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Weighted Modular Inequalities for Monotone Functions

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Weight characterizations of weighted modular inequalities for operators on the cone of monotone functions are given in terms of composition operators on arbitrary non-negative functions with changes in weights. The results extend to modular inequalities, those corresponding to weighted Lebesgue spaces given by E.T. Sawyer [15]. Application to Hardy and fractional integral operators on monotone functions are given.

Keywords: Weighted modular inequalities; Orlicz spaces; norms; duality; Hardy type inequalities.

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1 INTRODUCTION

The object of this note is to characterize weight functions w_i, $i = 0, 1$ and v, for which the modular inequality

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)\,Tf(x)]w_0(x)\,dx\right\} \le P^{-1}\left\{\int_0^\infty P[C\,f(x)]v(x)\,dx\right\} \tag{1.1}$$

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holds, where T is a linear operator defined on non-negative monotone functions. P and Q are N-functions, where no order relations – such as $Q \circ P^{-1}$ convex – are assumed, and $C > 0$ is a constant. The characterizations, via duality arguments are given in terms of modular inequalities of the form (1.1) where T is replaced by the composition operators $T \circ I$ and $T \circ I^*$, defined on arbitrary non-negative functions and with changes of weights. Here I and I^* are defined

$$(Ih)(x) = \int_0^x h, \qquad \text{and} \qquad (I^*h)(x) = \int_x^\infty h \qquad x > 0\,.$$

Our results generalize to modulars the duality principle for weighted L^p-spaces and operators defined on non-negative non-increasing functions given by E.T. Sawyer [15]. His results follow on taking $Q(x) = |x|^q/q$, $P(x) = |x|^p/p$, $1 < p, q < \infty$, while the case $p = q$ and T the Hardy operator yields a result of M. Ariño and B. Muckenhoupt [2].

Characterizations of weights for which (1.1) holds and T is a Volterra convolution type operator defined on monotone functions was given recently by J.Q. Sun [19] [20] in the case when $Q \circ P^{-1}$ is convex. Our methods are different from those and except for some special cases, we do not obtain these results for the general Volterra convolution operator. Conversely our results (Proposition 2.2) for the Riemann-Liouville operator on increasing functions and the Weyl fractional integral operator defined on decreasing functions does not follow from his.

Our main result (Theorem 2.3) depends strongly on a duality principle given in [7] and on an extension of a result by C. Herz [8] and S. Bloom and R. Kerman [3], establishing an equivalence of a weighted modular inequality and certain weighted Orlicz-Luxemburg norm inequality (Proposition 2.1). This result may be of independent interest.

We conclude this section by giving some definitions and notation required in the sequel.

DEFINITION 1.1 (a) A convex function $P : \mathbb{R}^+ \to \mathbb{R}^+$ is called a Young function if $P(0) = 0$ and $\lim_{x\to\infty} P(x) = \infty$.

(b) A continuous Young function P is called an N-function if it has the form

$$P(x) = \int_0^{|x|} p(t)\,dt\,,$$

where p is non-decreasing, right continuous, $p(0+) = 0$ and $p(\infty) = +\infty$, and $p(t) > 0$ if $t > 0$. If p^{-1} denotes the right continuous inverse of p, then the complementary function $\tilde{P}$ of P is defined by

$$\tilde{P}(x) = \int_0^{|x|} p^{-1}(t)\,dt\ .$$

If P is an N-function, so is $\tilde{P}$.

(c) If v is a non-negative measurable (weight) function defined on a measure space X and P is an N-function, then the Orlicz space $L_{P(v)}$ consists of those measurable f on X for which the (Luxemburg) norm

$$\|f\|_{P(v)} = \inf\left\{\lambda > 0 : \int_X P\left[\frac{|f(x)|}{\lambda}\right] v(x)\,dx \le 1\right\} \tag{1.2}$$

is finite. If $v(x)\,dx$ is replaced by a positive Borel measure μ we write also $P(d\mu)$, and if $v \equiv 1$ simply P, in (1.2).

For further properties of N-functions and Orlicz spaces we refer to [9, 10, 14].

Definition 1.2 (a) An N-function P satisfies the Δ_2 condition ($P \in \Delta_2$), if there is a constant $D > 0$, such that $P(2x) \le D\,P(x)$, for all $x > 0$.

(b) The order relation $\prec$ is defined by $P \prec Q$, (P, Q, N-functions) if there is a constant $C > 0$, such that

$$\sum_i Q \circ P^{-1}(a_i) \le C\,Q \circ P^{-1}\left(\sum_i a_i\right)$$

for every non-negative sequence $\{a_i\}$.

Note that if $Q \circ P^{-1}$ is convex, then $P \prec Q$. For other properties see [5, Lemma 1.1].

Finally we denote by χ_E the characteristic function of the set E, and define i_0 by $i_0(x) = x$. Constants are denoted by $A,\ B,\ C,\ D$ and inequalities, such as (1.1) are interpreted to mean that if the right side is finite, so is the left side and the inequality holds. Weight functions are non-negative measurable functions on a measure space X (usually $\mathbb{R}^+$) and are denoted by $w_0,\ w_1,\ w,\ u,\ v$. Non-negative non-increasing $\equiv$ decreasing, respectively, non-decreasing $\equiv$ increasing functions f are denoted by $0 \le f\downarrow$, respectively $0 \le f\uparrow$.

2 RESULTS AND APPLICATIONS

We require the following result whose special case $P = Q$ and $u = w = 1$ was proved by C. Herz in the preprint [8] (see also [3, Proposition 2.5]).

PROPOSITION 2.1 *Suppose $(X, d\mu)$, $(Y, d\nu)$ are σ-finite measure spaces and T is a linear operator mapping measurable functions on X to measurable functions on Y. If P and Q are N-functions, then the modular inequality*

$$Q^{-1}\left\{\int_Y Q[w(y)\,|(Tf)(y)|]d\nu(y)\right\} \leq P^{-1}\left\{\int_X P[Cu(x)\,|f(x)|]d\mu(x)\right\} \tag{2.1}$$

is satisfied, if and only if, for every $\varepsilon > 0$,

$$\|w\,Tf\|_{Q(\varepsilon_Q d\nu)} \leq C\|uf\|_{P(\varepsilon_P d\mu)} \tag{2.2}$$

holds, where $\varepsilon_Q = 1/Q(1/\varepsilon)$, $\varepsilon_P = 1/P(1/\varepsilon)$.

Proof Suppose (2.2) holds. Define ε by $P\left(\frac{1}{\varepsilon}\right) = \int_X P[uf]d\mu$, then

$$\begin{aligned}\|uf\|_{P(\varepsilon_P d\mu)} &= \inf\left\{\lambda > 0 : \int_X P\left[\frac{uf}{\lambda}\right]\varepsilon_P d\mu \leq 1\right\} \\ &= \inf\left\{\lambda > 0 : \int_X P\left[\frac{uf}{\lambda}\right]d\mu \leq \int_X P[uf]d\mu\right\} = 1\end{aligned}$$

and therefore $\|w\,Tf\|_{Q(\varepsilon_Q d\nu)} \leq C$. From the homogeneity of the norm and the linearity of T it follows that

$$\|w\,T(f/C)\|_{Q(\varepsilon_Q d\mu)} = \inf\left\{\lambda > 0 : \int_Y Q\left[\frac{w\,T(f/C)}{\lambda}\right]\varepsilon_Q d\nu \leq 1\right\} \leq 1\,.$$

Hence

$$\int_Y Q[w\,T(f/C)]d\nu \leq \frac{1}{\varepsilon_Q} = Q\left[P^{-1}\left(\int_X P[u\,f]d\mu\right)\right],$$

which implies (2.1) after replacing f/C by f.

Conversely, if (2.1) is satisfied, fix f and let $\alpha = \|u\,f\|_{P(\varepsilon_P d\mu)}$, then the homogeneity of the norms shows that

$$1 = \|u\,f/\alpha\|P(\varepsilon_P d\mu) = \inf\left\{\lambda > 0 : \int_X P\left[\frac{uf}{\lambda\alpha}\right]\varepsilon_P d\mu \leq 1\right\}$$

and therefore,

$$\int_X P\left[\frac{uf}{\alpha}\right] \varepsilon_P d\mu \leq 1 .$$

Now (2.1) and the linearity of T imply

$$\begin{aligned}\int_Y Q\left[\frac{w\,Tf}{\alpha C}\right] \varepsilon_Q d\nu &= \varepsilon_Q \int_Y Q[w\,T(f/\alpha C))]d\nu \\ &\leq \varepsilon_Q Q\left\{P^{-1}\left(\int_X P[Cu(f/(\alpha C))]d\mu\right)\right\} \\ &= \varepsilon_Q Q\left\{P^{-1}\left(\int_X P[uf/\alpha]d\mu\right)\right\} \\ &= \varepsilon_Q Q \circ P^{-1}\left(\frac{1}{\varepsilon_P}\right) \\ &= \varepsilon_Q Q\left(\frac{1}{\varepsilon}\right) = 1.\end{aligned}$$

Hence

$$\begin{aligned}\|w\,Tf\|_{Q(\varepsilon_Q d\nu)} &= \inf\left\{\lambda > 0 : \int_Y Q\left[\frac{w\,Tf}{\lambda}\right] \varepsilon_Q d\nu \leq 1\right\} \\ &\leq \alpha C = C\|uf\|_{P(\varepsilon_P d\mu)}\end{aligned}$$

and the result follows.

The (weighted) duality principle in Orlicz spaces may be written in the following form:

$$\sup_{0\leq f} \frac{\int_0^\infty f(x)g(x)\,dx}{\|f\|_{P(v)}} = \left\|\frac{g}{v}\right\|_{\tilde{P}(v)} , \tag{2.3}$$

where v is a weight function and $\tilde{P}$ is the complementary function of the N-function P.

We also require the following known duality principle for monotone functions:

THEOREM 2.1 ([7, Theorems 2.2, 2.3]) *Suppose P and $\tilde{P}$ are N-functions satisfying the Δ_2 condition.*

(i) If $(Ig)(x) = \int_0^x g$, $g \geq 0$; $(Iv)(x) = \int_0^x v$, with $(Iv)(\infty) = \infty$; then

$$\sup_{0\leq f\downarrow} \frac{\int_0^\infty f(x)g(x)\,dx}{\|f\|_{P(v)}} \approx \left\|\frac{Ig}{Iv}\right\|_{\tilde{P}(v)} \tag{2.4}$$

(ii) If $(I^*g)(x) = \int_x^\infty g$, $g \geq 0$; $(I^*v)(x) = \int_x^\infty v$, *with* $(I^*v)(0) = \infty$; *then*

$$\sup_{0 \leq f\uparrow} \frac{\int_0^\infty f(x)g(x)\,dx}{\|f\|_{P(v)}} \approx \left\| \frac{I^*g}{I^*v} \right\|_{\tilde{P}(v)} . \tag{2.5}$$

Here the symbol $\approx$ is defined to mean that the quotient of the left and right side of (2.4) and (2.5) is bounded above and below by positive constants.

Remark 2.1 It follows from (2.3) and (2.4) that if T is a linear operator defined on decreasing functions on $\mathbb{R}^+$ then the inequality $\|Tf\|_{Q(w)} \leq C\|f\|_{P(v)}$ $(0 \leq f\downarrow)$ is equivalent to

$$\left\| \frac{I(T^*g)}{Iv} \right\|_{\tilde{P}(v)} \leq C \left\| \frac{g}{w} \right\|_{\tilde{Q}(w)}, \qquad g \geq 0,$$

where T^* is the adjoint of T. Similarly, the above norm inequality for increasing functions is by (2.3) and (2.5) equivalent to

$$\left\| \frac{I^*(T^*g)}{I^*v} \right\|_{\tilde{P}(v)} \leq C \left\| \frac{g}{w} \right\|_{\tilde{Q}(w)}, \qquad g \geq 0.$$

Our main result is now the following:

THEOREM 2.2 *Suppose* P, $\tilde{P} \in \Delta_2$ *are N-functions, and* T *is a positive linear operator.*

(i) If $0 \leq f\downarrow$ *and* $(Iv)(\infty) = \infty$ *then the modular inequalities*

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)(Tf)(x)]w_0(x)dx\right\} \leq P^{-1}\left\{\int_0^\infty P[C\,f(x)]v(x)dx\right\} \tag{2.6}$$

and

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)T(I^*h)(x)]w_0(x)dx\right\} \leq P^{-1}\left\{\int_0^\infty P\left[\frac{C(Iv)(x)h(x)}{v(x)}\right]v(x)dx\right\} \tag{2.7}$$

$h \geq 0$, *are equivalent.*

(ii) If $0 \le f\uparrow$ *and* $(I^*v)(0) = \infty$, *then* (2.6) *and*

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)T(Ih)(x)]w_0(x)dx\right\} \le P^{-1}\left\{\int_0^\infty P\left[\frac{C(I^*v)(x)h(x)}{v(x)}\right]v(x)dx\right\} \tag{2.8}$$

$h \ge 0$, *are equivalent.*

Proof (i) By Proposition 2.1 with $w_0(x)dx = d\nu(x)$, $v(x)dx = d\mu(x)$, (2.6) is equivalent to $\|w_1 Tf\|_{Q(\varepsilon_Q w_0)} \le C\|f\|_{P(\varepsilon_P v)}$ where $\varepsilon > 0$. Writing $T_1 f = w_1 Tf$, this estimate has the form

$$\|T_1 f\|_{Q(\varepsilon_Q w_0)} \le C\|f\|_{P(\varepsilon_P v)} \qquad 0 \le f\downarrow\ . \tag{2.9}$$

But then by (2.3) and (2.4) (cf. Remark 2.1, with $w = \varepsilon_Q w_0$, $v = \varepsilon_P v$ so $(Iv)(x)$ becomes $\varepsilon_P(Iv)(x)$) (2.9) is equivalent to

$$\left\|\frac{I(T_1^* g)}{\varepsilon_P(Iv)}\right\|_{\tilde P(\varepsilon_P v)} \le C\left\|\frac{g}{\varepsilon_Q w_0}\right\|_{\tilde Q(\varepsilon_Q w)} \qquad g \ge 0, \tag{2.10}$$

where T_1^* is the adjoint of T_1. Now, for any linear operator T_2 and weight functions α, β, w and v

$$\|\alpha T_2 g\|_{\tilde P(w)} \le C\|\beta g\|_{\tilde Q(v)} \Longleftrightarrow \left\|\frac{T_2^* h}{v\beta}\right\|_{Q(v)} \le C\left\|\frac{h}{\alpha w}\right\|_{P(w)}, \tag{2.11}$$

where T_2^* is the adjoint of T_2.

Define T_2 by $T_2 g = I\,T_1^* g$, then $T_2^* = T_1 I^*$ and hence by (2.11) with $\alpha = 1/(\varepsilon_P Iv)$, $w = \varepsilon_P v$, $\beta = 1/(\varepsilon_Q w_0)$ and $v = \varepsilon_Q w_0$, (2.10) is equivalent to

$$\|T_2^* h\|_{Q(\varepsilon_Q w_0)} \le C\left\|\frac{(Iv)h}{v}\right\|_{P(\varepsilon_P v)}.$$

But by Proposition (2.1) this is equivalent to (2.7) since $T_2^* h = T_1(I^* h) = w_1 T(I^* h)$.

(ii) The proof for $0 \le f\uparrow$ is quite similar. Again we obtain (2.9) from (2.6) for $0 \le f\uparrow$. But then by (2.3) and (2.5) this is equivalent to

$$\left\| \frac{I^*(T_1^* g)}{\varepsilon_P(Iv)} \right\|_{\tilde{P}(\varepsilon_P v)} \leq C \left\| \frac{g}{\varepsilon_Q w_0} \right\|_{\tilde{Q}(\varepsilon_Q w_0)} \qquad g \geq 0\,. \tag{2.12}$$

Now, applying (2.11) with $T_2 g = I^* T_1^* g$, $\alpha = 1/(\varepsilon_P(I^* v))$, $w = \varepsilon_P v$, $\beta = 1/(\varepsilon_Q w_0)$ and $v = \varepsilon_Q w_0$, (2.12) is equivalent to

$$\|T_2^* h\|_{Q(\varepsilon_Q w_0)} \leq C \left\| \frac{(I^* v) h}{v} \right\|_{P(\varepsilon_P v)}\,.$$

By Proposition 2.1 this is equivalent to (2.8) since $T_2^* h = w_1 T(Ih)$.

Remark 2.2 As noted earlier, there are no order relations assumed on the N-functions P and Q in Theorem 2.2 and if for example $Q \prec P$, then in general weight characterizations for which the modular inequality (2.7) and (2.8) hold seem not to be known. However, if $P(x) = x^p/p$, $Q(x) = x^q/q$, $1 < p, q < \infty$ then Theorem 2.2 shows that $\|Tf\|_{q,w} \leq C\|f\|_{p,v}$ holds for $0 \leq f\downarrow$ if and only if $\|T(I^* h)\|_{q,w} \leq C\|\frac{h(Iv)}{v}\|_{p,v}$ holds for $0 \leq h$. But this norm inequality is equivalent to $\|\frac{I(T^* g)}{(Iv)}\|_{p',v} \leq C\|\frac{g}{w}\|_{q',w}$, where p' and q' are the conjugate indices of p and q. However, this is exactly the formulation obtained by E. Sawyer [15]. In the case $0 \leq f\uparrow$, one obtained in a similar way, using now Theorem 2.2(ii), the corresponding result of V.D. Stepanov [18].

If $P \prec Q$, then for suitable operators TI^* and TI weight characterizations for which (2.7) and (2.8) holds are known. We begin with our applications of Theorem 2.2 when T is the identity operator.

COROLLARY 2.1 *Suppose P and Q are N-functions, such that $P \prec Q$. Let $P, \tilde{P} \in \Delta_2$ and $(Iv)(\infty) = \infty$. If $0 \leq f\downarrow$, then the following are equivalent:*

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x) f(x)] w_0(x) dx\right\} \leq P^{-1}\left\{\int_0^\infty P[Cf(x)] v(x) dx\right\}. \tag{2.13}$$

There exists $B > 0$, such that for all $r > 0$, $\varepsilon > 0$

$$Q^{-1}\left\{\int_0^r Q\left[\frac{w_1(x)}{B}\left\|\frac{\chi_{(r,\infty)}}{\varepsilon(Iv)}\right\|_{\tilde{P}(\varepsilon v)}\right] w_0(x)\, dx\right\} \leq P^{-1}\left(\frac{1}{\varepsilon}\right) \tag{2.14}$$

holds.

There exists a $B > 0$, such that, for all $r > 0$, $\varepsilon > 0$,

$$Q^{-1}\left\{\int_0^r Q[\varepsilon w_1(x)]w_0(x)dx\right\} \le P^{-1}[P(B\varepsilon)(Iv)(r)] \tag{2.15}$$

holds.

Proof Let T be the identity operator, then $T(I^*h)(x) = (I^*h)(x) = \int_x^\infty h$. Hence by Theorem 2.2(i), (2.13) is equivalent to

$$Q^{-1}\left\{\int_0^\infty Q\left[w_1(x)\int_x^\infty h\right]w_0(x)dx\right\} \le P^{-1}\left\{\int_0^\infty P\left[\frac{C(Iv)(x)h(x)}{v(x)}\right]v(x)dx\right\}.$$

But by [5, Corollary 2.2] or [15, Proposition 3] this modular inequality is equivalent to (2.14).

That (2.13) and (2.15) are equivalent follows from a result of J.Q. Sun ([20, Theorem 3.4]).

If $0 \le f\uparrow$ a result similar to Corollary 2.1 holds. In fact if P and Q are as in Corollary 2.1 and $(I^*v)(0) = \infty$ then (2.13) with $0 \le f\uparrow$ is equivalent to:
There is a $B > 0$, such that for all $r > 0$, $\varepsilon > 0$,

$$Q^{-1}\left\{\int_r^\infty Q\left[\frac{w_1(x)}{B}\left\|\frac{\chi_{(0,r)}}{\varepsilon(I^*v)}\right\|_{\tilde{P}(\varepsilon v)}\right]w_0(x)\,dx\right\} \le P^{-1}(1/\varepsilon) \tag{2.16}$$

and there exists a $B > 0$, such that for all $r > 0$, $\varepsilon > 0$

$$Q^{-1}\left\{\int_r^\infty Q[\varepsilon w_1(x)]w_0(x)\,dx\right\} \le P^{-1}\{P(B\varepsilon)(I^*v)Ir)\}. \tag{2.17}$$

The proof of this follows (again) from Theorem 2.2(ii) and [5, Theorem 2.1] or [20, Theorem 3.4].

If $P(x) = x^p/p$, $Q(x) = x^q/q$, $1 < p \le q < \infty$, $w_1(x) = 1$, Corollary 2.1 reduces to a result of E. Sawyer [15], while the case for $0 \le f\uparrow$ may be found in [6].

Our next application considers the Hardy operator defined on decreasing functions.

COROLLARY 2.2 *Suppose P and Q are N-functions, P, $\tilde{P} \in \Delta_2$ and $P \prec Q$. If $(Iv)(\infty) = \infty$ and $0 \le f\downarrow$ then*

$$Q^{-1}\left\{\int_0^\infty Q\left[w_1(x)\int_0^x f\right]w_0(x)\,dx\right\} \le P^{-1}\left\{\int_0^\infty P[Cf(x)]v(x)\,dx\right\} \tag{2.18}$$

is satisfied, if and only if, there are constants $B_0, B_1 > 0$, such that for all $r > 0$, $\varepsilon > 0$

$$Q^{-1}\left\{\int_r^\infty Q\left[\frac{w_1(x)}{B_0}\left\|\frac{\chi(0,r)\,i_0}{\varepsilon(Iv)}\right\|_{\tilde{P}(\varepsilon v)}\right]w_0(x)\,dx\right\} \le P\left(\frac{1}{\varepsilon}\right)$$

and

$$Q^{-1}\left\{\int_0^r Q\left[\frac{w_1(x)x}{B_1}\left\|\frac{\chi(r,\infty)}{\varepsilon I(v)}\right\|_{\tilde{P}(\varepsilon v)}\right]w_0(x)\,dx\right\} \le P\left(\frac{1}{\varepsilon}\right).$$

(Recall that $i_0(x) = x$.)

Proof If $Tf(x) = \int_0^x f$, then $T(I^*h)(x) = \int_0^x t\,h(t)\,dt + x\int_x^\infty h(t)\,dt$. Hence by Theorem 2.2(i), (2.18) is equivalent to

$$Q^{-1}\left\{\int_0^\infty Q\left[w_1(x)\left(\int_0^x t\,h(t)\,dt + x\int_0^\infty h(t)\,dt\right)\right]w_0(x)\,dx\right\}$$
$$\le P^{-1}\left\{\int_0^\infty P\left[\frac{C(Iv)(x)h(x)}{v(x)}\right]v(x)\,dx\right\}.$$

But since $P \in \Delta_2$, the convexity of Q and P show that this is equivalent to

$$Q^{-1}\left\{\int_0^\infty Q\left[w_1(x)\int_0^x t\,h(t)\,dt\right]w_0(x)\,dx\right\}$$
$$\le P^{-1}\left\{\int_0^\infty P\left[\frac{C_1(Iv)(x)h(x)}{v(x)}\right]v(x)\,dx\right\}$$

and
$$Q^{-1}\left\{\int_0^\infty Q\left[w_1(x)x\int_x^\infty h(t)\,dt\right]w_0(x)\,dx\right\}$$
$$\leq P^{-1}\left\{\int_0^\infty P\left[\frac{C_2(Iv)(x)h(x)}{v(x)}\right]v(x)\,dx\right\}, \tag{2.19}$$
where $C_1, C_2 > 0$ and $h \geq 0$. Since $P \prec Q$ the weight characterizations for which (2.19) hold are shown in [5] or [19] to be equivalent to the conditions of the corollary.

Note that the proof of Corollary 2.1 shows that the second modular of (2.19) is also equivalent to

$$Q^{-1}\left\{\int_0^r Q[\varepsilon x w_1(x)]w_0(x)dx\right\} \leq P^{-1}\{P(B\varepsilon)(Iv)(r)\}$$

$r > 0, \varepsilon > 0$.

Remark 2.3 (i) The equivalence of (2.18) and (2.19) does not require the order condition $P \prec Q$. Hence if $P(x) = x^p/p$, $Q(x) = x^q/q$, $1 < p, q < \infty$ in (2.19) with $w_1(x) = \frac{1}{x}$ weight characterizations are known (cf. [12]) and one obtained the result of E. Sawyer [15] and for $p = q$ that of M. Ariño and B. Muckenhoupt [2].

(ii) Corollary 2.2 was proved by J.Q. Sun [19] by different methods.

(iii) For $0 \leq f\uparrow$, a result corresponding to Corollary 2.2 may also be given. Only now one applies Theorem 2.2(ii) and uses the modular estimates for

$$T(Ih)(x) = \int_0^x (x-s)h(s)\,ds$$

given in [2, 13, 16, 17]. The reduction to the weighted Lebesgue case yield the results of [1, 6, 11].

Our final example involves the Riemann-Liouville fractional integral operator of order α, $0 < \alpha < \infty$ defined by

$$(I_\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt$$

and the Weyl fractional integral

$$(I_\alpha^* f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty (t-x)^{\alpha-1}f(t)\,dt\,.$$

For the case $\alpha \geq 1$ weight characterizations for weighted modular inequalities or for weighted L^p-norm inequalities are known. For the case $0 < \alpha < 1$ a weighted L^p-norm characterization was given recently in [4] while weighted modular inequalities seem not to be available in the literature. It may therefore be surprising that in the monotone case more can be said.

PROPOSITION 2.2 *(i) Suppose P, Q are N-functions, $P, \tilde{P} \in \Delta_2$ and $P < Q$. If $(Iv)(\infty) = \infty$ and $0 \leq f\downarrow$ then for $0 < \alpha < \infty$*

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)(I_\alpha^* f)(x)]w_0(x)\,dx\right\} \leq P^{-1}\left\{\int_0^\infty P[C\,f(x)]v(x)\,dx\right\}, \tag{2.20}$$

if and only if for all $\varepsilon > 0$, $r > 0$,

$$Q^{-1}\left\{\int_0^r Q\left[\frac{w_1(x)}{C}\left\|\frac{(\cdot\, - r)^\alpha \chi_{(r,\infty)}}{\varepsilon(Iv)}\right\|_{\tilde{P}(\varepsilon v)}\right]w_0(x)\,dx\right\} \leq P^{-1}\left(\frac{1}{\varepsilon}\right)$$

and

$$Q^{-1}\left\{\int_0^r Q\left[\frac{w_1(x)}{C}\left\|\frac{\chi_{(r,\infty)}}{\varepsilon(Iv)}\right\|_{\tilde{P}(\varepsilon v)}(r-x)^\alpha\right]w_0(x)\,dx\right\} \leq P^{-1}\left(\frac{1}{\varepsilon}\right)$$

are satisfied.

(ii) If $(I^*v)(0) = \infty$ and $0 \leq f\uparrow$, then for $0 < \alpha < \infty$

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)(I_\alpha f)(x)]w_0(x)\,dx\right\} \leq P^{-1}\left\{\int_0^\infty P[C\,f(x)]v(x)\,dx\right\} \tag{2.21}$$

holds, if and only if, for all $r > 0$, $\varepsilon > 0$,

$$Q^{-1}\left\{\int_r^\infty Q\left[\frac{w_1(x)}{C}\left\|\frac{(r-\,\cdot)^\alpha \chi_{(0,r)}}{\varepsilon(I^*v)}\right\|_{\tilde{P}(\varepsilon v)}\right]w_0(x)\,dx\right\} \leq P^{-1}\left(\frac{1}{\varepsilon}\right)$$

and

$$Q^{-1}\left\{\int_r^\infty Q\left[\frac{w_1(x)}{C}\left\|\frac{\chi_{(0,r)}}{\varepsilon(I^*v)}\right\|_{\tilde{P}(\varepsilon v)}(x-r)^\alpha\right]w_0(x)\,dx\right\} \leq P^{-1}\left(\frac{1}{\varepsilon}\right).$$

Proof (i) By Theorem 2.2(i), (2.20) is equivalent to

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)I_\alpha^*(I^*h)(x)]w_0(x)\,dx\right\} \leq P^{-1}\left\{\int_0^\infty P\left[\frac{C(Iv)(x)h(x)}{v(x)}\right]v(x)\,dx\right\}$$

$h \geq 0$. But $I_\alpha^*(I^*h)(x) = (I_{\alpha+1}^*h)(x)$ and since $\alpha + 1 > 1$, Theorem 2.2 of [20] (see also [3, 13]) shows that this is equivalent to the conditions of the proposition.

(ii) By Theorem 2.2(ii), (2.21) is equivalent to

$$Q^{-1}\left\{\int_0^\infty Q[w_1(x)I_\alpha(Ih)(x)]w_0(x)\,dx\right\} \leq P^{-1}\left\{\int_0^\infty P\left[\frac{C(I^*v)(x)h(x)}{v(x)}\right]v(x)\,dx\right\},$$

where $h \geq 0$. But $I_\alpha Ih = I_{\alpha+1}h$, and since $\alpha + 1 > 1$ the result follows from [19, Theorem 2.1]. (See also [3, 13]).

For $\alpha > 1$ this result was also proved by J.Q. Sun [20].

Note that if $P(x) = x^p/p$, $Q(x) = x^q/q$, $1 < p \leq q < \infty$, $w_1(x) = 1$, then Proposition 2.2(i) shows that for $0 < \alpha < \infty$

$$\left(\int_0^\infty \left[\int_x^\infty (t-x)^{\alpha-1} f(t)\,dt\right]^q w_0(x)\,dx\right)^{1/q} \leq C\left(\int_0^\infty f(x)^p v(x)\,dx\right)^{1/p}$$

holds for all $0 \leq f\downarrow$, if and only if, for every $r > 0$

$$\left(\int_0^r w_0(x)\,dx\right)^{1/q}\left(\int_r^\infty (x-r)^{\alpha p'}\left(\int_0^x v\right)^{-p'} v(x)\,dx\right)^{1/p'} \leq C$$

and

$$\left(\int_0^r (r-x)^{\alpha q} w_0(x)\,dx\right)^{1/q}\left(\int_0^r v(x)\,dx\right)^{-1/p} \leq C\,.$$

A similar remark holds for the Riemann-Liouville operator defined on increasing functions.

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