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TWO-PARAMETER SEMIGROUPS, EVOLUTIONS AND THEIR APPLICATIONS TO MARKOV AND DIFFUSION FIELDS ON THE PLANE

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ABSTRACT

We study two-parameter coordinate-wise C_0-semigroups and their generators, as well as two-parameter evolutions and differential equations up to the second order for them. These results are applied to obtain the Hille-Yosida theorem for homogeneous Markov fields of the Feller type and to establish forward, backward, and mixed Kolmogorov equations for nonhomogeneous diffusion fields on the plane.

Key words: Two-Parameter Semigroup, Two-Parameter Evolution Operator, Markov Field, Diffusion Field.

AMS (MOS) subject classifications: 60G60, 60J25, 60J35, 60J60, 47D06, 47D07.

1. Introduction

Let T_{t_1,t_2} be a two-parameter coordinate-wise C_0-semigroup. The paper is organized as follows. In Section 2, we prove that $T_{t_1,t_2} = T_{1,t_1t_2} = T_{t_1t_2,1}$ and establish that its generator coincides with the generator of the one-parameter semigroup $T_{1,t}$. We also derive differential equations up to the second order to T_{t_1,t_2} and its resolvent and establish Hille-Yosida theorem for T_{t_1,t_2} (Remark 2). In the third section we consider two-parameter evolution operators, $T_{ss',tt'}$ up to the second order. In the fourth section we study $*$-Markov fields on the plane with transition functions and present the Hille-Yosida theorem for $*$-Markov fields of the Feller type. In the fifth section the class of diffusion fields is introduced. The form of generators and relations between them are established. Forward, backward, and mixed Kolmogorov equations of the second order for the densities of diffusion fields are presented. A partial case of backward Kolmogorov equations was considered in [3, 4].

2. Two-Parameter Semigroups, Their Generators, and Resolvents

Let X be a complex Banach space, $\mathfrak{L}(X)$ be a space of linear continuous operators from X to X, I be the identity operator on X, and $D(A)$ be the domain of operator A.

Denote $\mathbb{R}^2_+ = [0, +\infty)^2$ with partial ordering $\overline{s} < (\leq) \overline{t}$, if $\overline{s} = (s_1, s_2)$, $\overline{t} = (t_1, t_2)$ and $s_i < (\leq) t_i$, $i = 1, 2$.

Definition 1: The family $\{T_{t_1, t_2}, (t_1, t_2) \in \mathbb{R}^2_+\} \subset \mathfrak{L}(X)$ is called *a coordinate-wise two-parameter semigroup* if it satisfies the following two conditions.

(A) $a)$ $T_{0, t_2} = T_{t_1, 0} = I$.

$b)$ For any $t_i \geq 0$, $s_i \geq 0$, $i = 1, 2$,

$$T_{s_1 + t_1, t_2} = T_{s_1, t_2} T_{t_1, t_2} \text{ and } T_{s_1, s_2 + t_2} = T_{s_1, s_2} T_{s_1, t_2}.$$

Furthermore, we simply refer to the coordinate-wise two-parameter semigroup as *semigroup.*

Definition 2: A semigroup T_{t_1, t_2} is called a C_0*-semigroup* if for any $x \in X$, $\overline{s} \in \mathbb{R}^2_+$,

$$\lim_{\overline{t} \to \overline{s}} \| T_{\overline{s}} x - T_{\overline{t}} x \| = 0.$$

Lemma 1: *The following conditions are equivalent.*

(B_1) *For any* $x \in X$, $t_1 > 0$, $t_2 > 0$,

$$\lim_{t_2 \to 0} T_{t_1, t_2} x = \lim_{t_1 \to 0} T_{t_1, t_2} x = x.$$

(B_2) *For any* $x \in X$,

$$\lim_{t_1 \vee t_2 \to 0} T_{t_1, t_2} x = x.$$

The proof of Lemma 1 is similar to that of the classical theorem about continuity of separately continuous bilinear forms [8] when we replace functionals by operators, so it is omitted.

Lemma 2: *The semigroup* T_{t_1, t_2} *is a* C_0*-semigroup if and only if it satisfies one of the conditions* (B_1) *or* (B_2).

Proof: The necessity is obvious. Let us prove sufficiency. Suppose, for example (B_1) is satisfied. Then the one-parameter semigroups, $T_{t_1, \cdot}$, and $T_{\cdot, t_2}$ are continuous for any fixed t_1 and t_2. From known properties of one-parameter semigroups, for any $t_i \geq 0$, $i = 1, 2$, there exist constants $C_i = C_i(t_i) > 0$ and $a_i = a_i(t_i) \in \mathbb{R}$ such that $\| T_{t_1, u} \| \leq C_1 e^{a_1 u}$ and $\| T_{u, t_2} \| \leq C_2 e^{a_2 u}$ for any $u \geq 0$. Now let $\overline{s}$ be fixed with $\overline{t} > \overline{s}$. Then, from Lemma 1, we find that

$$\begin{aligned}
\lim_{\overline{t} \to \overline{s}} \| T_{\overline{t}} x - T_{\overline{s}} x \| &= \lim_{t \to s} \| T_{\overline{s}} T_{s_1, t_2 - s_2} T_{t_1 - s_1, s_2} T_{t_1 - s_1, t_2 - s_2} x - T_{\overline{s}} x \| \\
&\leq \lim_{\overline{t} \to \overline{s}} \| T_{\overline{s}} \| \, \| T_{s_1, t_2 - s_2} T_{t_1 - s_1, s_2} T_{t_1 - s_1, t_2 - s_2} x - x \| \\
&\leq \lim_{\overline{t} \to \overline{s}} \| T_{\overline{s}} \| (\| T_{s_1, t_2 - s_2} T_{t_1 - s_1, s_2} \| \, \| T_{t_1 - s_1, t_2 - s_2} x - x \| \\
&\qquad + \| T_{s_1, t_2 - s_2} \| \, \| T_{t_1 - s_1, s_2} x - x \| \\
&\quad + \| T_{s_1, t_2 - s_2} x - x \|) \leq \lim_{\overline{t} \to \overline{s}} \| T_{\overline{s}} \| (C_1 C_2 e^{a_1(t_1 - s_1)} e^{a_2(t_2 - s_2)} \\
&\times \| T_{t_1 - s_1, t_2 - s_2} x - x \| + C_2 e^{a_2(t_2 - s_2)} \| T_{t_1 - s_1, s_2} x - x \| + \| T_{s_1, t_2 - s_2} x - x \|) = 0.
\end{aligned}$$

Other version of the arrangement of the point $\overline{t}$ with respect to $\overline{s}$ are considered similarly. □

Thus, Definition 2 can be weakened to condition (B_1).

Lemma 3: *Let* $T_{\overline{t}}$ *be a* C_0*-semigroup. Then, for any* $\overline{t} = (t_1, t_2) \in \mathbb{R}^2_+$,

$$T_{\overline{t}} = T_{1, t_1 t_2} = T_{t_1 t_2, 1}. \tag{1}$$

Proof: Note that for any $\overline{t} \in \mathbb{R}^2_+$ and $n \in \mathbb{N}$,

$$T_{t_1, nt_2} = (T_{\overline{t}})^n = T_{nt_1, t_2}.$$

Furthermore, for any $t_1 \geq 0$, there exists a sequence $[u_n, n \geq 1\} \subset Q^+$ such that $u_n = (p_n/q_n) \to t_1$ as $n \to \infty$. Then for any $x \in X$,

$$T_{\overline{t}}\, x = \lim_{n\to\infty} T_{u_n, t_2} x = \lim_{n\to\infty} T_{(p_n/q_n),(t_2 q_n/q_n)} x$$

$$= \lim_{n\to\infty} (T_{(1/q_n),(t_2/q_n)})^{p_n q_n} x = \lim_{n\to\infty} T_{1, t_2(p_n/q_n)} x = T_{1, t_1 t_2} x.$$

Hence, we also have $T_{t_1, t_2} = T_{t_1 t_2, 1}$. □

Remark 1: Let $\overline{t}, \overline{s} \in \mathbb{R}^2_+$. Then,

$$T_{\overline{t}}\, T_{\overline{s}} = T_{1, t_1 t_2} T_{1, s_1 s_2} = T_{1, t_1 t_2 + s_1 s_2} = T_{1, s_1 s_2} T_{1, t_1 t_2} = T_{\overline{s}}\, T_{\overline{t}}\,.$$

Definition 3: 1. The *generator* A of C_0-semigroup T_t is defined by

$$Ax := \lim_{\overline{t}\to 0} \frac{1}{t_1 t_2}(T_{\overline{t}}\, x - x),$$

whenever the limit exists.

2. The *i-generators* $(i = 1,2)$ of C_0-semigroup $T_{\overline{t}}$ are defined by

$$A^2_{t_1} x := \lim_{t_2 \to 0} \frac{1}{t_2}(T_{\overline{t}}\, x - x) \text{ and } A^1_{t_2} x := \lim_{t_1 \to 0} \frac{1}{t_1}(T_{\overline{t}}\, x - x),$$

whenever the limits exist.

Theorem 1: *Let $T_{\overline{t}}$ be a C_0-semigroup. Then the following hold:*

1) $$A = A^1_1 = A^2_1 \text{ and } A^j_{t_i} = t_i A, \quad t_i > 0,\ i = 1,2,\ j = 1,2,\ i \neq j. \tag{2}$$

2) *For any $x \in D(A)$,*

$$T_{\overline{t}}\, Ax = AT_{\overline{t}}\, x.$$

3) *For any $\overline{t} = (t_1, t_2) \in \mathbb{R}^2_+$ and $x \in D(A)$,*

$$\frac{\partial T_{\overline{t}}}{\partial t_1 t_j} = t_i T_{\overline{t}}\, Ax = T_{\overline{t}}\, A^j_{t_i} x.$$

For any $x \in D(A^2)$,

$$\frac{\partial^2 T_{\overline{t}}}{\partial t_1 \partial t_2} x = AT_{\overline{t}}\, x + t_1 t_2 A^2 T_{\overline{t}}\, x.$$

Proof: 1. Let $x \in D(A^2_1)$. It follows from Lemma 3 that

$$\lim_{\overline{t}\to 0} \frac{T_{t_1, t_2} - I}{t_1 t_2}\, x = \lim_{\overline{t}\to 0} \frac{T_{1, t_1 t_2} - I}{t_1 t_2}\, x$$

and

$$\lim_{t_1 \to 0} \frac{T_{t_1, t_2} - I}{t_1}\, x = t_2 \lim_{t_1 \to 0} \frac{T_{1, t_1 t_2} - I}{t_1 t_2}\, x$$

exist or do not exist simultaneously. Therefore, according to Definition 3, $D(A_1^2) = D(A_{t_1}^2)$, $D(A_1^2) \subset D(A)$ and $A_{t_1}^2 x = t_1 Ax$. From the same arguments applied to $x \in D(A_1^2)$ and $x \in D(A)$, we have $D(A_1^1) \subset D(A) \subset D(A_1^1) = D(A_1^2)$, and, consequently, $D(A_1^1) = D(A_1^2) = D(A)$. Therefore, the equalities (2) hold.

2. Operators A_1^1 and $T_{t,1}$ commute on $D(A_2^1)$ (this follows from the corresponding properties of one-parameter semigroups); therefore, $A = A_1^1$ and $T_{\overline{t}} = T_{t_1 t_2, 1}$ commute on $D(A) = D(A_1^1)$.

3. This statement can be obtained by direct calculations. □

Suppose the semigroup $T_{\overline{t}}$ is not continuous on the whole space X. In this case, let us consider the linear manifold,

$$X_0 = \{ \mid x \in X \mid \lim_{u \to 0} T_{u, t_2} x = \lim_{v \to 0} T_{t_1, v} x = x \text{ for } t_1, t_2 \geq 0 \}.$$

Lemma 4: 1) *X_0 is a subspace in X.*

2) *Operators $T_{\overline{t}}$ act from X_0 to X_0.*

The proof follows from equality (1) and similar results for one-parameter semigroups. □

Theorem 2: *The linear operator A is a generator of a coordinate-wise C_0-semigroup if and only if it is a generator of a one-parameter C_0-semigroup.*

Proof: Let A be a generator of the coordinate-wise C_0-semigroup $T_{\overline{t}}$. Then from Lemma 3 and Theorem 1, A is a generator of the one-parameter C_0-semigroup $U(t) = T_{1,t}$. Conversely, assume that A generates a one-parameter semigroup $U(t)$. Set $T_{\overline{t}} = U(t_1 t_2)$. Then $T_{\overline{t}}$ is coordinate-wise C_0-semigroup, the limits

$$\lim_{\overline{t} \to 0} \frac{T_{t_1, t_2} x - x}{t_1 t_2} \text{ and } \lim_{t_1 t_2 \to 0} \frac{U(t_1 t_2) x - x}{t_1 t_2}$$

exist or do not exist simultaneously, and for $x \in D(A)$

$$Ax = \lim_{t_1, t_2 \to 0} \frac{U(t_1 t_2) x - x}{t_1 t_2} = \lim_{\overline{t} \to 0} \frac{T_{t_1, t_2} x - x}{t_1 t_2}.$$

Therefore, A is a generator of coordinate-wise semigroup. □

Remark 2: It follows from Theorem 2 that the conditions of the well-known Hille-Yosida theorem are necessary and sufficient for the closed operator A with $\overline{D(A)} = X$ to generate coordinate-wise semigroup.

Remark 3: The statement similar to Theorem 2 for an n-parameter coordinate-wise semigroup is true and would have the same proof.

It is well known that in the one-parameter case, the Laplace transform of semigroup is a resolvent of its generator, defined in the appropriate half-plane of $\mathbb{C}$. Analogously, in the case of the multiplicative semigroup $\widehat{T}_{t_1, t_2}$, given by equations,

$$\widehat{T}_{t_1, t_2} = T_1(t_1) T_2(t_2) \text{ and } T_1(t_1) T_2(t_2) = T_2(t_2) T_1(t_1),$$

where $T_i(t_i)$ for $i = 1, 2$ is a one-parameter semigroup, the two-dimensional Laplace transform of $\widehat{T}(t_1, t_2)$ is decomposed into a product of one-dimensional transforms and is the product of resolvents of semigroup generators. There are no such simple relations for coordinate-wise semigroups. In this vein, we can obtain only the following result.

Theorem 3: *Let $\{T_{\overline{t}}, \overline{t} \in \mathbb{R}_+^2\}$ be a contractive coordinate-wise semigroup (this assumption is made for the sake of simplicity), and let $L_{z,w} = L_{z,w}(f)$ and $L_z = L_z(g)$ be two- and one-dimen-*

sional Laplace transform of functions f and g, respectively. Then, the following hold.

1) *For any $z, w > 0$,*

$$L_{z,w}(T_{t_1,t_2}) = L_{1,zw} = L_1(R_{(zw)/t_2}),$$

where R is a resolvent of the generator of the semigroup $T_{1,t}$.

2) *For any $x \in D(A^2)$,*

$$A^2\frac{\partial^2 L_{z,w}}{\partial z \partial w}\,x + AL_{z,w}x = zwL_{z,w}x - x,$$

where $L_{z,w}x = L_{z,w}(T_{t_1,t_2}x)$.

3) *For any $x \in D(A^2)$,*

$$A^2(L'(u)u)'x + AL(u)x = uL(u)x - x,$$

where $L(u)x = L_{1,u}x = L_{1,u}(t_{t_1,t_2}x)$.

Proof: 1) From equality (1) for $z, w > 0$ and $x \in X$,

$$L_{z,w}x = \int\limits_{R^2_+} e^{-zt_2 - wt_1}T_{1,t_2t_1}x\,dt_2dt_1 = \int\limits_{R^2_+} e^{-z(t_2'/z) - wzt_1'}T_{1,(t_2'/z)zt_1'}x\tfrac{1}{z}dt_2'zdt_1'$$

$$= \int\limits_{R^2_+} e^{-z(t_2'/z) - wzt_1'}T_{1,t_2t_1}x\,dt_2dt_1$$

$$= L_{1,zw}x = \int_0^\infty e^{-1\cdot t_2}\left(\int_0^\infty e^{-zwt_1}T_{1,t_1t_2}x\,dt_1\right)dt_2 = L_1\left(R_{zw/t_2}x\right).$$

2) Let $x \in D(A^2)$. Then for any $u, v > 0$,

$$\frac{1}{uv}(T_{u,v} - I)L_{z,w}x = \widehat{A}_{u,v}x - T_{u,v}\int\limits_{R^2_+} e^{-zt_1 - wt_2}\frac{T_{t_1,v} - I}{v}\,\frac{T_{u,t_2} - I}{u}$$

$$\times T_{t_1,t_2}x\,dt_2dt_2 - \frac{T_{u,v} - I}{v}\int\limits_{R^2_+} e^{-zt_1 - wt_2}\frac{T_{t_1,v} - I}{v}\,T_{t_1,t_2}x\,dt_1dt_2$$

$$-\frac{T_{u,v} - I}{u}\int\limits_{R^2_+} e^{-zt_1 - wt_2}\frac{T_{t_1,v} - I}{v}T_{t_1,t_2}dt_1dt_2, \tag{3}$$

where

$$\widehat{A}_{u,v}x = \frac{1}{uv}\Big((e^{zu} - 1)(e^{wv} - 1)L_{z,w}x - e^{zu}(e^{wv} - 1)\int_0^u\int_0^\infty e^{-zt_1 - wt_2}T_{t_1,t_2}x\,dt_1dt_2$$

$$-e^{wv}(e^{zu} - 1)\int_0^\infty\int_0^v e^{-zt_1 - wt_2}T_{t_1,t_2}x\,dt_1dt_2 + e^{zu+wv}\int_0^u\int_0^v e^{-zt_1 - wt_2}T_{t_1,t_2}x\,dt_1dt_2\Big).$$

Obviously,

$$\widehat{A}_{uv}x \to zwL_{z,w}x - \left(w\int_0^\infty e^{-wt_2}dt_2x\right) - \left(z\int_0^\infty e^{-zt_1}dt_1x\right) + x = zwL_{z,w}x - x.$$

Note that

$$\left\| \frac{T_{t_1,v}-I}{v}\frac{T_{u,t_2}-I}{u}x \right\| = \left\| \frac{1}{v}\int_0^{sv} T_{1,z}\left(\frac{1}{u}\int_0^{ut} T_{1,p}Axdp\right)dz \right\|$$

$$= \left\| \frac{1}{v}\frac{1}{u}\int_0^{t_1v}\int_0^{t_2u} T_{1,z}T_{1,p}A^2xdpdz \right\| \le t_1t_2 \| A^2x \|$$

in view of the contractive property of semigroups. Existence of the integrated majorant implies that

$$T_{u,v}\int_{R^2_+} e^{-zt_1-wt_2}\frac{T_{t_1,v}-I}{v}\frac{T_{u,t_2}-I}{u}T_{t_1,t_2}xdt_1dt_2$$

$$\rightarrow t_1t_2\int_{R^2_+} e^{-zt_1-wt_2}T_{t_1,t_2}A^2xdt_1dt_2, \quad u,v\rightarrow 0. \tag{4}$$

Again, from the existence of the integrated majorant, the last integral equals

$$t_1t_2\int_{R^2_+} e^{-zt_1-wt_2}\lim_{u\vee v\rightarrow 0}\frac{1}{u}(T_{1,u}-I)\frac{1}{v}(T_{v,1}-I)T_{t_1t_2}xdt_1dt_2$$

$$=\lim_{u\vee v\rightarrow 0}\frac{T_{1,u}-I}{u}\frac{T_{v,1}-I}{v}\int_{R^2_+} e^{-zt_1-wt_2}t_1t_2T_{t_1,t_2}xdt_1dt_2 = A^2\frac{\partial^2 L_{z,w}}{\partial z\partial w}x. \tag{5}$$

Furthermore,

$$\lim_{u\vee v\rightarrow 0}\left\| \frac{T_{u,v}-I}{v}\int_{R^2_+} e^{-zt_1-wt_2}t_2\frac{T_{u,t_2}-I}{t_2u}T_{t_1t_2}xdt_1dt_2 \right\|$$

$$=\lim_{u\vee v\rightarrow 0}\left\| u\int_{R^2_+} e^{-zt_1-wt_2}t_2\frac{T_{u,t_2}-I}{t_2u}T_{t_1t_2}\frac{T_{u,v}-I}{vu}xdt_1dt_2 \right\| = 0. \tag{6}$$

Analogously,

$$\lim_{u\vee v\rightarrow 0}\left\| \frac{T_{u,v}-I}{u}\int_{R^2_+} e^{-zt_1-wt_2}t_1\frac{T_{t_1,v}-I}{t_1v}T_{t_1,t_2}xdt_1dt_2 \right\| = 0. \tag{7}$$

Equation from the statement 2) follows from (3) through (7).

3) Finally, from the equality $L(u) = L_{1,u}$ with $u = zw$, we obtain that

$$\frac{\partial^2 L_{z,w}}{\partial z\partial w} = \frac{\partial}{\partial z}(L'(u)\cdot z) = L''(u)\cdot u + L'(u).$$

Equation from the statement 3) follows. □

3. Two-Parameter Evolution Operators and Their Generators

Let us consider the family of operators,

$$\{T_{ss',tt'} 0 \le s \le s', 0 \le t \le t'\} \subset \mathfrak{L}(X), \quad s', t' \in \mathbb{R}_+,$$

satisfying the following conditions.

(C) $a)$ For any $0 \le s \le s' \le s''$ and $0 \le t \le t'$,

$$T_{ss',tt'} T_{ss',t't''} = T_{ss',tt''}.$$

$b)$ For any $0 \le s \le s'$ and $0 \le t \le t' \le t''$,

$$T_{ss',tt'} T_{ss',tt'} = T_{ss',t''} = T_{ss',tt''}.$$

$c)$ For any $0 \le s \le s'$ and $0 \le t \le t'$,

$$T_{ss,tt'} = T_{ss',tt} = I.$$

We call any operator, $T_{ss',tt'}$, in this family a two-parameter evolution operator (or simply an evolution).

Definition 4: The family of evolutions is said to be *continuous* if, for any $0 \le s < s'$ and $0 \le t < t'$,

$$\lim_{h\to 0} T_{s+hs',tt'} = \lim_{h\to 0} T_{ss'+h,tt'} = \lim_{h\to 0} T_{ss',t+ht'} = \lim_{h\to 0} T_{ss',tt'+h} = T_{ss',tt'}$$

in the sense of strong convergence in X.

Further, we consider only continuous families of evolutions. Let us denote

$$(u,a)^+ = (u, u+a), (u,a)^- = (u-a, u), \quad a > 0;$$

$$\Box T^{\pm\pm}_{s,t,h,k} = T_{(s,h)^\pm (t,k)^\pm} - I; \quad \Delta^1 T^{\pm}_{h,s,t,t'} = T_{(s,h)^\pm, tt'} - I;$$

$$\Delta^2 T^{\pm}_{k,s,s',t} = T_{ss',(t,k)^\pm} - I; \quad \frac{\partial^2}{\partial s \partial t} = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial t}\right).$$

Definition 5: 1) The elements of the family of operators $\{A^{\pm\pm}_{s,t}\}$, defined as

$$A^{\pm\pm}_{s,t} x := \lim_{h,k\to 0} \frac{1}{hk} \Box T^{\pm\pm}_{s,t,h,k} x,$$

considered on the sets where corresponding limits exist, are called *generators* of evolutions.

2) The elements of the family of operators $\{A^{1,\pm}_{stt'}, A^{2,\pm}_{ss't}\}$, defined as

$$A^{1\pm\pm}_{s,t,t'} x := \lim \frac{1}{h} \Delta^1 T^{\pm}_{h,s,t,t'} x \text{ and } A^{2,\pm}_{ss't} x := \lim \frac{1}{k} \Delta^2 T^{\pm}_{k,s,s',t} x,$$

considered on the sets where corresponding limits exist, are called *i-generators* $(i = 1, 2)$ of evolutions. If $A^{i,+} = A^{i,-}$ or $A^{++} = A^{+-} = A^{-+} = A^{--}$, then we denote the common value as A^i or A respectively.

Definition 6: *Right and left derivatives* of evolutions are defined as

$$\frac{\partial^+ T_{ss',tt'}}{\partial s'} = \lim \frac{1}{h}(T_{ss'+h,tt'} - T_{ss',tt'}) \text{ and } \frac{\partial^- T_{ss',tt'}}{\partial s} = -\lim \frac{1}{h}(T_{s-hs',tt'} - T_{ss',tt'})$$

respectively.

In similar ways, one can define right and left derivatives of other families of operators, depending on s, s', t and t'.

Lemma 5: 1) *If* $T_{ss',tt'} x \in D(A^{1,-}_{stt'})$, *then*

$$\frac{\partial^- T_{ss',tt'}}{\partial s} x = -A^{1,-}_{stt'} T_{ss',tt'} x.$$

2) *If* $x \in D(A^{1,+}_{s'tt'})$, *then*

$$\frac{\partial^+ T_{ss',tt'}}{\partial s'}x = T_{ss',tt'}A^{1,+}_{s'tt'}x.$$

3) *If* $T_{ss',tt'}x \in D(A^{2,-}_{ss't})$, *then*

$$\frac{\partial^- T_{ss',tt'}}{\partial t}x = -A^{2,-}_{ss't}T_{ss',tt'}x.$$

4) *If* $x \in D(A^{2,+}_{ss't'})$, *then*

$$\frac{\partial^+ T_{ss',tt'}}{\partial t'}x = T_{ss',tt'}A^{2,+}_{ss't'}x.$$

Proof: Let us prove 1). (The other equalities are proved similarly). If $T_{ss',tt'}x \in D(A^{1,-}_{stt'})$, then

$$\frac{\partial^- T_{ss',tt'}}{\partial s}x = -\lim\frac{1}{h}(T_{s-hs',tt'} - T_{ss',tt'})x$$

$$= -\lim\frac{1}{h}(T_{s-hs',tt'} - I)T_{ss'tt'}x = -A^{1,-}_{stt'}T_{ss'tt'}x. \qquad \square$$

Lemma 6: 1) *Let* $x \in D(A^{\pm\,-}_{st}) \cap D(A^{1,\pm}_{svt'})$ *and* $v \in [t-\delta, t]$ *for some* $\delta > 0$. *Let also*

$$\lim\frac{1}{hk}\square T^{\pm,-}_{s,t,h,k}\cdot\Delta^1 T^{\pm}_{h,s,t,t'}x = 0. \tag{8}$$

Then

$$\frac{\partial^- A^{1,\pm}_{stt'}}{\partial t}x = -A^{\pm\,-}_{st}x.$$

2) *Let* $x \in D(A^{\pm,+}_{st'}) \cap D(A^{1,\pm}_{stv})$ *and* $v \in [t', t'+\delta]$ *for some* $\delta > 0$. *Then*

$$\frac{\partial^+ A^{1,\pm}_{stt'}}{\partial t'}x = -A^{\pm,+}_{st'}x.$$

3) *Let* $x \in D(A^{-,\pm}_{st'}) \cap D(A^{2,\pm}_{us't})$ *and* $u \in [s-\delta, s]$ *for some* $\delta > 0$. *Let also*

$$\lim\frac{1}{hk}\square T^{+,\pm}_{s,t,h,k}\cdot\Delta^2 T^{\pm}_{k,s,s't}x = 0. \tag{9}$$

Then

$$\frac{\partial^- A^{2,\pm}_{ss't}}{\partial s}x = -A^{-,\pm}_{st'}x.$$

4) *Let* $x \in D(A^{+,\pm}_{s't}) \cap D(A^{2,\pm}_{sut})$ *and* $u \in [s', s'+\delta]$ *for some* $\delta > 0$. *Then*

$$\frac{\partial^+ A^{2,\pm}_{ss't}}{\partial s'}x = A^{+,\pm}_{s't}x.$$

Proof: Let us prove the 1). (The other equalities are proved similarly.) If condition (8) holds then there exists double limit:

$$\lim -\frac{1}{k}\left(\frac{T_{(s,h)\pm,t-kt'} - I}{h} - \frac{T_{(s,h)\pm,tt'} - I}{h}\right)x$$

$$= -\lim\frac{1}{hk}(T_{(s,h)\pm,(t,k)^-} - I)(T_{(s,h)\pm,tt'} - I)x - \lim\frac{1}{hk}(T_{(s,h)\pm,(t,k)^-} - I)x = A^{\pm}_{st}x.$$

Moreover, there exist inner limits

$$\lim\frac{T_{(s,h)\pm,t-kt'} - I}{h}x = A^{1,\pm}_{s,t-k,t'}x \text{ and } \lim\frac{T_{(s,h)\pm,tt'} - I}{h}x = A^{1,\pm}_{stt'}x$$

for $0 < k < \delta$. So, the repeated limit exists and equals the double limit:

$$\lim -\frac{1}{k}(A^{1,\pm}_{s,t-k,t'} - A^{1,\pm}_{s,t,t'})x = -A^{\pm\,-}_{st}x. \qquad \square$$

Remark 4: The following conditions are sufficient for (8).

(D) $a)$ The function $T_{us',tt'}x$ is continuously differentiable in $u \in [s-\delta, s+\delta]$ for some $\delta > 0$, and

$$A^{1,+}_{utt'} = A^{1,-}_{utt'}.$$

$b)$ For any $u \in [s-\delta, s+\delta]$,

$$A^1_{utt'} T_{us',tt'}x \in D(A^{\pm\,-}_{st}).$$

$c)$ There exists $c > 0$ such that

$$\frac{1}{hk} \| \square T^{\pm\,\pm}_{s,t,h,k} A^1_{utt'} T_{us',tt'}x \| \le c, \quad 0 < h < \delta, \ 0 < k < \delta.$$

Indeed, if (D) $a)$ holds, then

$$(T_{(s,h)^-,tt'} - I)x = \int_{s-h}^{s} A^1_{utt'} T_{us',tt'}x du.$$

Therefore, by (D) $b)$ and (D) $c)$ in view of the existence of the integrated majorant and the equality,

$$\lim \frac{1}{hk}(T_{(s,h)^-,(t,k)^-} - I) A^1_{utt'} T_{us',tt'}x = A^{-\,-}_{st} A^1_{utt'} T_{us',tt'}x,$$

we have

$$\lim \frac{1}{hk}(T_{(s,h)^-,(t,k)^-} - I)(T_{(s,h)^{\pm},tt'} - I)x$$

$$= \lim \int_{s-h}^{s} \frac{1}{hk}(T_{(s,h)^-,(t,k)^-} - I) A^1_{utt'} T_{us',tt'}x du$$

$$= \lim \int_{s-\delta}^{s} 1\{\delta - h \le u \le s\} \frac{1}{hk}(T_{(s,h)^-,(t,k)^-} - I) A^1_{utt'} T_{us',tt'}x du = 0.$$

Sufficient conditions for (9) can be formulated in a similar way.

Remark 5: Let $A^{i,+} = A^{i,-}$, $i = 1,2$ and let families of operators $\{A^1_{stt'}, (s,t,t') \in \mathbb{R}^3_+\}$ and $\{A^2_{ss't}(s,s',t) \in \mathbb{R}^3_+\}$ be continuously differentiable in (t,t') and (s,s') respectively on the set

$$X^* = \bigcap_{s,s',t,t'} [D(A^1_{stt'}) \cap D(A^2_{ss't}) \cap D(A_{st})].$$

Then one can write equalities 1) through 4) of Lemma 6 in the form:

$$A^1_{stt'} = \int_t^{t'} A_{sv} dv, \quad A^2_{ss't} = \int_s^{s'} A_{ut} du.$$

Theorem 4: 1) *Let the following conditions hold.*

(E_1) *a)* $T_{us',tt'}x \in D(A^{1,-}_{stt'}) \cap D(A^{2,-}_{us't}) \cap D(A_{st})$ *for any* $u \in [s-\delta, s]$ *and some* $\delta > 0$.

b) The operator $A^{2,-}_{ss't}$ *is closed.*

c) There exists the limit, $\lim \frac{1}{h} A^{2,-}_{ss't} (\Delta^1 T^-_{h,s,t,t'}) T_{ss',tt'}x$.

d) $\lim \frac{1}{h}[(-A^{2,-}_{s-hs't} + A^{2,-}_{ss't})(\Delta^1 T^-_{h,s,t,t'}) T_{ss',tt'}x] = 0$.

Then

$$\frac{\partial^{2,--} T_{ss',tt'}}{\partial t \partial s} x = A^{-\,-}_{st} T_{ss',tt'}x + A^{1,-}_{stt'} A^{2,-}_{ss't} T_{ss',tt'}x.$$

2) *Let the following conditions hold.*

(E_2) *a)* $T_{sv,tt'}x \in D(A^{2,-}_{ss't}) \cap D(A^{1,-}_{svt'}) \cap D(A^{-}_{st}{}^{-})$ *for any* $v \in [t-\delta, t]$ *and some* $\delta > 0$.

b) The operator $A^{1,-}_{stt'}$ *is closed.*

c) There exists the limit, $\lim \frac{1}{h} A^{1,-}_{stt'}(\Delta^2 T^{-}_{k,s,s',t}) T_{ss',tt'}x$.

d) $\lim \frac{1}{h}[(-A^{1,-}_{s,t-k,t'} + A^{1,-}_{stt'})(\Delta^2 T^{-}_{k,s,s',t} T_{ss',tt'}x)] = 0$.

Then

$$\frac{\partial^{2,--} T_{ss',tt'}}{\partial s \partial t} x = A^{-}_{st}{}^{-} T_{ss',tt'}x + A^{2,-}_{ss't} A^{1,-}_{stt'} T_{ss',tt'}x.$$

Remark 6: The following conditions are sufficient for (E_1) *d*).

(E_3) *a*) The function $A^{2,-}_{us't}$ is continuous differentiable in $u \in [s-\delta, s]$ for some $\delta > 0$.
b) There exists $c > 0$ such that, for all $u \in [s-\delta, s]$,

$$\| A^{-}_{ut}{}^{-}\Big(\frac{1}{h}\Delta^1 T^{-}_{h,s,t,t'}\Big) T_{ss',tt'}x \| \leq c.$$

Indeed, in that case,

$$\lim \| \frac{1}{h}[(-A^{2,-}_{s-hs't} + A^{2,-}_{ss't})(\Delta^1 T^{-}_{hs,tt'} T_{ss',tt'}x)] \|$$

$$\leq \lim \int_{s-h}^{s} \| A^{-}_{ut}{}^{-}\Big(\frac{1}{h}\Delta^1 T^{-}_{hs,tt'}\Big) T_{ss',tt'}x \| \, du = 0.$$

The conditions sufficient for (E_2) can be formulated similarly.

Remark 7: Let us assume that there exists the derivative

$$\frac{\partial^{2,--} T_{ss',tt'}}{\partial t \partial s} x,$$

and that conditions (E_1) *a*), (E_1) *b*) and (E_1) *d*) hold. Then, obviously, condition (E_1) *c*) holds and statement 1) of Theorem 4 is true.

Remark 8: Let $B(t,s): R^2_+ \to X$ be a twice continuously differentiable function on some $D \subset \mathbb{R}^2_+$. Then in the usual way, using corresponding results for the functions from $\mathbb{R}^2_+$ to $\mathbb{C}$ and the Hahn-Banach theorem, we obtain for any $x \in D$ that $(\partial^2 B/\partial s \partial t)x = (\partial^2 B/\partial t \partial s)x$. A similar result is true for one-sided derivatives. Thus, if operators $A^{-}_{st}{}^{-} T_{ss',tt'}$, $A^{2,-}_{ss't} A^{1,-}_{stt'} T_{ss',tt'}$ and $A^{1,-}_{stt'} A^{2,-}_{ss't} T_{ss',tt'}$ are continuous as functions of $(s,s',t,t') \in \mathbb{R}^4_+$ on the $\mathbb{R}^4_+$, then

$$\frac{\partial^{2,--} T}{\partial s \partial t} x = \frac{\partial^{2,--} T}{\partial t \partial s} x$$

and

$$A^{2,-}_{ss't} A^{1,-}_{stt'} T_{ss',tt'}x = A^{1,-}_{stt'} A^{2,-}_{ss't} T_{ss',tt'}x.$$

Proof: We prove only the statement 1) of Theorem 4. Let condition (E_1) *a*) be satisfied. Then from Lemma 5,

$$\frac{\partial^{2,--} T_{ss',tt'}}{\partial s \partial t} x = \lim \frac{1}{h}\Big(\frac{\partial^{-} T_{s-hs,tt'}}{\partial t} - \frac{\partial^{-} T_{ss',tt'}}{\partial t}\Big)x$$

$$= \lim \frac{1}{h}[-A_{s-hs't} T_{s-hs',tt'}x + A^{2,-}_{ss't} T_{ss',tt'}x]$$

$$= \lim[\frac{1}{h}(-A^{2,-}_{s-hs't} + A^{2,-}_{ss't})(\Delta^1 T^-_{h,s,t,t'} - I)T_{ss',tt'}x + \frac{1}{h}(-A^{2,-}_{s-h,s't} + A^{2,-}_{ss't})T_{ss',tt'}x$$

$$-\frac{1}{h}A^{2,-}_{ss't}(\Delta^1 T^-_{h,s,t,t'}T_{ss',tt'}x)] = \lim(S^h_1 x + S^h_2 x + S^h_3)x.$$

(E_1) $a)$ implies that $\lim S^h_2 x = A^{-\,-}_{st} T_{ss',tt'}x$, and

$$\lim\frac{1}{h}(\Delta^1 T^-_{h,s,t,t'})T_{ss',tt'}x = A^{1,-}_{stt'} T_{ss',tt'}x.$$

(E_1) $b)$ and (E_1) $c)$ imply that $S^h_3 x = A^{2-}_{ss't}A^{1,-}_{stt'} T_{ss',tt'}x$. Condition (E_1) $d)$ ensures the equality $\lim S^h_1 x = 0$. Hence, the proof follows. □

The following statements are proved analogously to the proof of Theorem 4.

Theorem 5: 1) *Let the following conditions hold.*

(E_4) $a)$ $T_{su,tt'}x \in D(A^2_{vtt'}) \cap D(A^{+\,-}_{s't})$ *for* $u, v \in [s, s+\delta]$ *for some* $\delta > 0$; $x \in D(A^{1,+}_{s'tt'})$.

$b)$ *The operator* $A^{2,-}_{ss't}$ *is closed.*

$c)$ *There exists the limit,* $\lim\frac{1}{h}A^{2,-}_{ss't} T_{ss',tt'}\Delta^1 T^+_{h,s',t,t'}x$.

$d)$ $\lim\frac{1}{h}(-A^{2,-}_{s,s'+h,t} + A^{2,-}_{ss,t})T_{ss',tt'}\Delta^1 T^+_{h,s',t,t'}x = 0$.

Then

$$\frac{\partial^{2,+-}T_{ss',tt'}}{\partial s'\partial t}x = A^{+\,-}_{s't} T_{ss'tt'}x - A^{2,-}_{ss't} T_{ss',tt'}A^{1,+}_{s'tt'}x.$$

2) *Let the following conditions hold.*

(E_5) $a)$ *Condition* 1) *of Lemma* 6 *holds.*

$b)$ $T_{ss',tt'}A^{1,+}_{stt'}x \in D(A^{2,-}_{ss't})$.

$c)$ *There exists* $C > 0$ *such that* $\| T_{ss',vt'} \| \leq C$ *while* $v \in [t-\delta, t]$ *for some* $\delta > 0$.

Then

$$\frac{\partial^{2,-+}T_{ss',tt'}}{\partial t\partial s'}x = -T_{ss',tt'}A^{+\,-}_{s't}x - A^{2,-}_{ss't} T_{ss',tt'}A^{1,+}_{s'tt'}x.$$

3) *Let the following conditions hold.*

(E_6) $a)$ $T_{ss',tu} \in D(A^{1,-}_{ss'v}) \cap D(A^{-\,+}_{st'})$ *with* $u, v \in [t', t'+\delta]$ *for some* $\delta > 0$; $x \in D(A^{2,+}_{ss't'})$.

$b)$ *The operator* $A^{1,-}_{ss't'}$ *is closed.*

$c)$ *There exists the limit,* $\lim\frac{1}{k}A^{1,-}_{stt'} T_{ss',tt'}\Delta^{2,+}T_{k,s,s',t'}x$.

$d)$ $\lim\frac{1}{k}(A^{1,-}_{s,t,t'+k} - A^{1,-}_{stt'})T_{ss',tt'}\Delta^{2,+}T_{k,s,s',t'}x = 0$.

Then

$$\frac{\partial^{2,+-}T_{ss',tt'}}{\partial t'\partial s}x = -A^{-\,+}_{st'}T_{ss',tt'}x - A^{1,-}_{stt'} T_{ss',tt'}A^{2,+}_{ss't'}x.$$

4) *Let the following conditions hold.*

(E_7) $a)$ *Condition* 3) *of Lemma* 6 *hold.*

$b)$ $T_{ss',tt'}A^{2,+}_{ss't}x \in D(A^{1,-}_{stt'})$.

$c)$ *There exists* $C > 0$ *such that* $\| T_{us'tt} \| \leq C$ *while* $u \in [s-\delta, s]$ *for some* $\delta > 0$.

Then

$$\frac{\partial^{2,-+} T_{ss',tt'}}{\partial s \partial t'} x = -T_{ss',tt'} A^{-,+}_{st'} x - A^{1,-}_{stt'} T_{ss',tt'} A^{2,+}_{ss't'} x.$$

5) *Let the following conditions hold.*

(E_8) *a) Condition* 2) *of Lemma* 6 *holds.*
b) $A^{1,+}_{s'tt'} x \in D(A^{2,+}_{ss't'})$.
c) There exists $C > 0$ *such that* $\| T_{ss,'tv} \| \leq C$ *while* $v \in [t', t' + \delta]$ *for some* $\delta > 0$.

Then

$$\frac{\partial^{2,++} T_{ss',tt'}}{\partial t' \partial s'} x = T_{ss',tt'} A^{+,+}_{s't'} x + T_{ss',tt'} A^{2,+}_{ss't'} A^{1,+}_{s'tt'} x.$$

6) *Let the following conditions hold.*

(E_9) *a) Condition* 4) *of Lemma* 6 *holds.*
b) $A^{2,+}_{ss't'} x \in D(A^{1,+}_{stt'})$.
c) There exists $C > 0$ *such that* $\| T_{su,'tt'} \| \leq C$ *while* $u \in [s', s' + \delta]$ *for some* $\delta > 0$.

Then

$$\frac{\partial^{2,++} T_{ss',tt'}}{\partial s' \partial t'} x = T_{ss',tt'} A^{+,+}_{s't'} x + T_{ss',tt'} A^{1,+}_{s'tt'} A^{2,+}_{ss't'} x.$$

4. Markov Fields and Semigroups

Let (Ω, F, P) be a complete probability space; let $(E, \mathcal{E})$ be a measurable space; let $X = \{X_{\overline{t}}, \overline{t} \in \mathbb{R}^2\}$ be a stochastic field with the values in E that is constant on the set $(\mathbb{R}^2 \backslash R^2_+) \cup \{[0,\infty) \times \{0\}\} \cup \{\{0\} \times [0,\infty)\}$. Put $F_{\overline{t}} = \sigma\{x_{\overline{s}}, \overline{s} \leq \overline{t}\} \vee N$, $F^i_{t_i} = \bigvee_{t_j \geq 0} F_{\overline{t}}$ and $F^*_{\overline{t}} = F^1_{t_1} \vee F^2_{t_2}$ where N is the class of P-zero sets of F.

Definition 7: The field X is called an $*$-*Markov field* if for any $\overline{s} \leq \overline{t}$ and $B \in \mathcal{E}$

$$P\{X_{\overline{t}} \in B / F^*_s\} = P\{X_{\overline{t}} \in B / X_s, X_{s_1 t_2}, X_{t_1 s_2}\} \qquad [5,11].$$

Definition 8: The function $P\{\overline{s}, \overline{t}, x, y, z, B\}$, with $\overline{s} \in R^2_+, \overline{t} \in \mathbb{R}^2_+$, $x, y, z \in E$ and $B \in \mathcal{E}$ is called *transition function on* $(E, \mathcal{E})$ if

1) it is a probability measure on $(E, \mathcal{E})$ when $x, y, z \in E$ are fixed;
2) it is an $\mathcal{E}^3$-measurable function when $B \in \mathcal{E}$ is fixed;
3) for any $x, y, z, \xi \in E$, $B \in \mathcal{E}$, and $\overline{t} < \overline{u}$

$$P\{\overline{s}, (u_1, t_2), x, y, z, B\} = \int P\{\overline{s}, \overline{t}, x, \xi, z, d\eta\} P\{(t_1, s_2), (u_1, t_2), \xi, \eta, z, B\}$$

and

$$P\{\overline{s}, (t_1, u_2), x, y, z, B\} = \int P\{\overline{s}, \overline{t}, x, \xi, z, d\eta\} P\{(s_1, t_2), (t_1, u_2), \xi, y, \eta, B\} \qquad [5,11].$$

Definition 9: X is called an $*$-*Markov field with transition function* P, if for any $m \geq 1$ and $n \geq 1$, with $B_{ij} \in \mathcal{E}$ for $i = \overline{1,m}$ and for $j = \overline{1,n}$, with $(s_i, t_j) \in \mathbb{R}^2_+$, we have

$$P\{\cap_{i=1}^m \cap_{j=1}^n (X_{s_i t_j} \in B_{ij})\} = \int \ldots \int \prod_{i=1}^m \prod_{j=1}^n I_{B_{ij}}(x_{ij}) \times P\{(s_{i-1}, t_{j-1}), (s_i, t_j), x_{i-1 j-1}, x_{i-1 j}, x_{i j-1}, dx_{ij}\}.$$

It follows from [11] that any $*$-Markov field with transition function is a Markov field.

Now we define the families of functions $\{P^{1t}\{s,y,s_1,B\},\ 0 \le s < s_1, B \in \mathcal{E}, t \ge 0, s_1 > 0\}$ and $\{P^{2s}\{t,y,t_1,B\}, 0 \le t < t_1, B \in \mathcal{E}, s \ge 0, t_1 > 0\}$ of the following kind:

$$P^{10}\{s,y,s_1,B\} = P^{20}\{t,y,t_1,B\} = I_B(y), \tag{10}$$

$$P^{1t}\{s,y,s_1,B\} = P\{(s,0),(s_1,t),x,y,x,B\}, \tag{11}$$

and

$$P^{2s}\{t,y,t_1,B\} = P\{(0,t),(s,t_1),x,x,y,B] \tag{12}$$

(under the assumption that the right-hand sides do not depend on x). In this case, the collection $(P^{10}, P^{20}, P^{1t}, P^{2s})$ is called an $*$-transition function on $(E,\mathcal{E})$.

The following equalities are true for any $*$-Markov field with a transition function:

$$P\{X_{s+u,t} \in B/F_s^1\} = P^{1t}\{s, X_{s,t}, s+u, B\} \text{ a.s.}$$

and

$$P\{X_{s,t+v} \in B/F_s^2\} = P^{2s}\{t, X_{s,t}, t+v, B\} \text{ a.s.,}$$

for any $s,t,u,v \ge 0$ and $B \in \mathcal{E}$. Let $f: E \to R$ be a bounded measurable function. Set

$$T^1_{s,s_1,t} f(x) = \int f(y) P^{1t}\{s,x,s_1,dy\}$$

and

$$T^2_{t,t_1,s} f(x) = \int f(y) P^{2s}\{t,x,t_1,dy\}.$$

Then it follows from (10) through (12) that, for $0 \le s < s_1 < s_2$ and $0 \le t < t_1 < t_2$,

$$T^1_{s_1,s_2,t} f(x) = T^1_{ss_1,t} T^1_{s_1,s_2,t} f(x),$$

$$T^2_{t,t_2,s} f(x) = T^2_{t,t_1,s} T^2_{t_1,t_2,s} f(x).$$

Now, consider the case of the homogeneous $*$-Markov field x, for which

$$P\{\bar{s}, \bar{t}, x,x,x,B\} = P\{0,(t_1-s_1, t_2-s_2),x,x,x,B\} =: \widetilde{P}(t_1-s_1, t_2-s_2, x, B)$$

while

$$P^{1t}\{s,y,s_1,B\} = \widetilde{P}(s_1-s,t,x,B) \text{ and } P^{2s}\{t,y,t_1,B\} = \widetilde{P}(s,t_1-t,x,B).$$

Then, $T^1_{s,s_1,t} f(x) = \widetilde{T}^1_{s_1-s,t} f(x)$ and $T^2_{s,t_1,t} f(x) = \widetilde{T}^2_{s,t_1-t} f(x)$, where $\widetilde{T}^1_{s,t} f(x) = \widetilde{T}^2_{s,t} f(x)$. We denote their common value as $T_{s,t} f(x)$. Then $T_{s,t} f(x)$ is a coordinate-wise contractive semigroup on the space $B(E)$ of bounded measurable functions $f: E \to R$. Further, we consider only homogeneous fields.

Definition 10: Transition function $\widetilde{P}(s,t;x,B)$ is said to be *continuous in probability* (P-continuous), if for any $\epsilon > 0$, $\lim_{s \vee t \to 0} \widetilde{P}(s,t,x,U_\epsilon(x)) = 1$, where $U_\epsilon(x) \in \mathcal{E}$ is any ϵ-neighborhood of x. $*$-Markov field with a P-continuous transition function will be called a P-continuous field. The index ˜ will be omitted.

Let us denote $C_B(E) \subset B(E)$ as the space of continuous bounded functions on E.

Lemma 7: *The following conditions are equivalent.*

(F_1) *The field X is P-continuous.*

(F_2) $\lim_{u \to 0} P(u, v_0, x, U_\epsilon(x)) = \lim_{v \to 0} P(u_0, v, x, U_\epsilon(x)) = 1$, *for any $u_0, v_0, \epsilon > 0$ and $x \in E$.*

(F_3) $\lim_{u \vee v \to 0} T_{u,v} f(x) = f(x)$ *for any $f \in C_B(E)$ and $x \in E$.*

(F_4) $\quad \lim_{u \to 0} T_{u,v_0} f(x) = \lim_{v \to 0} T_{u_0,v} f(x) = f(x)$ *for any* $f \in C_B(E)$ *and* $x \in E$.

Proof: Let us show (F_1) implies (F_3). If $\lim_{u \vee v \to 0} P(u,v,x,U_\epsilon(x)) = 1$, then

$$\lim_{u \vee v \to 0} | T_{uv} f(x) - fx) | = \lim_{u \vee v \to 0} \left| \int_E (f(y) - f(x)) P(u,v,x,dy) \right|$$

$$\leq \lim_{u \vee v \to 0} \| f \| P(u,v,x,E \backslash U_\epsilon(x)) + \sup_{z \in U_\epsilon(x)} | f(z) - f(x) | = \sup_{z \in U_\epsilon(x)} | f(z) - f(x) |.$$

Since $\epsilon > 0$ was arbitrary, we obtain (F_3). The implication, $(F_3) \Rightarrow (F_4)$ follows from Lemma 1. Let us show that (F_4) implies (F_2). Consider the function $f_\epsilon \in C_B(E)$ such that $f_\epsilon(y) \geq \alpha > 0$ for $y \in E \backslash U_\epsilon(x)$ and $f_\epsilon(x) = 0$. Then from (F_4), we have

$$\lim_{u \to 0} P(u, v_0, x, E \backslash U_\epsilon(x)) \leq \alpha^{-1} \lim_{u \to 0} \int_{E \backslash U_\epsilon(x)} f_\epsilon(y) P(u, v_0, x, dy)$$

$$\leq \alpha^{-1} \lim_{u \to 0} T_{uv_0} f_\epsilon(x) = \alpha^{-1} f_\epsilon(x) = 0,$$

i.e., (F_2) holds.

The implication, $(F_2) \Rightarrow (F_4)$, has a proof similar to the proof that $(F_1) \Rightarrow (F_3)$. The implication $(F_4) \Rightarrow (F_3)$ follows from Lemma 1. The implication, $(F_3) \Rightarrow (F_1)$ has a proof similar to the proof that $(F_4) \Rightarrow (F_2)$.

Definition 11: Transition function $P(s,t,x,B)$ is said to be *Feller*, if for any $\overline{t} \in \mathbb{R}^2_+$, $T_{\overline{t}}(C_B) \subset C_B$. The corresponding $*$-Markov field will be called a *Feller field.* (Note that if E is a compact set, then $C_B(E) = C(E)$, where $C(E)$ is the space of continuous functions.)

Theorem 6: 1) *Let* X *be a P-continuous field. Then* $T_{\overline{t}} = T_{1,t_1t_2} = T_{t_1t_2,1}$ *on* $C_B(E)$.

2) *Let* E *be a compact set and* X *be a P-continuous Feller field. Then* $T_{\overline{t}}$ *is a* C_0*-semigroup on* $C(E)$.

Proof: 1) According to Lemma 7, for any $x \in E$ and $f \in C_B(E)$, with $u_0, v_0 > 0$,

$$\lim_{u \downarrow 0} T_{u,v_0} f(x) = \lim_{u \downarrow 0} T_{u_0,v} f(x) = f(x).$$

Therefore from the boundedness of f and Lebesgue convergence theorem,

$$\lim_{t_1 \downarrow s_1} T_{t_1,s_2} f(x) = \lim_{u \downarrow 0} T_{\overline{s}} T_{u,s_2} f(x) = T_{\overline{s}} f(x),$$

$$\lim_{t_2 \downarrow s_2} T_{s_1,t_2} f(x) = \lim_{v \downarrow 0} T_{\overline{s}} T_{s_1,v} f(x) = T_{\overline{s}} f(x).$$

Now, let $u_n = \frac{p_n}{q_n} \in Q^+$, $\frac{p_n}{q_n} \downarrow s_1$ as $n \to \infty$. Then for any $x \in E$ and $f \in C(E)$,

$$T_{\overline{s}} f(x) = \lim_{n \to \infty} T_{u_n,s_2} f(x) = \lim_{n \to \infty} T_{1,u_n s_2} f(x) = T_{1,s_1 s_2} f(x).$$

Therefore, $T_{\overline{s}} = T_{1,s_1,s_2}$ on $C_B(E)$. Similarly, $T_{\overline{s}} = T_{s_1 s_2,1}$ on $C_B(E)$.

2) Taking into account statement 1), we obtain that $\widetilde{X}_t := X_{1,t}$ is a homogeneous Markov P-continuous, Feller process. According to famous results for Feller processes, the semigroup $\widetilde{T}_t := T_{1,t}$, $t \geq 0$, is continuous on $C(E)$. Therefore,

$$\lim_{\overline{t} \to \overline{s}} \| T_{t_1,t_2} - T_{s_1,s_2} \| = \lim_{\overline{t} \to \overline{s}} \| T_{1,t_1t_2} f - T_{1,s_1s_2} f \| = 0, \text{ for } f \in C(E),$$

i.e., T_{t_1,t_2} is a C_0-semigroup. □

The Hille-Yosida theorem for Feller fields on compact sets is similar to the one-parameter case.

Theorem 7: *An operator A with domain $D(A)$ that is dense in $C(E)$ generates a P-continuous Feller field on the compact set E if and only if the following conditions are satisfied.*

(G) a) *There exists $\lambda > 0$ such that $(\lambda I - A)(D(A)) = C(E)$.*
b) *If $f \in D(A)$ and $f(x_0) \geq f(x)$, then $Af(x) \leq 0$.*

Proof: If A generates a P-continuous Feller field, then from Theorem 1, A is a generator of a Feller one-parameter semigroup, $\widetilde{T}_t = T_{1,t}$, and necessity follows. If assumption (G) is satisfied, then from Theorem 1, there exists a semigroup $\widetilde{T}_t$, $t \geq 0$, such that $\widetilde{T}_t(C(E)) \subset C(E)$ and $\widetilde{T}_t f \rightarrow f$ as $t \rightarrow 0$ for any $f \in C(E)$ [2, p. 167]. Let $T_{s,t} = \widetilde{T}_{st}$. Then $T_{s,t}(C(E)) \subset C(E)$. Since $T_{s,t}f(x)$, for any s,t and x fixed, is a linear functional on $C(E)$, then there exists a measure $P(s,t,x,B)$ on $\mathcal{E}$ such that $T_{s,t}f(x) = \int f(y)P(s,t,x,B)$. Moreover, $P(s,t,x,E) = 1$ and P is a transition function by the semigroup property of $T_{s,t}$. Now, as with the proof of Lemma 7, choose $f_\epsilon(y) \geq \alpha > 0$ with $y \in E \backslash U_\epsilon(x)$, $f(x) = 0$ and $f \in C(E)$. Then

$$P(s,t,x,\overline{U_\epsilon(x)}) \leq \alpha^{-1}T_{s,t}f(x) = \alpha^{-1}T_{1,st}f(x) \rightarrow \alpha^{-1}f(x) = 0 \text{ as } s \vee t \downarrow 0,$$

i.e., the transition function is P-continuous. The construction of an $*$-Markov field with transition function P, under the assumptions of its Feller property and P-continuity, is realized in [6]. □

5. Diffusion Fields and Evolutions

Let $(E, \mathcal{E}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, $\bar{B} = R^n \backslash B$, $\Delta s = s_2 - s_1$, and $\Delta t = t_2 - t_1$.

Definition 12: An $*$-Markov field with transition function,

$$P(s,t,s',t',x,B) := P\{(s,t),(s',t'),x,x,x,B\}$$

with $(s,t),(s',t') \in \mathbb{R}^2_+$, $(s,t) \leq (s',t')$, $x \in \mathbb{R}^n$, and $B \in \mathcal{B}(\mathbb{R}^n)$,

is called a *diffusion field*, if the following conditions are true for any $\epsilon > 0$ uniformly in $x \in K$ where K is any compact set, $K \subset \mathbb{R}^n$.

(H) a)
$$P(s_1,t_1,s_2,t_2,x,\overline{U_\epsilon(x)}) = o(\Delta s \Delta t),$$
$$P(s_1,t,s_2,t',x,\overline{U_\epsilon(x)}) = o(\Delta s)$$
and
$$P(s,t_1,s',t_2,x,\overline{U_\epsilon(x)}) = o(\Delta t).$$

b)
$$\int_{U_\epsilon(x)} (y^i - x^i)P(s_1,t_1,s_2,t_2,x,dy) = b_0^i(s_1,t_1,x)\Delta s\Delta t + o(\Delta s\Delta t),$$
$$\int_{U_\epsilon(x)} (y^i - x^i)P(s_1,t,s_2,t',x,dy) = b_1^i(s_1,t,t',x)\Delta s + o(\Delta s),$$
and
$$\int_{U_\epsilon(x)} (y^i - x^i)P(t_1,s,t_2,s',x,dy) = b_2^i(s,s',t,t_1,x)\Delta t + o(\Delta t).$$

c)
$$\int_{U_\epsilon(x)} (y^i - x^i)(y^j - x^j)P(s_1,t_1,s_2,t_2,x,dy) = a_0^{ij}(s_1,t_1,x)\Delta s\Delta t + o(\Delta s\Delta t),$$

$$\int\limits_{U_\epsilon(x)} (y^i - x^i)(y^j - x^j)P(s_1, t, s_2, t', x, dy) = a_1^{ij}(s_1, t, t', x)\Delta s + o(\Delta s),$$

and
$$\int\limits_{U_\epsilon(x)} (y^i - x^i)(y^j - x^j)P(t_1, s, t_2, s', x, dy) = a_2^{ij}(s, s', t_1, x)\Delta t + o(\Delta t),$$

as
$$\Delta s \to 0 \text{ and } \Delta t \to 0.$$

Here,

$$\{b_0^i, a_0^{ij}, i, j = \overline{1,n}\} \subset C(\mathbb{R}_+^2 \times R^n) \text{ and } \{b_k^i, a_k^{ij}, i, j = \overline{1,n}] \subset C(\mathbb{R}_+^3 \times R^n) \text{ for } k = 1, 2.$$

Remark 9: Different classes of diffusion fields on the plane were considered in [1, 3, 4, 7], similar class of diffusion processes were considered in [9, 10].

Let us introduce the notations,

$$(b, \nabla f) = \sum_{i=1}^n b_i \frac{\partial f}{\partial x_i} \text{ and } (a\nabla, \nabla f) = \sum_{i,j=1}^n a^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

Consider the families of differential operators

$$\mathcal{L}_0(s, t, x)f = \tfrac{1}{2}(a_0 \nabla, \nabla f) + (b_0, \nabla f),$$

$$\mathcal{L}_1(s, t, t', x)f = \tfrac{1}{2}(a_1 \nabla, \nabla f) + (b_1, \nabla f)$$

and
$$\mathcal{L}_2(s, s', t, x)f = \tfrac{1}{2}(a_2 \nabla, \nabla f) + (b_2, \nabla f), \text{ where } f \in C^2(\mathbb{R}^n).$$

Note that the following family of evolutions, $T_{ss'tt'}$, is connected in a natural way with the diffusion field

$$T_{ss'tt'}f(x) = \int\limits_{R^n} f(y)P(s, t, s', t', x, dy),$$

where f is a bounded measurable function.

Denote $C_{fin}^2 = C_{fin}^2(\mathbb{R}^n) \subset C^2(\mathbb{R}^n)$ as the space of functions with compact support.

Theorem 8: *Let the diffusion field X satisfy the condition*

(*I*) *for any compact set $K \subset \mathbb{R}^n$ there exists a compact set $K' \supset K$ such that*

$$P(s_1, t_1, s_2, t_2, x, K) = o(\Delta s \Delta t), \quad P(s_1, t, s_2, t', x, K) = o(\Delta s),$$

and
$$P(s, t_1, s', t_2, x, K) = o(\Delta t) \text{ as } \Delta s \to 0 \text{ and } \Delta t \to 0$$

uniformly in $x \in \mathbb{R}^n \backslash K'$. Then $C_{fin}^2 \subset D(A_{st}^{\pm\,\pm}) \cap D(A_{stt'}^{1,\,\pm}) \cap D(A_{ss't}^{2,\,\pm})$ for any $(s, s', t, t') \subset \mathbb{R}_+^4$ and the following equalities hold on C_{fin}^2: $A_{st}^{\pm\,\pm} = \mathcal{L}_0(s, t)$, $A_{stt'}^{1,\,\pm} = \mathcal{L}^1(s, t, t')$ and $A_{ss't}^{2,\,\pm} = \mathcal{L}^2(s, s', t)$.

Proof: Consider one of the generators, $A_{st}^{+\,+}$, for example. Let $f \in C_{fin}^2$, $f = 0$ if $x \overline{\in} K$. It follows from (*I*) that

$$\sup_{x \overline{\in} K'} \left| \frac{1}{hk}(T_{(s,h)\,+\,(t,k)\,+}f(x) - f(x)) - \mathcal{L}_0(s, t, x)f \right|$$

$$= \sup_{x \overline{\in} K'} (hk)^{-1} \left| \int_K f(y) P(s,t,s+h,t+k,x,dy) \right|$$

$$\leq \| f \| \sup_{x \overline{\in} K'} (hk)^{-1} P(s,t,s+h,t+k,x,K) \to 0, \text{ as } h \vee k \downarrow 0. \tag{13}$$

Furthermore, the functions $f, \frac{\partial f}{\partial x_i}$ and $\frac{\partial^2 f}{\partial x_i \partial x_j}$ are bounded and uniformly continuous on the set $K_\delta = \bigcup_{x \in K'} U_\delta(x)$. For any $\epsilon > 0$ the value $\delta > 0$ can be chosen in such a way that

$$\left| \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \frac{\partial^2 f}{\partial x_i \partial x_j}(y) \right| < \epsilon \text{ if } x, y \in K_\delta, \quad | x - y | < \delta.$$

Now choose $\lambda > 0$ in such a way that all $o(\ )$ in (H) be less then $\epsilon \Delta s \Delta t$ for $\Delta s \vee \Delta t < \lambda$ and for any $x \in K'$. Put the Taylor expansion,

$$f(y) = f(x) + (f', y - x) + \tfrac{1}{2}(y-x)^T f''(y-x) + \alpha \mid y - x^2 \mid,$$

where $| \alpha | = | \alpha(x,y) | \leq \frac{1}{2} n^2 \epsilon$ for $| y - x | < \delta$, into the following estimations

$$\sup_{x \in K'} | (hk)^{-1} (T_{(s,h)} + {}_{(t,k)} + f(x) - f(x)) - \mathcal{L}_0(s,t,x) f |$$

$$\leq \sup_{x \in K'} \left| (hk)^{-1} \left(\int_{U_\delta(x)} f(y) P(s,t,s+h,t+k,x,dy) - f(x) \right) - \mathcal{L}_0(s,t,x) f \right|$$

$$+ \sup_{x \in K'} (hk)^{-1} \int_{U_\delta(x)} | f(y) - f(x) | P(s,t,s+h,t+k,x,dy)$$

$$\leq \sup_{x \in K'} (hk)^{-1} \int_{U_\delta(x)} | \alpha(x,y) | \, | y - x |^2 P(s,t,s+h,t+k,x,dy) + C\epsilon$$

$$\leq \frac{1}{2} n^2 \epsilon \sup_{x \in K'} \left| \sum_{i=1}^{n} a_0^{ii}(s,t,x) \right| + C\epsilon$$

where

$$C = 2 \| f \| + n \max_i \| \frac{\partial f}{\partial x_i} \| + \frac{n^2}{2} \max_{ij} \| \frac{\partial^2 f}{\partial x_i \partial x_j} \|. \tag{14}$$

The proof follows from (13) and (14).

Corollary 1: *Let the functions b_1^i and a_1^{ij}, be continuously differentiable in t and t' and let b_2^i and a_2^{ij} be continuous differentiable in s and s'. Then from Remark 5, we obtain that, on the space C_{fin}^2,*

$$\mathcal{L}_1(s,t,t',x) f = \int_t^{t'} \mathcal{L}_0(s,v,x) f dv_1 \text{ and } \mathcal{L}_2(s,s',t,x) f = \int_s^{s'} \mathcal{L}_0(u,t,x) f du.$$

Furthermore, we consider the condition

(J) *the transition function P has a density*

$$P(s,t,s',t',x,B) = \int_B p(s,t,s',t',x,y) dy, \text{ for } B \in \mathfrak{B}(\mathbb{R}^n).$$

Suppose that the conditions (I) and (J) are satisfied. Consider each operator $\mathcal{L}_k^*$ that is formally adjoint of $\mathcal{L}_k$:

$$\mathcal{L}_k^* = \frac{1}{2}\nabla\ \nabla(a_k f) - \nabla(b_k f),\ \text{for } k = 0,1,2$$

where

$$\nabla\ \nabla(af) = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial y_i \partial y_j}(a^{ij} f)$$

and

$$\nabla(bf) = \sum_{i=1}^{n} \frac{\partial}{\partial y_i}(b^i f).$$

Let $C > 0$, $\varphi > 0$, $C = C(s,t,s',t',y) \in C(\mathbb{R}_+^4 \times \mathbb{R}^n)$ and $\varphi = \varphi(x) \to 0$ while $|x| \to \infty$. We say that the function g satisfies the (C,φ) condition, if $|g| \le C\varphi$. We say that the set E of functions satisfies (C,φ)-condition if every element of it satisfies this condition. Let also,

$$T = \{f \in C^2(\mathbb{R}^n): \max(|f|, \left|\frac{\partial f}{\partial x_i}\right|, \left|\frac{\partial^2 f}{\partial x_i \partial x_j}\right|) \le \varphi(x)\}.$$

Theorem 9: 1) *Let* $\{p, \frac{\partial p}{\partial s}, \frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial y_i \partial y_i}\} \subset C(\mathbb{R}_+^4 \times \mathbb{R}^{2n})$. *Then* $\frac{\partial p}{\partial s'} = \mathcal{L}_{1(y)}^* p$ *and* $\frac{\partial p}{\partial t'} = \mathcal{L}_{2(y)}^* p$, *where index* y *means that the operator* $\mathcal{L}_1^*$ *is applied to* p *as a function of* y *under fixed* x.

2) *Let* $A^{i,\pm} = \mathcal{L}_i$, $i = 1,2$ *on the set* T; *let*

$$\left\{p, \frac{\partial p}{\partial s}, \frac{\partial p}{\partial t}, \frac{\partial^2 p}{\partial x_i \partial x_i}\right\} \subset C(\mathbb{R}_+^4 \times \mathbb{R}^{2n})$$

and let $p, \frac{\partial p}{\partial x_i}$ *and* $\frac{\partial^2 p}{\partial x_i \partial x_j}$ *satisfy the* (C,φ)*-condition. Then* $\frac{\partial p}{\partial s} = -\mathcal{L}_{1(x)} p$ *and* $\frac{\partial p}{\partial t} = -\mathcal{L}_{2(x)} p$.

Proof: The scheme of the proof is the same in both cases, so we prove only 2). Let $f \in C_{fin}^{(2)}$, with

$$T_{ss',tt'} f(x) = \int_{R^n} p(s,t,s',t',x,y) f(y) dy. \tag{15}$$

The (C,φ)-condition permits us to differentiate (15) under the integral sign:

$$\frac{\partial T}{\partial s} = \int_{R^n} \frac{\partial p}{\partial s} f(y) dy,\ \frac{\partial T}{\partial x_i} = \int_{R^n} \frac{\partial p}{\partial x_i} f(y) dy,\ \frac{\partial^2 T}{\partial x_i \partial x_j} = \int_{R^n} \frac{\partial^2 p}{\partial x_i \partial x_j} f(y) dy.$$

The (C,φ)-condition also ensures that $T_{ss',tt'} f \in T$, whence

$$A_{st'}^1 T_{ss'tt'} f(x) = \int_{R^n} \mathcal{L}_1(s,t,t',x) p(s,t,s',t',x,y) f(y) dy.$$

Since $f \in C_{fin}^2$ is arbitrary, we obtain that $\frac{\partial p}{\partial s} = -\mathcal{L}_1(s,t,t',x)p$. (The second equality in 2) is proved the same way.)

Denote $D_x^k g$ as the family of partial derivatives of g in x of order k.

Theorem 10: 1) *Suppose the following conditions are satisfied.*

(K_1) *a)* $\{a_l^{ij}, p\} \subset C^{(4)}(\mathbb{R}_+^4 \times \mathbb{R}^n)$,

$\{a_0^{ij}, b_l^i\} \subset C^{(2)}(\mathbb{R}_+^4 \times \mathbb{R}^n)$ *and* $b_0^i \in C^{(1)}(\mathbb{R}_+^4 \times \mathbb{R}^n)$ *for* $l = 1,2$ *and* $i,j = \overline{1,n}$.

b) The set

$$\left\{\frac{\partial p}{\partial s'},\frac{\partial p}{\partial t'},\frac{\partial^2 p}{\partial s'\partial t'},\frac{\partial^2 p}{\partial t'\partial s'}\right\}\subset C(\mathbb{R}^4_+\times\mathbb{R}^{2n})$$

and satisfies (C,φ)-condition.
Then

$$\frac{\partial^2 p}{\partial s'\partial t'}=\mathfrak{L}^*_{0(y)}(s',t')p+\mathfrak{L}^*_{1(y)}(s',t,t')\mathfrak{L}^*_{2(y)}(s,s',t')p. \tag{16}$$

2) *Suppose the following conditions are satisfied.*

(K_2) a) $A^i=\mathfrak{L}^i$ *on the set* T *for* $i=0,1,2.$
b) $\left\{\frac{\partial z}{\partial t},\frac{\partial z}{\partial t'},\frac{\partial u}{\partial s},\frac{\partial u}{\partial s'},\ z=a_1^{ij},b_1^i,u=a_2^{ij},b_2^i\right\}\subset C(\mathbb{R}^4_+\times\mathbb{R}^{2n}).$
c) *The set* $\left\{p,\frac{\partial p}{\partial s},\frac{\partial p}{\partial t},\frac{\partial^2 p}{\partial s\partial t},\frac{\partial^2 p}{\partial t\partial s}\right\}\subset C(\mathbb{R}^4_+\times\mathbb{R}^{2n})$ *and satisfies (C,φ)-condition.*
d) *The set* $\left\{D_x^k p,\ k=\overline{0,4}\right\}\subset C(\mathbb{R}^4_+\times\mathbb{R}^{2n})$ *and satisfies (C,φ)-condition.*
e) *The set*

$$\left\{D_x^k\left(a_0^{ij}\frac{\partial^2 p}{\partial x_i\partial x_j}\right),D_x^k\left(b_0^i\frac{\partial p}{\partial x_i}\right),\ D_x^k(a_0^{ij}),D_x^k(b_0^i),\ k=0,1,2\right\}\subset C(\mathbb{R}^4_+\times\mathbb{R}^{2n})$$

and satisfies (C,φ)-condition.
Then

$$\begin{aligned}\frac{\partial^2 p}{\partial s\partial t}&=\mathfrak{L}_{0(x)}(s,t)p+\mathfrak{L}_{1(x)}(s,t,t')\mathfrak{L}_{2(x)}(s,s',t)p\\&=\mathfrak{L}_{0(x)}(s,t)p+\mathfrak{L}_{2(x)}(s,s',t)\mathfrak{L}_{1(x)}(s,t,t')p.\end{aligned} \tag{17}$$

3) *Suppose the following conditions are satisfied.*

(K_3) a) $A^i=\mathfrak{L}^i$ *on the set* T *for* $i=0,1.$
b) *The set* $\left\{p,\frac{\partial p}{\partial s},\frac{\partial p}{\partial t'},\frac{\partial^2 p}{\partial t'\partial s},\frac{\partial^2 p}{\partial s\partial t'}\right\}\subset C(\mathbb{R}^4_+\times\mathbb{R}^{2n})$ *and satisfies (C,φ)-condition.*
c) $\{D_x^k p,D_y^k p,D_x^k D_y^l(a_2^{ij}p),D_x^k D_y^m(b_2^i p),\ k=\overline{0,2},\ l=2,\ m=1,$

$$D_x^k(a_0^{ij}),D_x^k(b_0^i\}\subset C(\mathbb{R}^4_+\times\mathbb{R}^{2n})$$

and satisfies (C,φ)-condition.
Then

$$\frac{\partial^2 p}{\partial t'\partial s}=-\mathfrak{L}^*_{0(y)}(s,t')p-\mathfrak{L}_{1(x)}(s,t,t')\mathfrak{L}^*_2(y)(s,s',t')p. \tag{18}$$

If, in addition, (K_2) b), d) and e) hold, then

$$\frac{\partial^2 p}{\partial t'\partial s}=-\mathfrak{L}_{0(x)}(s,t')p-\mathfrak{L}_{1(x)}(s',t,t')\mathfrak{L}^*_{2(y)}(s,s',t')p. \tag{18'}$$

4) *Under conditions (K_3) where we change $i=0,1$ to $i=0,2$ in (K_3) a), if we change $\frac{\partial}{\partial s}$ on $\frac{\partial}{\partial s'}$ and $\frac{\partial}{\partial t'}$ on $\frac{\partial}{\partial t}$ in (K_3) b) and if we change a_2^{ij} and b_2^i to a_1^{ij} and b_1^i, then we have that*

$$\frac{\partial^2 p}{\partial t\partial s'}=-\mathfrak{L}^*_{0(y)}(s',t)p-\mathfrak{L}_{2(x)}(s,s',t)\mathfrak{L}^*_{1(y)}(s,t,t')p. \tag{19}$$

If, in addition, (K_2) b), d) and e) hold, then

$$\frac{\partial^2 p}{\partial t\partial s'}=-\mathfrak{L}_{0(x)}(s',t)p-\mathfrak{L}_{2(x)}(s,s',t)\mathfrak{L}^*_1(y)(s,t,t')p. \tag{19'}$$

Proof: We prove only 1) and 2) (the other parts are proved in the same way).

1) Let $f \in C^2_{fin}$. Then assumption (K_1) $b)$ implies that

$$\frac{\partial^2 T}{\partial s' \partial t'} = \int_{R^n} \frac{\partial^2 p}{\partial s' \partial t'} f(y) dy.$$

Assumption (I) ensures that

$$A_{s't'} f(x) = \mathfrak{L}_0(s', t', x) f = (a_0 \nabla, \nabla f) + (b_0, \nabla f),$$

$$A^1_{s'tt'} f(x) = \mathfrak{L}_1(s', t, t', x) f = (a_1 \nabla, \nabla f) + (b_1, \nabla f),$$

and the last expression belongs to C^2_{fin}. Therefore,

$$A^2_{ss't'} A^1_{s'tt'} f(x) = \mathfrak{L}_2(s, s', t', x) \mathfrak{L}_1(s', t, t', x) f$$

$$= (a_2 \nabla, \nabla \mathfrak{L}_1 f) + (b_2, \nabla \mathfrak{L}_1 f).$$

Fulfillment of the conditions (E_8) and (E_9) is obvious. Furthermore, from statements 5) and 6) of Theorem 5 and from the assumptions (K_1) $a)$,

$$\int_{R^n} \frac{\partial^2 p}{\partial s' \partial t'} f(y) dy = \int_{R^n} p(s, t, s', t', x, y) \left[\sum_{i,j} a_0^{ij}(s', t', y) \frac{\partial^2 f(y)}{\partial y_i \partial y_i} + \sum_i b_0^i(s', t', y) \frac{\partial f(y)}{\partial y_i} \right] dy$$

$$+ \int_{R^n} p(s, t, s', t', x, y) [\sum_{i,j} a_2^{ij}(s, s', t', y) \frac{\partial^2 [\mathfrak{L}_1(s', t, t', y) f(y)]}{\partial y_i \partial y_j}$$

$$+ \sum_i b_2^i(s, s', t', y) \frac{\partial [\mathfrak{L}_1(s', t, t', y) f(y)]}{\partial y_i}] dy$$

$$\int_{R^n} [\sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} (a_0^{ij}(s', t', y) p) - \sum_i \frac{\partial}{\partial y_i} (b_0^i(s', t', y) p)] f(y) dy$$

$$+ \int_{R^n} [\sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_i} (a_2^{ij}(s, s', t', y) p) \mathfrak{L}_1(s', t, t', y) f$$

$$- \sum_i \frac{\partial}{\partial y_i} (b_2^i(s, s', t', y) p) \mathfrak{L}_1(s', t, t', y) f] dy$$

$$= \int_{R^n} [\mathfrak{L}^*_{0(y)}(s', t') p] f(y) dy + \int_{R^n} [\mathfrak{L}^*_{2(y)}(s, s', t') p] \mathfrak{L}_1(s', t, t', y) f dy. \tag{20}$$

The second integral in (20) can also be transformed by integration by parts, and we obtain that

$$\int_{R^n} \frac{\partial^2}{\partial s' \partial t} f(y) dy = \int_{R^n} [\mathfrak{L}^*_{0(y}(s', t') p] f(y) dy + \int_{R^n} [\mathfrak{L}^*_{1(y)}(s', t, t') \mathfrak{L}^*_{2(y)}(s', s', t')] f(y) dy.$$

By virtue of an arbitrary choice of $f \in C^4_{fin}$ and by (K_1) $b)$ and Remark 8, we obtain (16).

2) Note that conditions (K_2) $a)$ and $b)$, Remark 5 and Corollary 1 imply that $\mathfrak{L}^1(s, t, t') =$

$\int_t^{t'} \mathfrak{L}_0(s,v)dv$ and $\mathfrak{L}^2(s,s',t) = \int_s^{s'} \mathfrak{L}_0(u,t)du$ on T. Condition (K_2) 3) implies that

$$\frac{\partial^2 T}{\partial s \partial t} = \int\limits_{R^n} \frac{\partial^2 p}{\partial s \partial t} f(y)dy$$

for any C^2_{fin}. Also from $(K_2)c)$, $T_{ss',tt'}f \in T$. Condition (K_2) $d)$ gives us that

$$A_{st}T_{ss'tt'}f(x) = \mathfrak{L}_0(s,t,x)T_{ss',tt'}f = \int\limits_{R^n} [\mathfrak{L}_0(s,t,x)p]f(y)dy;$$

and also,

$$A^i Tf(x) = \mathfrak{L}^i Tf = \int\limits_{R^n} [\mathfrak{L}^i p]f(y)dy = \begin{cases} \int\limits_{R^n} [\int_s^{s'} \mathfrak{L}_0(u,t)pdu]f(y)dy, i = 2, \\ \int\limits_{R^n} [\int_t^{t'} \mathfrak{L}_0(s,v)pdv]f(y)dy, \quad i = 1. \end{cases}$$

Under conditions (K_2) $d)$ and $e)$, $A^i Tf \in T$; therefore,

$$A^1_{stt'}A^2_{ss't}T_{ss',tt'}f(x) = \int\limits_{R^n} [\mathfrak{L}_1(s,t,t')\mathfrak{L}_2(s,s',t)p]f(y)dy.$$

We must verify conditions (E_1) and (E_2). (E_1) $a)$ and $b)$ are evident. Since the derivatives $\frac{\partial^2 T}{\partial s \partial t}$ and $\frac{\partial^2 T}{\partial t \partial s}$ exist, it follows from Remark 7 that we must verify only (E_1) $d)$. From Remark 6, it is sufficient to verify (E_3). (E_3) $a)$ follows from (K_2) $b)$. To show (E_3) $b)$, let $g(x) = T_{ss',tt'}f(x)$, $g_1(x) = T_{us',tt'}f(x)$. Then from (K_2) $d)$ and $e)$ and Lemma 5, part 1), we get that

$$\| A_{ut}\Big(\frac{1}{h}\Delta^1 T_{hs,tt'}g\Big)\| = \| A_{ut}\left(\frac{1}{h}\int\limits_{s-h}^{s} A^1_{vtt'}T_{us,tt'}g dv\right)\|$$

$$\leq \sup_{v \in [s-h,s]} \| A_{ut}A^1_{vtt'}g_1 \| .$$

From (K_2) $a)$, $b)$ and $e)$, Remarks 5 and Corollary 1,

$$A_{ut}A^1_{vtt'}g_1 = \int\limits_t^{t'} A_{ut}A_{vv'}g_1 dv'$$

$$- \int\limits_t^{t'} \left(\sum_{ij} a_0^{ij}(u,t,x)\frac{\partial^2}{\partial x_i \partial_j} + \sum_i b_0^i(u,t,x)\frac{\partial}{\partial x_i}\right)$$

$$\times\left(\sum_{k,l} a_0^{kl}(v,v',x)\frac{\partial^2 g_1(x)}{\partial x_k \partial x_l} + \sum_k b_0^k(v,v',x)\frac{\partial g_1(x)}{\partial x_k}\right)dv'. \tag{21}$$

Since $f \in C^2_{fin}$ and because condition (K_2) $d)$ is satisfied, each derivative $D^k g_1(x)$ for $k = \overline{1,4}$ is bounded in the following sense:

$$| D^k g_1(x) | = \left| \int\limits_K D^k pf(y)dy \right| \leq \varphi(x) \int\limits_K C(u,t,s',t',y)f(y)dy \leq C_1(t,s',t'), \tag{22}$$

where $u \in [s-\delta, s]$, $s > 0$ and $K \supset \operatorname{supp} f$.

Any term in (21) has the form,

$$c_0^{ij}(u,t,x)D^p d_0^{kl}(v,v',x)D^r g_1(x),$$

where $p = 1,2$, $r = \overline{1,4}$, and $c_0, d_0 = a_0$ or b_0. Then from (K_2) $e)$ and (22), each of them is uniformly bounded in x. Hence (E_3) $b)$ follows. Condition (E_2) is verified in the same way. From Theorem 4, our proof follows.

Remark 10: As in the one-parameter case [10], under the assumption that $A^k = \mathfrak{L}^k$ on C^2_{fin} and

$$a_k^{ij}\varphi(x) \to 0,\ b_k^i\varphi(x) \to 0, \text{ as } |x| \to \infty, \text{ for } k = 0,1,2,$$

it follows that $A^k = \mathfrak{L}_k$ for such f that each of the functions f, $\frac{\partial f}{\partial x}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$ is majorized by $C\varphi(x)$ for sufficiently large x.

Remark 11: A particular case of equation (16) was considered in [3, 4].

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