)$ and this means that the solution y of (3.5) given by $y(t_0) = 0$, $y'(t_0) = 1$ has a zero in (t_0, ∞). Now, by Corollary 3.3(i), the solution of (1.1) given by $x(t_0) = 0$, $x'(t_0) = 1$ has a zero in (t_0, ∞) as well. □

The existence of a pair or conjugate points relative to (1.1) in a given interval (the previous theorem gives a sufficient condition for the existence of such a pair of points) play an important role in the investigation positivity of the p-degree functional

$$\mathcal{F}_0(y; a, b) = \int_a^b \{r(t)|y'|^p - c(t)|y|^p\}\,dt \tag{4.5}$$

over the class of C^1 functions satisfying $y(a) = 0 = y(b)$, see [8, 9]. If we consider a more general functional

$$\mathcal{F}(y; a, b) = \gamma\,|y(a)|^p + \mathcal{F}_0(y; a, b), \tag{4.6}$$

where γ is a real constant, over the class of functions satisfying only one-side restriction $y(b) = 0$, the positivity of $\mathcal{F}$ can be characterized via nonexistence of a zero point of the solution of (1.1) given by the initial condition $x(a) = 1$, $r(a)\Phi(x'(a)) = \gamma$. The following statement deals with a problem of this kind.

THEOREM 4.2. *Suppose that $c(t) > \tilde{c}(t)$ for $t \in (t_0,\infty)$ and there exist $t_1,t_2 \in [t_0,\infty)$, $t_1 < t_2$, such that*

$$\frac{p}{2}\int_{t_0}^{t_1}[c(t)-\tilde{c}(t)]h^p(t)dt > \left(\int_{t_1}^{t_2}\frac{dt}{r(t)h^2(t)\,|\,h'(t)\,|^{p-2}}\right)^{-1}. \tag{4.7}$$

Then the solution x of (1.1) given by the initial condition

$$x(t_0) = 1, \qquad x'(t_0) = w_h(t_0), \tag{4.8}$$

where w_h is given by (3.3), has a zero in $(t_0,t_2]$.

Proof. Similarly as in the previous proof, the transformation of independent variable $s = \int_{t_0}^{t} R^{-1}(\tau)\,d\tau$, transforms this equation into (4.2). Denoting $s_1 = \int_{t_0}^{t_1} R^{-1}(t)\,dt$, $s_2 = \int_{t_1}^{t_2} R^{-1}(t)\,dt$, similarly as in the previous proof, inequality (4.7) can he written in the form

$$\frac{1}{s_2} < \frac{p}{2}\int_{s_0}^{s_1} C(t(s))R(t(s))\,ds. \tag{4.9}$$

Now consider the solution X of (4.2) given by the initial condition $X(0) = 1$, $X'(0) = 0$. Then (4.9) and [12, Theorem 4] imply that this solution has a zero in $[0,s_1+s_2)$ and this means that the solution y of (3.5) given by $y(t_0) = 1$, $y'(t_0) = 0$ has a zero in (t_0,t_2) as well. Let $u(t) = 2R(t)y'/py$, then u is a solution of (3.6) and $u(t_0) = 0$, $u(t_3-) = -\infty$ for some $t_3 \in (t_0,t_2)$ (where t_3 is the zero of y). Let $v(t) = h^{-p}u + w_h$, since $1 < p \le 2$, by Lemma 2.2, we have

$$0 = u' + C(t) + \frac{p}{2}\frac{u^2}{R(t)} \le u' + C(t) + pr^{1-q}(t)h^p(t)P(\Phi_q(w_h),v). \tag{4.10}$$

and $v(t_0) = w_h(t_0) = r(t_0)\Phi(h'(t_0))/\Phi(h(t_0))$. Substituting for v in (4.10), we get (after a short computation)

$$v' + c(t) + (p-1)r^{1-q}(t)|v|^q \ge 0. \tag{4.11}$$

Denote $\tilde{C} = -v' - (p-1)r^{1-q}|v|^q$. Then

$$v' + \tilde{C}(t) + (p-1)r^{1-q}(t)|v|^q = 0 \tag{4.12}$$

and $\tilde{C}(t) \le c(t)$ for $t \in [t_0,t_3)$. Consider the solution w of (2.1) given by the initial condition $w(t_0) = v(t_0) = w_h(t_0)$. By Lemma 2.3 $w(t) \le v(t)$ for $t \ge t_0$, that is, $w(t_4-) = -\infty$ for some $t_4 \in (t_0,t_3]$ which means that the solution of (1.1) given by $x(t_0) = 0$, $x'(t_0) = w_h(t_0)$ has a zero at $t_4 \in (t_0,t_2)$. □

Now we turn our attention to the case $p \ge 2$. In this case, disconjugacy of (3.5) in a given interval implies the same property of (1.1). The classical linear Lyapunov criterion states that (1.2) is disconjugate in $[a,b]$ provided

$$\int_a^b c_+(t)dt < \frac{4}{\int_a^b r^{-1}(t)dt}, \tag{4.13}$$

where $c_+ = \max\{0, c\}$. This criterion has been extended to (1.1) in several papers, see, for example, [4, 13], and states that (1.1) is disconjugate in $[a,b]$ if

$$\int_a^b c_+(t)dt < \frac{2^p}{\left(\int_a^b r^{1-q}(t)dt\right)^{p-1}}. \tag{4.14}$$

In the previous inequality, (1.1) is viewed as a perturbation of the one-term equation $(r(t)\Phi(x'))' = 0$. The below given corollary deals with the case when (1.1) is viewed as a perturbation of disconjugate equation (3.1). The first part of the statement follows immediately from Theorem 3.1 and (4.13). The second part is the application of the first one to the case when $[a,b] = [0, \pi_p/2]$, where π_p is given by (2.9), and $\tilde{c}(t) = ((p-1-\delta)/(p-1))^{1/p}$ with $0 < \delta < p-1$. In the third part, we consider (3.16) as a perturbation of Euler equation (3.17) and we apply (4.13) to (3.5) with $C(t) = (c(t) - \tilde{c}(t))t^{p-1}$, $R(t) = ((p-1)/p)^{p-2}t^{-1}$.

Corollary 4.3. *Let $p \geq 2$.*

(i) *Suppose that*

$$\frac{p}{2}\int_a^b \left(c(t) - \tilde{c}(t)\right)_+ h^p(t) < \frac{4}{\int_a^b r(t)h^2(t)\left|h'(t)\right|^{p-2}dt}, \tag{4.15}$$

then (1.1) is disconjugate on $[a,b]$.

(ii) *Suppose that there exists $0 < \delta < p-1$ and $0 < \beta < (1-\alpha_p)(\pi_p/2)$, where $\alpha_p = ((p-1-\delta)/(p-1))^{1/p}$. If*

$$\frac{p}{2}\int_0^{\pi_p/2} C(t)dt < \frac{4}{\int_0^{\pi_p/2} R^{-1}(t)dt}, \tag{4.16}$$

where

$$\begin{aligned} R(t) &= \left(\frac{p-1-\delta}{p-1}\right)^{(p-2)/p} \sin_p^2(\alpha_p t + \beta)\cos_p^{p-2}(\alpha_p t + \beta), \\ C(t) &= \left[c(t) - \left(\frac{p-1-\delta}{p-1}\right)^{1/p}\right]\sin_p^p(\alpha_p t + \beta), \end{aligned} \tag{4.17}$$

then (3.16) is disconjugate in $[0, \pi_p/2]$.

(iii) *Suppose that*

$$\lg\frac{b}{a}\int_a^b \left(c(t) - \frac{\gamma_p}{t^p}\right)t^{p-1}dt < \frac{8}{p}\left(\frac{p-1}{p}\right)^{p-2}, \tag{4.18}$$

then (3.16) is disconjugate in $[a,b] \subset (0,\infty)$.

Remark 4.4. Throughout the paper, the function h is a solution of (3.1). In the recent paper [11] oscillatory properties of (3.16) are investigated and this equation is viewed as a perturbation of (3.17). In the oscillation criteria of that paper, the function $h(t) = t^{(p-1)/p}\lg^{2/p} t$ appears, which *is not* a solution of (3.17), but it only close to a solution of this equation, in a certain sense. This suggest another idea for the next investigation, to formulate the results of our paper in the situation when the function h which appears in (3.4) is not a solution of (3.1) but it close to it, in a certain sense. This problem is also a subject of the present investigation.

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