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Notes to G. Bennett's Problems

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G. Bennett showed, by elementary proof, that if $p \le q$ then (1.1) holds, and the constant p/s is best possible; and if $p \ge q$ then (1.2) is valid. The reversed inequalities have remained open problems. As a first step into the converse direction, what seems to be very intricate without additional assumptions, we prove the inverse inequalities under slight restrictions on the monotonicity of the parameters appearing in the problem.

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1 INTRODUCTION

In [1] G. Bennett described a new way of looking at inequalities and proved several interesting theorems. The new way means "the factorization of inequalities". For precise definition and explanation of the great advantage of the factorization we refer to [1]. Now we only observe that Bennett's results provide the best possible version of several classical and latest inequalities, for they replace inequalities by equalities appearing in the factorization.

As one of the advantages of the factorization of inequalities we mention that using the new factorizing results being in the work mentioned above, G. Bennett proved, among others, that the following two conditions of quite different type

$$\sigma_1 := \sum_{k=1}^{\infty} a_k \Big(\sum_{n=k}^{\infty} \frac{|x_n|^s}{A_n} \Big)^{p/s} < \infty$$

and

$$\sigma_2 := \sum_{k=1}^{\infty} |x_k|^s \Big(\frac{1}{A_k}\sum_{n=1}^{k}|x_n|^s\Big)^{p/q} < \infty$$

are equivalent, where $0 < p, q < \infty$, $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$ and $\mathbf{a} := \{a_n\}$ is a fixed sequence of non-negative terms with $a_1 > 1$ so that the partial sums

$$A_n := \sum_{k=1}^{n} a_k$$

never vanish, furthermore $\mathbf{x} := \{x_n\}$ is an arbitrary sequence of real (or complex) numbers.

To give a direct proof of this equivalence it seems to be a very troublesome task, see also the comment in [1], p. 24. But in section 10 of his cited work G. Bennett showed that if $p \le q$ then

$$\sigma_1 \le \frac{p}{s}\sigma_2 \tag{1.1}$$

holds, and the constant p/s is best possible; and if $p \ge q$ then

$$\sigma_2 \le \sigma_1. \tag{1.2}$$

The reversed inequalities have remained open problems.

After several attempts we are still unable to prove (1.1) for $p > q$, or (1.2) if $p < q$ without additional assumption; however under slight restriction on the monotonicity of the sequences $\{A_n\}$ and $\{t_n\}$, where

$$t_n := t_n(\mathbf{a}, \mathbf{x}, s) := \sum_{k=n}^{\infty} \frac{|x_k|^s}{A_k},$$

we can verify that if $p > q$ then

$$\sigma_1 \le K\sigma_2, \tag{1.3}$$

and if $p < q$ then

$$\sigma_2 \le K\sigma_1 \tag{1.4}$$

hold.

We hope that the "blocking-method" to be used in our proofs, or its refined version, will be transmittable for catching the original problem, that is, to prove (1.3) and (1.4) without conditions for the sequences $\mathbf{x}$. In the forthcoming inequalities, unfortunately, by means of $\{t_n\}$ we have to assume certain smoothness on the sequences $\mathbf{x}$, too.

To formulate our results we have to present some furher notions and notations.

We shall say that a sequence $\gamma := \{\gamma_n\}$ of positive terms is *quasi β-power-monotone increasing (decreasing)* if there exists a constant $K := K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \quad (n^\beta \gamma_n \leq Km^\beta \gamma_m)$$

holds for any $n \geq m$. Here and in the sequel, K and K_i denote positive constants, not necessarily the same at each occurence. If we wish to express the dependence explicitly, we write K in the form $K(\alpha, \dots)$.

Furthermore we shall say that a sequence γ of positive terms is *quasi geometrically increasing (decreasing)* if there exist a natural number μ and a constant $K := K(\gamma) \geq 1$ such that

$$\gamma_{n+\mu} \geq 2\gamma_n \quad \textit{and} \quad \gamma_n \leq K\gamma_{n+1} \quad (\gamma_{n+\mu} \leq \frac{1}{2}\gamma_n \quad \textit{and} \quad \gamma_{n+1} \leq K\gamma_n)$$

hold for all natural numbers n.

We shall also use the following notations:

$$\alpha_m := \sum_{n=2^m}^{2^{m+1}-1} a_n,$$

$$X_m := \sum_{k=2^m}^{2^{m+1}-1} x_k,$$

and

$$X_{m,\mu} := \sum_{n=m}^{m+\mu-1} X_n.$$

Now we establish our results.

THEOREM *(i) Using the notations given above, if $p > q$, the sequence $\{A_n\}$ satisfies the condition*

$$A_{2n} \leq A_n, \tag{1.5}$$

and the sequence $\{t_n\}$ is quasi β-power-monotone decreasing with some positive β, then inequality (1.3) holds.

(ii) If $p < q$, (1.5) holds and the sequence $\{t_n A_n^{2-p/s}\}$ is quasi β-power-monotone decreasing with some positive β, then inequality (1.4) holds.

(iii) If $p < q$ and the sequence $\{t_{2^m}\alpha_m\}$ is quasi geometrically increasing, furthermore

$$\alpha_{m+1} \leq K\alpha_m \tag{1.6}$$

stays, then (1.4) newly holds.

Comparing the conditions of (ii) and (iii), it is easy to see, roughly speaking, that one of the conditions of (ii) restricts the growth of the terms t_n, while the condition on $\{t_{2^m}\alpha_m\}$ in (iii) limits the decline of the terms t_n.

We would like to emphasize that in the most commonly used case, that is, if $a_n \equiv 1$, then restrictions (1.5) and (1.6) are automatically satisfied. Then, in each case, already one condition assures the suitable inequality.

2 LEMMAS

We need the following lemmas.

LEMMA 1 *([3]). For any positive sequence $\gamma = \{\gamma_n\}$ the inequalities*

$$\sum_{n=m}^{\infty} \gamma_n \leq K\gamma_m \quad (m = 1, 2, \dots; K \geq 1), \tag{2.1}$$

or

$$\sum_{n=1}^{m} \gamma_n \leq K\gamma_m \quad (m = 1, 2, \dots; K \geq 1) \tag{2.2}$$

hold if and only if the sequence γ is quasi geometrically decreasing or increasing, respectively.

LEMMA 2 *([4]). If a positive sequence γ is quasi β-power-monotone increasing (decreasing) with a certain negative (positive) exponent β, then the sequence $\{\gamma_{2^n}\}$ is quasi geometrically increasing (decreasing).*

LEMMA 3 *([2]) If $\{c_k\}$ is a sequence of non-negative numbers and* $0 < r < R$*, then*

$$\Big(\sum_{k=1}^{n} c_k^R\Big)^{1/R} \le \Big(\sum_{k=1}^{n} c_k^r\Big)^{1/r}. \tag{2.3}$$

3 PROOF OF THEOREM

First we prove the special cases $s = 1$ of our inequalities, namely it is easy to see that if inequalities (1.3) and (1.4) hold in the special case $s = 1$, then the general cases follow effortlessly replacing $|x_k|$ by $|x_k|^s$, p by p/s and q by q/s both in the inequalities and in the assumptions.

For simplifying the writing we shall write x_n instead of $|x_n|$.

Proof of the case (i). The assumptions $s = 1$ and $p > q$ imply that $p > 2$. Since the sequence $\{t_n\}$ is quasi β-power-monotone decreasing with some positive β, so, by Lemma 2, the sequence $\{t_{2^n}\}$ is quasi geometrically decreasing, that is, there exists a natural number μ so that

$$t_{2^n} \ge 2t_{2^{n+\mu}}$$

holds for all n. Hence we get that

$$\sum_{k=2^n}^{2^{n+\mu}-1} \frac{x_k}{A_k} \ge t_{2^{n+\mu}}$$

and consequently

$$X_{n,\mu} = \sum_{k=2^n}^{2^{n+\mu}-1} x_k \ge A_{2^n} t_{2^{n+\mu}}. \tag{3.1}$$

Since $p/q = p - 1$, thus an elementary consideration gives that for any N

$$\sum_{n=1}^{N} x_n \le \Big\{\sum_{n=1}^{N} \frac{x_n^p}{A_n^{p-1}}\Big\}^{1/p} K(N, \mathbf{a}) \le K(N, \mathbf{a})\sigma_2^{1/p}. \tag{3.2}$$

Thus (3.2) clearly implies that

$$t_1 - t_{2^\mu} \le K_1(\mu, \mathbf{a})\sigma_2^{1/p}. \tag{3.3}$$

Hence and from (3.1) with $n = 0$ we get that

$$t_{2^\mu} = \sum_{n=2^\mu}^{\infty} \frac{x_n}{A_n} \le A_1^{-1} \sum_{k=1}^{2^\mu-1} x_k \le K_2(\mu, \mathbf{a})\sigma_2^{1/p} \tag{3.4}$$

Now we can start to verify (1.3).

$$\sigma_1 = \sum_{n=1}^{\infty} a_n \Big(\sum_{k=n}^{\infty} \frac{x_k}{A_k}\Big)^p = \Big(\sum_{n=1}^{2^\mu} + \sum_{n=2^\mu+1}^{\infty}\Big) a_n\, t_n^p =: S_1 + S_2. \tag{3.5}$$

Since

$$S_1 = \sum_{n=1}^{2^\mu} a_n t_n^p \le t_1^p A_{2^\mu},$$

so, by (3.3) and (3.4), we obtain that

$$S_1 \le K(\mu, \mathbf{a})\sigma_2. \tag{3.6}$$

Using (1.5) and (3.1) we easily get that

$$\begin{aligned} S_2 &= \sum_{n=2^\mu+1}^{\infty} a_n\, t_n^p \le \sum_{m=\mu}^{\infty}\Big(\sum_{n=2^m+1}^{2^{m+1}} a_n\Big) t_{2^m}^p \le \sum_{m=\mu}^{\infty} A_{2^{m+1}}\, t_{2^m}^p \le \\ &\le K \sum_{m=\mu}^{\infty} A_{2^{m-\mu}}\, t_{2^m}\, t_{2^m}^{p-1} \le K \sum_{m=\mu}^{\infty} X_{m-\mu,\mu}\, t_{2^m}^{p-1} = \\ &= K \sum_{m=0}^{\infty} X_{m,\mu}\, t_{2^{m+\mu}}^{p-1} \le K \sum_{m=0}^{\infty} X_{m,\mu}\, t_{2^m}^{p-1} = \\ &= K\Big(\sum_{m=0}^{\mu} + \sum_{m=\mu+1}^{\infty}\Big) X_{m,\mu}\, t_{2^m}^{p-1} = \\ &=: S_3 + S_4. \end{aligned} \tag{3.7}$$

Here S_3 can be estimated as S_1 above, consequently we know that

$$S_3 \le K_1(\mu, \mathbf{a})\sigma_2. \tag{3.8}$$

Finally, again utilizing (3.1) and (1.5), we estimate S_4 as follows:

$$\begin{aligned} S_4 &\le K \sum_{m=\mu+1}^{\infty} X_{m,\mu}\Big(A_{2^{m-\mu}}^{-1} \sum_{i=2^{m-\mu}}^{2^m-1} x_i\Big)^{p-1} \le \\ &\le K_1 \sum_{m=\mu+1}^{\infty} \sum_{k=2^m}^{2^{m+\mu}-1} x_k \Big(\frac{1}{A_k} \sum_{i=1}^{k} x_i\Big)^{p-1} \le K_1\sigma_2. \end{aligned} \tag{3.9}$$

Summing up the partial results (3.5) - (3.9) we have proved (1.3) for $s = 1$.

Proof of the case (ii). An easy calculation gives that

$$\sum_{n=1}^{m} x_n = \sum_{n=1}^{m} a_n t_n - A_n t_{n+1},$$

so

$$\begin{aligned}\sigma_2 &\le \sum_{k=1}^{\infty} x_k \Big(\frac{1}{A_k}\sum_{n=1}^{k} a_n t_n\Big)^{p-1} \le \\ &\le \sum_{m=0}^{\infty}\Big(\sum_{k=2^m}^{2^{m+1}-1} x_k\Big)\Big(\frac{1}{A_{2^m}}\sum_{i=0}^{m}\sum_{n=2^i}^{2^{i+1}-1} a_n t_n\Big)^{p-1} =: S_5.\end{aligned} \tag{3.10}$$

Now we apply Lemma 3, so by $p-1<1$,

$$\begin{aligned}S_5 &\le \sum_{m=0}^{\infty} X_m\, A_{2^m}^{1-p}\sum_{i=0}^{m} t_{2^i}^{p-1}\alpha_i^{p-1} \le \\ &\le \sum_{i=0}^{\infty} t_{2^i}^{p-1}\alpha_i^{p-1}\sum_{m=i}^{\infty} A_{2^m}^{1-p} X_m \le \\ &\le \sum_{i=0}^{\infty} t_{2^i}^{p-1} A_{2^{i+1}}^{p-1}\sum_{m=i}^{\infty} A_{2^m}^{1-p} A_{2^{m+1}}\, t_{2^m} =: S_6.\end{aligned} \tag{3.11}$$

By (1.5)

$$\sum_{m=i}^{\infty} A_{2^m}^{1-p} A_{2^{m+1}}\, t_{2^m} \le K \sum_{m=i}^{\infty} A_{2^m}^{2-p}\, t_{2^m}. \tag{3.12}$$

Since the sequence $\{A_n^{2-p} t_n\}$ is quasi β-power-monotone decreasing with some positive β, therefore, by Lemma 2, the sequence $\{A_{2^m}^{2-p}\, t_{2^m}\}$ is quasi geometrically decreasing, what, by Lemma 1, implies that

$$\sum_{m=i}^{\infty} A_{2^m}^{2-p} t_{2^m} \le K A_{2^i}^{2-p} t_{2^i}. \tag{3.13}$$

Using this, (3.12) and (1.5), we obtain that

$$\begin{aligned}S_6 &\le K\sum_{i=0}^{\infty} t_{2^i}^{p} A_{2^i} \le K_1\Big(A_1 t_1^p + \sum_{i=1}^{\infty} t_{2^i}^{p} A_{2^{i-1}}\Big) \le \\ &\le K_1\Big\{a_1 t_1^p + \sum_{i=1}^{\infty} t_{2^i}^{p}\Big(a_1 + \sum_{m=1}^{i-1}\alpha_m\Big)\Big\} = \\ &= K_1\Big(a_1\sum_{i=0}^{\infty} t_{2^i}^{p} + \sum_{m=1}^{\infty}\alpha_m\sum_{i=m+1}^{\infty} t_{2^i}^{p}\Big).\end{aligned} \tag{3.14}$$

Since $p < 2$ and $\{A_n\}$ is increasing, thus (3.13) clearly implies that

$$\sum_{m=i}^{\infty} t_{2^m} \leq K t_{2^i}$$

also holds. Thus, by Lemma 1, the sequence $\{t_{2^n}\}$ is also quasi geometrically decreasing, and therefore the sequence $\{t_{2^n}^p\}$ $(p > 1)$ is again quasi geometrically decreasing, consequently

$$\sum_{m=i}^{\infty} t_{2^m}^p \leq K t_{2^i}^p$$

also holds.

Putting this into (3.14) we get that

$$S_6 \leq K_2 \sum_{n=1}^{\infty} a_n t_n^p = K_2 \sigma_1. \tag{3.15}$$

Collecting the estimations (3.10), (3.11) and (3.15) the statement (ii) in the special case $s = 1$ is proved.

Proof of the case (iii). The first part of the proof is the same as in the case (ii). The deviation appears in the estimation of S_5. Then first we shall use the estimation

$$\sum_{i=0}^{m} t_{2^i} \alpha_i \leq K t_{2^m} \alpha_m, \tag{3.16}$$

coming, by Lemma 1, from the assumption that the sequence $\{t_{2^m} \alpha_m\}$ is quasi geometrically increasing. Later on we shall apply inequality (1.6) and the inequality

$$(A_{2^m} \leq) \sum_{i=0}^{m} \alpha_i \leq K \alpha_m, \tag{3.17}$$

what plainly follows from (3.16).

Thus we obtain that

$$\begin{aligned} S_5 &\le \sum_{m=0}^{\infty} X_m \Big(\frac{1}{A_{2^m}} \sum_{i=0}^{m} \alpha_i t_{2^i} \Big)^{p-1} \le \\ &\le K \sum_{m=0}^{\infty} X_m \, A_{2^m}^{1-p} \, \alpha_m^{p-1} \, t_{2^m}^{p-1} \le \\ &\le K_1 \sum_{m=0}^{\infty} t_{2^m} \, A_{2^{m+1}} \, A_{2^m}^{1-p} \, \alpha_m^{p-1} \, t_{2^m}^{p-1} \le \\ &\le K_2 \sum_{m=0}^{\infty} t_{2^m}^{p} \, \alpha_{m-1} \le K_2 \sum_{n=1}^{\infty} a_n \, t_n^p = K_2 \sigma_1 . \end{aligned}$$

This and (3.10) imply the statement (iii) for $s = 1$.

As we have asserted above the general cases when s is given by

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q} \quad (0 < p, q < \infty$$

can be derived from the special cases $s = 1$ by replacing the terms $|x_k|$, p and q by $|x_k|^s$, p/s and q/s, respectively.

Thus our theorem is completed.

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