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NONNEGATIVE SOLUTIONS TO SUPERLINEAR PROBLEMS OF GENERALIZED GELFAND TYPE

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ABSTRACT

Existence of nonnegative solutions to superlinear second order problems of the form $y'' + \mu q(t) g(t,y) = 0$ is discussed in this paper. Here $\mu \geq 0$ is a parameter.

Key words: Boundary Value Problems, Superlinear, Generalized Gelfand Problems.

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1. Introduction

This paper has two main objectives. In section 2 we establish existence of a nonnegative solution to

$$\begin{cases} y'' + \mu q(t) g(t,y) = 0, 0 < t < T \\ y(0) = a \geq 0 \\ y(T) = b \geq 0 \end{cases} \tag{1.1}$$

where $\mu \geq 0$ is a constant suitably chosen. We are interested mostly in the case when g is superlinear. Problems of the form (1.1) have been examined by many authors, see [1-7, 12] and their references. Usually it is shown that (1.1) has a nonnegative solution for $0 \leq \mu < \mu_0$ where $\mu_0 \in (0, \infty]$. For example, in [4] Erbe and Wang show that (1.1) with $g(t,y) \equiv g(y)$ and $a = b = 0$, has a nonnegative solution for all $\mu \geq 0$ if

$$\lim_{y \to 0} \frac{g(y)}{y} = 0 \text{ and } \lim_{y \to \infty} \frac{g(y)}{y} = \infty.$$

This paper presents a new existence argument [2], based on showing that no solutions of an appropriate family of problems lie on the boundary of a suitably open set, to problems of the form (1.1). This argument differs from the usual *a priori* bound type argument [7]. It has connections with the "forbidden interval" type approach introduced in [1]. In particular, we will show in this paper that (1.1) with $g(t,y) = g(y)$ and $a = b = 0$, has a solution for all $\mu \geq 0$ if

$$\sup_{[0,\infty)} \frac{x}{g(x)} = \infty.$$

In section 3 we examine boundary value problems on the semi-infinite interval, namely

$$\begin{cases} y'' + \mu q(t)g(t,y) = 0, 0 < t < \infty \\ y(0) = a \geq 0 \\ y \text{ bounded on } [0,\infty) \text{ or } \lim_{t\to\infty} y(t) \text{ exists or } \lim_{t\to\infty} y'(t) = 0. \end{cases} \tag{1.2}$$

Very little seems to be known about (1.2) when $g(t,a) \geq 0$ for $t \in (0,\infty)$ and g is superlinear; see [13, 14] for some initial results. To discuss (1.2) we will use the ideas in section 2, the Arzela-Ascoli theorem and a diagonalization argument. This diagonalization type argument has been applied before in a variety of situations; see [8, 10] and their references.

The arguments in this paper are based on the following fixed point theorem.

Theorem 1.1: (Nonlinear Alternative [6, 8]). *Assume U is a relatively open subset of a convex set K in a normed linear space E. Let $N: \bar{U} \to K$ be a compact map with $p \in U$. Then either*
(*i*) *N has a fixed point in $\bar{U}$; or*
(*ii*) *there is a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u = \lambda N u + (1-\lambda)p$.*

Remark: By a map being *compact* we mean it is continuous with relatively compact range. For later purposes, a map is *completely continuous* if it is continuous and the image of every bounded set in the domain is contained in a compact set in the range.

2. Finite Interval Problem

This section establishes the existence and nonexistence for the second order boundary value problem

$$\begin{cases} y'' + \mu q(t)g(t,y) = 0, 0 < t < T \\ y(0) = a \geq 0 \\ y(T) = b \geq a. \end{cases} \tag{2.1}$$

Here $\mu \geq 0$ is a constant.

Remark: For convenience, in writing we assume $b \geq a$ in (2.1). However, in general, it is enough to assume $b \geq 0$.

By a solution to (2.1) we mean a function $y \in C^1[0,T] \cap C^2(0,T)$ which satisfies the differential equation on $(0,T)$ and the stated boundary data. We begin by presenting two general existence results for problems of the form (2.1).

Theorem 2.1: *Assume*

$$q \in C(0,T) \text{ with } q > 0 \text{ on } (0,T) \text{ and } \int_0^T q(s)ds < \infty \tag{2.2}$$

and

$$\begin{cases} g: [0,T] \times [a,\infty) \to [0,\infty) \text{ is continuous and there exists} \\ \text{a continuous nondecreasing function } f: [a,\infty) \to [0,\infty) \text{ such that} \\ f(u) > 0 \text{ for } u > a \text{ and } g(x,u) \leq f(u) \text{ on } (0,T) \times (a,\infty) \end{cases} \tag{2.3}$$

are satisfied.

Case (*a*): *Suppose*

$$q \text{ is bounded on } [0,T]. \tag{2.4}$$

Let

$$K_0 = \sup_{c \in (b,\infty)} \left\{ \int_a^c \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} + \int_b^c \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} \right\} \tag{2.5}$$

where

$$F(u) = \int_0^u f(x)dx \tag{2.6}$$

and

$$\mu_0 = \frac{K_0^2}{2T^2[\sup_{[0,T]} q(t)]}.$$

If $0 \le \mu < \mu_0$ *then* (2.1) *has a nonnegative solution.*

Case (*b*): *Suppose*

$$q \text{ is nonincreasing on } (0,T). \tag{2.7}$$

Let

$$K_1 = \sup_{c \in (b,\infty)} \left\{ \int_a^c \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} \right\} \tag{2.8}$$

where F is as in (2.6), *and*

$$\mu_1 = \frac{K_1^2}{2\left(\int_0^T \sqrt{q(x)}ds \right)^2}.$$

If $0 \le \mu < \mu_1$ *then* (2.1) *has a nonnegative solution.*

Case (*c*): *Suppose*

$$q \text{ is nondecreasing on } (0,T). \tag{2.9}$$

Let

$$K_2 = \sup_{c \in (b,\infty)} \left(\int_b^c \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} \right) \tag{2.10}$$

where F is as in (2.6), *and*

$$\mu_2 = \frac{K_2^2}{2\left(\int_0^T \sqrt{q(x)}dx \right)^2}.$$

If $0 \le \mu < \mu_2$ *then* (2.1) *has a nonnegative solution.*

Remark: The supremum in (2.5), (2.8), (2.10) is allowed to be infinite.

Proof: Consider the family of problems

$$\begin{cases} y'' + \lambda\mu q(t)g^*(t,y) = 0, 0 < t < T \\ y(0) = a \ge 0, y(T) = b \ge a \end{cases} \tag*{$(2.11)_\lambda$}$$

for $0 < \lambda < 1$. Here $g^*:[0,T] \times \boldsymbol{R} \to [0,\infty)$ is defined by

$$g^*(t,y) = \begin{cases} g(t,a) + a - y, y < a \\ g(t,y), y \geq a. \end{cases}$$

We first show that any solution y to $(2.11)_\lambda$ satisfies

$$y(t) \geq a \text{ for } t \in [0,T]. \tag{2.12}$$

To see this suppose $y - a$ has a negative minimum at $t_0 \in (0,T)$. Then $y'(t_0) = 0$ and $y''(t_0) \geq 0$. However

$$y''(t_0) = -\lambda\mu q(t_0) g^*(t_0, y(t_0)) = -\lambda\mu q(t_0)[g(t_0,a) + a - y(t_0)] < 0,$$

a contradiction. Thus (2.12) is true.

For notational purposes let

$$y_0 = \sup_{[0,T]} y(t).$$

Case (*a*): Suppose (2.4) is satisfied.

Fix $\mu < \mu_0$. Then there exists $M_0 > b$ with

$$\mu < \frac{\left(\int_a^{M_0} \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} + \int_b^{M_0} \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} \right)^2}{2T^2[\sup_{[0,T]} q(t)]} \equiv \gamma_0 \leq \mu_0. \tag{2.13}$$

Suppose the absolute maximum of y occurs at $t_0 \in [0,T]$. If $t_0 = 0$ or T we have $y_0 \leq b$. Next consider the case when $t_0 \in (0,T)$ and $y_0 > b$. In this case $y'(t_0) = 0$ with $y' \geq 0$ on $(0,t_0)$ and $y' \leq 0$ on $(t_0, 1)$ (since $y'' \leq 0$ on $(0,T)$). Now for $t \in (0,t_0)$ we have

$$-y'y'' = \lambda\mu q(t) g(t,y) y'$$

and integration from $t (t < t_0)$ to t_0 yields

$$[y'(t)]^2 \leq 2\mu[\max_{[0,T]} q(x)] \int_{y(t)}^{y(t_0)} f(u)du.$$

Hence,

$$\frac{y'(t)}{[F(y_0) - F(y(t))]^{\frac{1}{2}}} \leq \sqrt{2\mu[\max_{[0,T]} q(x)]} \text{ for } t \in (0,t_0)$$

and integration from 0 to t_0 yields

$$\int_a^{y_0} \frac{du}{[F(y_0) - F(u)]^{\frac{1}{2}}} \leq t_0 \sqrt{2\mu[\max_{[0,T]} q(x)]}. \tag{2.14}$$

On the other hand, for $t \in (t_0, T)$ we have

$$y'y'' = \lambda\mu q(t)g(t,y)(-y').$$

Integrate from t_0 to t and then from t_0 to T to obtain

$$\int_b^{y_0} \frac{du}{[F(y_0)-F(u)]^{\frac{1}{2}}} \leq (T-t_0)\ \sqrt{2\mu[\max_{[0,T]} q(x)]}. \tag{2.15}$$

Combine (2.14) and (2.15) and we obtain

$$\int_a^{y_0} \frac{du}{[F(y_0)-F(u)]^{\frac{1}{2}}} + \int_b^{y_0} \frac{du}{[F(y_0)-[F(u)]^{\frac{1}{2}}} \leq T\sqrt{2\mu[\max_{[0,T]} q(x)]}. \tag{2.16}$$

Let

$$U = \{u \in C[0,T]\colon |u|_0 < M_0\}, E = K = C[0,T]$$

where $|u|_0 = \sup_{[0,T]}|u(t)|$. Now solving $(2.11)_1$ is equivalent to finding a fixed point of N: $C[0,T] \to C[0,T]$ where

$$Ny(t) = a + \frac{(b-a)t}{T} + \mu\int_0^T G(t,s)q(s)g^*(s,y(s))ds$$

with

$$G(t,s) = \begin{cases} \frac{t(T-s)}{T}, & 0 \leq t \leq s \leq T \\ \frac{s(T-t)}{T}, & 0 \leq s \leq t \leq T. \end{cases}$$

Notice $N\colon C[0,T] \to C[0,T]$ is continuous and completely continuous (by the Arzela-Ascoli theo rem). If condition (*ii*) of Theorem 1.1 holds, then there exists $\lambda \in (0,1)$ and $y \in \partial U$ with $y = \lambda Ny + (1-\lambda)p$; here $p = a + \frac{(b-a)t}{T}$. Thus y is a solution of $(2.11)_\lambda$ satisfying $|y|_0 = M_0$ i.e., $y_0 = M_0$. Now since $M_0 > b$, (2.16) implies

$$\int_a^{M_0} \frac{du}{[F(y_0)-F(u)]^{\frac{1}{2}}} + \int_b^{M_0} \frac{du}{[F(y_0)-F(u)]^{\frac{1}{2}}} \leq T\sqrt{2\mu[\max_{[0,T]} q(x)]},$$

a contradiction since $\mu < \gamma_0$. Hence N has a fixed point in U by Theorem 1.1. Thus $(2.11)_1$ has a solution $y \in C[0,T]$ with $a \leq y(t) \leq M_0$ for $t \in [0,T]$. It follows easily that $y \in C^1[0,T] \cap C^2(0,T)$. Hence y is a solution of (2.1).

Case (*b*): Suppose (2.7) is satisfied.

Fix $\mu < \mu_1$. There there exists $M_1 > b$ with

$$\mu < \frac{\left(\int_a^{M_1} \frac{du}{[F(c)-F(u)]^{\frac{1}{2}}}\right)^2}{2\left(\int_0^T \sqrt{q(x)}dx\right)^2} \equiv \gamma_1 \leq \mu_1.$$

Suppose the absolute maximum of y occurs at $t_0 \in (0,T)$ and $y_0 > b$. Then $y'(t_0) = 0$. For $t \in (0, t_0)$ we have

$$-y'y'' = \lambda\mu q(t)g(t,y)y'$$

and integration from $t(t < t_0)$ to t_0 yields

$$[y'(t)]^2 \le 2\mu q(t) \int_{y(t)}^{y(t_0)} f(u)du$$

since (2.7) holds (and $y' \ge 0$ on $(0, t_0)$). Hence

$$\frac{y'(t)}{[F(y_0) - F(y(t))]^{\frac{1}{2}}} \le \sqrt{2\mu q(t)} \text{ for } t \in (0, t_0)$$

and integration from 0 to t_0 yields

$$\int_0^{y_0} \frac{du}{[F(y_0) - F(u)]^{\frac{1}{2}}} \le \sqrt{2\mu} \int_0^T \sqrt{q(x)}dx.$$

Let

$$U = \{u \in C[0,T]: |u|_0 < M_1\}, \ \ E = K = C[0,T].$$

Essentially the same reasoning as in case (a) guarantees the existence of a solution y to (2.1) with $a \le y(t) \le M_1$ for $t \in [0,T]$.

Case (c): Suppose (2.9) is satisfied.

Fix $\mu < \mu_2$. Then there exists $M_2 > b$ with

$$\mu < \frac{\left(\int_b^{M_2} \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} \right)^2}{2 \left(\int_0^T \sqrt{q(x)}dx \right)^2} \equiv \gamma_2 \le \mu_2.$$

Suppose the absolute maximum of y occurs at $t_0 \in (0,T)$ and $y_0 > b$. Multiply the differential equation by y', integrate from t_0 to $t(t > t_0)$ and then from t_0 to T to obtain

$$\int_b^{y_0} \frac{du}{[F(y_0) - F(u)]^{\frac{1}{2}}} \le \sqrt{2\mu} \int_0^T \sqrt{q(x)}dx.$$

As in case (a), there exists a solution y to (2.1) with $a \le y(t) \le M_2$ for $t \in [0,T]$. □

Remark: Notice in the proof of Theorem 2.1 we only showed that any solution to $(2.11)_\lambda$ satisfies $y_0 \ne M_0$. We do **not** claim (and indeed it is not true in general) that any solution of $(2.11)_\lambda$ satisfies $y_0 \le M_0$.

Theorem 2.2: *Assume* (2.2) *and*

$$\begin{cases} g:[0,T]\times[a,\infty)\to \boldsymbol{R} \text{ is continuous, } g(t,a)\geq 0 \text{ for} \\ t\in(0,T) \text{ and there exists a continuous nondecreasing function} \\ f:[a,\infty)\to[0,\infty) \text{ such that } f(u)>0 \text{ for } u>a \\ \text{and } g(t,u)\leq f(u) \text{ on } (0,T)\times(a,\infty) \end{cases} \tag{2.17}$$

are satisfied. Let

$$Q_T=\sup_{t\in[0,T]}\left(\frac{(T-t)}{T}\int_0^t sq(s)ds+\frac{t}{T}\int_t^T (T-s)q(s)ds\right)$$

and let μ_0 *satisfy*

$$\sup_{c\in(b,\infty)}\left(\frac{c}{b+\mu_0 f(c)Q_T}\right)>1. \tag{2.18}$$

If $\mu\leq\mu\leq\mu_0$ *then* (2.1) *has a nonnegative solution.*

Remark: The supremum in (2.18) is allowed to be infinite.

Proof: Let y be a solution to $(2.11)_\lambda$. Exactly the same reasoning as in Theorem 2.1 yields $y(t)\geq a$ for $t\in[0,T]$. Fix $\mu\leq\mu_0$. Let $M_0>b$ satisfy

$$\frac{M_0}{b+\mu f(M_0)Q_T}>1. \tag{2.19}$$

Suppose the absolute maximum of y occurs at t_0. If $t_0=0$ or T we have $y_0\leq b$. Next consider the case when $t_0\in(0,T)$ and $y_0>b$. For $t\in[0,T]$ we have

$$y(t)=a+\frac{(b-a)t}{T}+\lambda\mu\left(\frac{(T-t)}{T}\int_0^t sq(s)g^*(s,y(s))ds+\frac{t}{T}\int_t^T (T-s)q(s)g^*(s,y(s))ds\right)$$

$$\leq b+\mu Q_T f(y_0).$$

Consequently,

$$\frac{y_0}{b+\mu f(y_0)Q_T}\leq 1. \tag{2.20}$$

Let

$$U=\{u\in C[0,T]: |u|_0<M_0\}, E=K=C[0,T].$$

Essentially the same reasoning as in Theorem 2.1, case (a) guarantees the existence of a solution y to (2.1) with $a\leq y(t)\leq M_0$ for $t\in[0,T]$. □

Example 2.1: Suppose (2.2) holds. In addition, assume (2.17) is satisfied with f either $f(y)=e^{-\frac{1}{y}}$ (see [9]) or $f(y)=e^{\frac{\alpha y}{\alpha+y}}$ where $\alpha>0$ is a constant (see [11]) or $f(y)=Ay^\beta+B$ where $A>0$, $B\geq 0$ and $0\leq\beta<1$ are constants. Then (2.1) has a nonnegative solution for **all** $\mu\geq 0$. This follows immediately from Theorem 2.2 since for any $\mu_0>0$ we have

$$\sup_{c\in(b,\infty)}\left(\frac{c}{b+\mu_0 f(c)Q_T}\right)=\infty>1.$$

Example 2.2: The boundary value problem

$$\begin{cases} y'' + \mu(y^{\alpha} + \epsilon) = 0, 0 < t < T \\ y(0) = y(T) = 0, \alpha > 1 \text{ and } \epsilon > 0 \end{cases}$$

has a nonnegative solution if

$$0 \le \mu < \frac{8}{\alpha T^2}\left(\frac{\alpha - 1}{\epsilon}\right)^{\frac{\alpha - 1}{\alpha}} \equiv r_0.$$

This follows immediately from theorem 2.2 since

$$\sup_{c \in (0,\infty)} \left(\frac{c}{\mu_0[c^{\alpha} + \epsilon]Q_T}\right) = \frac{8}{T^2 \mu_0} \sup_{c \in (0,\infty)} \frac{c}{[c^{\alpha} + \epsilon]} = \frac{8}{\alpha T^2 \mu_0}\left(\frac{\alpha - 1}{\epsilon}\right)^{\frac{\alpha - 1}{\alpha}} > 1$$

if $\mu_0 < r_0$.

Example 2.3: Suppose (2.2) and (2.7) holds. In addition, assume (2.3) is satisfied with $a = 0$, $b > 0$ and $f(y) = y^{\alpha}$, $\alpha > 1$. Then the boundary value problem (2.1) with $a = 0$, $b > 0$ has a nonnegative solution for all

$$0 \le \mu < \frac{\left(\int_0^1 \frac{dw}{[1 - w^{\alpha + 1}]^{\frac{1}{2}}}\right)^2}{2[\alpha + 1]b^{\alpha - 1}\left(\int_0^T \sqrt{q(x)}dx\right)^2}.$$

This follows immediately from Theorem 2.1 case (b), since

$$K_1 = \sup_{c \in (b,\infty)} \left\{ \int_a^c \frac{du}{[F(c) - F(u)]^{\frac{1}{2}}} \right\} = \sup_{c \in (b,\infty)} \left\{ \frac{1}{[\alpha + 1]^{\frac{1}{2}} c^{\frac{\alpha - 1}{2}}} \int_0^1 \frac{dw}{[1 - w^{\alpha + 1}]^{\frac{1}{2}}} \right\}$$

$$= \frac{1}{[\alpha + 1]^{\frac{1}{2}} b^{\frac{\alpha - 1}{2}}} \int_0^1 \frac{dw}{[1 - w^{\alpha + 1}]^{\frac{1}{2}}}.$$

To conclude this section we present a nonexistence result for the boundary value problem

$$\begin{cases} y'' + \mu q(t) g(t, y) = 0, 0 < t < T \\ y(0) = 0 \\ y(T) = b \ge 0. \end{cases} \tag{2.21}$$

Theorem 2.3: *Assume* (2.2) *and*

$$\begin{cases} g: [0,T] \times [0,\infty) \to (0,\infty) \text{ is continuous and there exists} \\ \text{a continuous nondecreasing function } f: [0,\infty) \to (0,\infty) \text{ such that} \\ g(t,y) \ge f(y) \text{ on } [0,T] \times [0,\infty) \end{cases} \tag{2.22}$$

are satisfied. In addition, assume μ *satisfies*

$$\mu\int_{\frac{T}{2}}^{T}(T-x)q(x)dx \geq \int_{b}^{\infty}\frac{du}{f(u)} \quad \textit{and} \quad \mu\int_{0}^{\frac{T}{2}} xq(x)dx \geq \int_{0}^{\infty}\frac{du}{f(u)}.$$

Then (2.21) *does **not** have a nonnegative solution on* $[0,T]$.

Proof: Suppose (2.21) has a nonnegative solution y on $[0,T]$. Then since $y'' \leq 0$ on $(0,T)$ we have either $y' \geq 0$ on $(0,T)$ or there exists $\tau \in (0,T)$ with $y' \geq 0$ on $(0,\tau)$ and $y' \leq 0$ on (τ, T).

Case (*a*): $y' \geq 0$ on $(0,T)$.

For $x \in (0,T)$ we have

$$y''(x) = (-\mu)q(x)g(x,y(x)) \leq (-\mu)q(x)f(y(x)). \tag{2.23}$$

Integrate from t to T to obtain (since $y' \geq 0$ on $(0,T)$),

$$y'(T)-y'(t) \leq (-\mu)\int_{t}^{T} q(x)f(y(x))dx \leq (-\mu)f(y(t))\int_{t}^{T} q(x)dx$$

and so

$$\frac{-y'(t)}{f(y(t))} \leq (-\mu)\int_{t}^{T} q(x)dx.$$

Thus for $t \in (0,T)$ we have

$$\frac{y'(t)}{f(y(t))} \geq \mu\int_{t}^{T} q(x)dx$$

and integration from 0 to T yields

$$\int_{0}^{b}\frac{du}{f(u)} \geq \mu\int_{0}^{T} xq(x)dx,$$

a contradiction.

Case (*b*): $y' \geq 0$ on $(0,\tau)$ and $y' \leq 0$ on (τ,T).

Integrate (2.23) from τ to $t(t > \tau)$ to obtain

$$y'(t) \leq (-\mu)\int_{\tau}^{t} q(x)f(y(x))dx \leq (-\mu)f(y(t))\int_{\tau}^{t} q(x)dx$$

and so

$$\frac{-y'(t)}{f(y(t))} \geq \mu\int_{\tau}^{t} q(x)dx \quad \text{for} \quad t \in (\tau,T).$$

Integration from τ to T yields

$$\int_{b}^{y(\tau)}\frac{du}{f(u)} \geq \mu\int_{\tau}^{t}(T-x)q(x)dx. \tag{2.24}$$

On the other hand integrate (2.23) from $t(t < \tau)$ to τ to obtain

$$-y'(t) \leq (-\mu)\int_{t}^{\tau} q(x)f(y(x))dx \leq (-\mu)f(y(t))\int_{t}^{\tau} q(x)dx$$

and so

$$\frac{y'(t)}{f(y(t))} \geq \mu \int_t^\tau q(x)dx \text{ for } t \in (0,\tau).$$

Integration from 0 to τ yields

$$\int_0^{y(\tau)} \frac{du}{f(u)} \geq \mu \int_0^\tau xq(x)dx. \tag{2.25}$$

Now either $\tau \leq \frac{T}{2}$ or $\tau \geq \frac{T}{2}$. If $\tau \leq \frac{T}{2}$ then (2.24) implies

$$\int_b^\infty \frac{du}{f(u)} > \int_b^{y(\tau)} \frac{du}{f(u)} \geq \mu \int_\tau^T (T-x)q(x)dx \geq \mu \int_{\frac{T}{2}}^T (T-x)q(x)dx,$$

a contradiction. On the other hand, if $\tau \geq \frac{T}{2}$, then (2.25) implies

$$\int_0^\infty \frac{du}{f(u)} > \int_0^{y(\tau)} \frac{du}{f(u)} \geq \mu \int_0^\tau xq(x)dx \geq \mu \int_0^{\frac{T}{2}} xq(x)dx,$$

a contradiction. □

3. Semi-infinite Interval Problem

The ideas in section 2 together with a diagonalization argument enable us to treat various problems defined on semi-infinite intervals. We begin by considering two such problems, namely,

$$\begin{cases} y'' + \mu q(t)g(t,y) = 0, 0 < t < \infty \\ y(0) = a \geq 0 \\ y \text{ bounded on } [0,\infty) \end{cases} \tag{3.1}$$

and

$$\begin{cases} y'' + \mu q(t)g(t,y) = 0, 0 < t < \infty \\ y(0) = a \geq 0 \\ \lim_{t\to\infty} y(t) \text{ exists.} \end{cases} \tag{3.2}$$

Two existence results are presented.

Theorem 3.1: *Choose $b \geq a$ and fix it. Suppose*

$$\begin{cases} q \in C(0,\infty) \text{ with } q > 0 \text{ nonincreasing on } (0,\infty) \\ \text{and } \int_0^\infty \sqrt{q(x)}ds < \infty \end{cases} \tag{3.3}$$

and

$$\begin{cases} g\colon [0,\infty) \times [a,\infty) \to [0,\infty) \text{ is continuous and there exists} \\ \text{a continuous nondecreasing function } f\colon [a,\infty) \to [0,\infty) \text{ such that} \\ f(u) > 0 \text{ for } u > a \text{ and } g(x,u) \leq f(u) \text{ on } (0,\infty) \times (a,\infty) \end{cases} \tag{3.4}$$

are satisfied. Define

$$K_{\infty} = \sup_{c \in (b,\infty)} \left\{ \int_a^c \frac{du}{[F(c)-F(u)]^{\frac{1}{2}}} \right\}$$

where F is as in (2.6), and

$$\mu_{\infty} = \frac{K_{\infty}^2}{2\left(\int_0^{\infty} \sqrt{q(x)}dx \right)^2}. \tag{3.5}$$

If $0 \le \mu < \mu_{\infty}$ *then (3.1) and (3.2) have a nonnegative solution* $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

Proof: Fix $n \in N^+ = \{1,2,\ldots\}$. Consider the family of problems

$$\begin{cases} y'' + \lambda\mu q(t)g^*(t,y) = 0, 0 < t < n \\ y(0) = a, y(n) = b \end{cases} \tag{$3.6)_\lambda^n$}$$

for $0 < \lambda < 1$; here g^* is as defined in theorem 2.1.

Fix $\mu < \mu_{\infty}$. Then there exists $M_{\infty} > b$ with

$$\mu < \frac{\left(\int_a^{M_{\infty}} \frac{du}{[F(c)-F(u)]^{\frac{1}{2}}} \right)^2}{2\left(\int_0^{\infty} \sqrt{q(x)}dx \right)^2} \equiv \gamma_{\infty} \le \mu_{\infty}. \tag{3.7}$$

Let y be any solution of $(3.6)_\lambda^n$. Then as in Theorem 2.1 we have $y(t) \ge a$ for $t \in [0,n]$. For notational purposes, let $y_{0,n} = \sup_{[0,n]} y(t)$. Suppose the absolute maximum of y occurs at $t_0 \in (0,n)$ and $y_{0,n} > b$. Essentially the same reasoning as in Theorem 2.1 case (b) yields

$$\int_a^{y_{0,n}} \frac{du}{[F(y_{0,n})-F(u)]^{\frac{1}{2}}} \le \sqrt{2\mu} \int_0^n \sqrt{q(x)}dx < \sqrt{2\mu} \int_0^{\infty} \sqrt{q(x)}dx.$$

Thus as in Theorem 2.1 there exists a solution y_n to $(3.6)_1^n$ with

$$a \le y_n(t) \le M_{\infty} \text{ for } t \in [0,n]. \tag{3.8}$$

In particular, $y_n \in C^1[0,n] \cap C^2(0,n)$ is a solution of

$$\begin{cases} y'' + \mu q(t)g(t,y) = 0, 0 < t < n \\ y(0) = a, y(n) = b. \end{cases} \tag{3.9}$$

Let

$$R_0 = \sup_{[0,\infty) \times [a, M_{\infty}]} g(t,u),$$

and for $t \in [0,n]$ we have

$$|\, y_n''(t) \,| \le \mu R_0 q(t). \tag{3.10}$$

Now (3.8) together with the mean value theorem implies that there exists $\tau \in (0,1)$ with $|y_n'(\tau)| = |y_n(1) - y_n(0)| \le M_\infty$. Consequently, for $t \ge \tau$ we have

$$|y_n'(t)| \le |y_n'(\tau)| + \int_\tau^t |y_n''(x)|\,dx$$

and so

$$|y_n'(t)| \le M_\infty + \mu R_0 \int_0^t q(x)dx. \tag{3.11a}$$

On the other hand, for $t < \tau$ we have

$$|y_n'(t)| \le M_\infty + \int_t^\tau |y_n''(x)|\,dx \le M_\infty + \mu R_0 \int_0^1 q(x)dx \equiv R_1. \tag{3.11b}$$

Now (3.11a) and (3.11b) imply

$$|y_n'(t)| \le R_1 + \mu R_0 \int_0^t q(x)dx \text{ for } t \in (0,n)$$

so for $t, s \in [0,n]$ we have

$$|y_n(t) - y_n(s)| \le R_1|t-s| + \mu R_0 \left| \int_s^t \int_0^x q(u)du dx \right|. \tag{3.12}$$

A standard diagonalization type argument [8, 10] will now complete the proof. Define

$$u_n(x) = \begin{cases} y_n(x), x \in [0,n] \\ b, x \in [n,\infty). \end{cases}$$

Then, u_n is continuous on $[0,\infty)$ and $a \le u_n(t) \le M_\infty$, $t \in [0,\infty)$. Also for $t, s \in [0,\infty)$ it is easy to check that

$$|u_n(t) - u_n(s)| \le R_1|t-s| + \mu R_0 \left| \int_s^t \int_0^x q(u)du dx \right|.$$

Using the Arzela-Ascoli theorem [8] we obtain for $k = 1,2,\ldots$ a subsequence $N_k \subseteq N^+$ with $N_k \subseteq N_{k-1}$ and a continuous function z_k on $[0,k]$ with $u_n \to z_k$ uniformly on $[0,k]$ as $n \to \infty$ through N_k. Also $z_k = z_{k-1}$ on $[0,k-1]$.

Define a function y as follows. Fix $x \in [0,\infty)$ and let $k \in N^+$ with $x \le k$. Define $y(x) = z_k(x)$. Notice $y \in C[0,\infty)$ and $a \le y(t) \le M_\infty$ for $t \in [0,\infty)$.

Fix x and choose $k > x$, $k \in N^+$. Then for $n \in N_k$ we have

$$u_n(x) = \frac{tu_n(k)}{k} + a + \frac{(b-a)t}{k} + \frac{\mu(k-t)}{k} \int_0^t sq(s)g(s,u_n(s))ds + \frac{\mu t}{k} \int_t^k (k-s)q(s)g(s,u_n(s))ds.$$

Let $n \to \infty$ through N_k to obtain

$$z_k(x) = \frac{tz_k(k)}{k} + a + \frac{(b-a)t}{k} + \frac{\mu(k-t)}{k}\int_0^t sq(s)g(s,z_k(s))ds$$
$$+\frac{\mu t}{k}\int_t^k (k-s)q(s)g(s,z_k(s))ds.$$

Thus

$$y(x) = \frac{ty(k)}{k} + a + \frac{(b-a)t}{k} + \frac{\mu(k-t)}{k}\int_0^t sq(s)g(s,y(s))ds$$
$$+\frac{\mu t}{k}\int_t^k (k-s)q(s)g(s,y(s))ds$$

which implies $y \in C^1[0,\infty) \cap C^2(0,\infty)$ with $y''(x) = -\mu q(x)g(x,y(x))$ for $0 < x < \infty$. Consequently y is a solution of (3.1). To show y is a solution of (3.2) we claim

$$y'(t) > 0 \text{ for } t \in (0,\infty). \tag{3.13}$$

If this is not true then there exists $x_0 \geq 0$ with $y'(x_0) < 0$. Then for $x > x_0$ we have

$$y'(x) = y'(x_0) - \mu\int_{x_0}^x q(s)g(s,y(s))ds \leq y'(x_0).$$

Hence for $x > x_0$ we have

$$y(x) - y(x_0) \leq y'(x_0)(x - x_0) \to -\infty \text{ as } x \to \infty.$$

This contradicts $a \leq y(t) \leq M_\infty$ for $t \in [0,\infty)$. Hence (3.13) is true i.e., y is nondecreasing on $(0,\infty)$. This together with $a \leq y(t) \leq M_\infty$ for $t \in [0,\infty)$ implies $\lim_{t\to\infty} y(t)$ exists. □

Theorem 3.2: *Let $N^+ = \{1,2,\ldots\}$. Suppose*

$$q \in C(0,\infty) \textit{ with } q > 0 \textit{ on } (0,\infty) \tag{3.14}$$

$$Q_\infty = \sup_{n \in N+}\left(\sup_{t \in [0,n]}\left\{\frac{(n-t)}{n}\int_0^t sq(s)ds + \frac{t}{n}\int_t^n (n-s)q(s)ds\right\}\right) < \infty \tag{3.15}$$

$$\textit{for } 0 \leq t < \infty \textit{ and } u \geq a \textit{ in a bounded set then } |g(t,u)| \textit{ is bounded} \tag{3.16}$$

and

$$\begin{cases} g:[0,\infty)\times[a,\infty)\to \mathbf{R} \textit{ is continuous, } g(t,a) \geq 0 \textit{ for} \\ t \in (0,\infty) \textit{ and there exists a continuous nondecreasing function} \\ f:[a,\infty)\to[0,\infty) \textit{ such that } f(u) > 0 \textit{ for } u > a \\ \textit{and } g(t,u) \leq f(u) \textit{ on } (0,\infty)\times(a,\infty) \end{cases} \tag{3.17}$$

are satisfied. Choose $b \geq a$ and fix it. Let μ_∞ satisfy

$$\sup_{c \in (b,\infty)} \left(\frac{c}{b+\mu_\infty f(c)Q_\infty}\right) > 1. \tag{3.18}$$

If $0 \le \mu \le \mu_\infty$ *then* (3.1) *has a nonnegative solution* $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

Proof: Fix $\mu \le \mu_\infty$. Let $M_\infty > b$ satisfy

$$\frac{M_\infty}{b+\mu f(M_\infty)Q_\infty} > 1. \tag{3.19}$$

Fix $n \in N^+$ and let y be any solution of $(3.6)^n_\lambda$. As in Theorem 2.1 we have $y(t) \ge a$ for $t \in [0,n]$. For notational purposes, let $y_{0,n} = \sup_{[0,n]} y(t)$. Suppose the absolute maximum of y occurs at $t_0 \in (0,n)$ and $y_{0,n} > b$. For $t \in [0,n]$ we have, as in Theorem 2.2,

$$y(t) \le b + \mu f(y_{0,n})\left(\frac{(n-t)}{n}\int_0^t sq(s)ds + \frac{t}{n}\int_t^n (n-s)q(s)ds\right)$$

$$\le b + \mu Q_\infty f(y_{0,n}).$$

Consequently,

$$\frac{y_{0,n}}{b+\mu Q_\infty f(y_{0,n})} \le 1$$

and the argument in Theorem 2.1 implies that $(3.6)^n_1$ has a solution $y_n \in C^1[0,n] \cap C^2(0,n)$ with $a \le y_n(t) \le M_\infty$ for $t \in [0,n]$.

Essentially the same reasoning as in Theorem 3.1 (from (3.10) onwards) implies that (3.1) has a solution $y \in C^1[0,\infty) \cap C^2(0,\infty)$ with $a \le y(t) \le M_\infty$ for $t \in [0,\infty)$. □

Remarks: (*i*) Suppose the conditions in Theorem 3.2 hold and in addition, $g(x,u) > 0$ for $(x,u) \in (0,\infty) \times (a,\infty)$. Then the argument in Theorem 3.1 implies that (3.2) has a nonnegative solution.

(*ii*) As an example, if $q(t) = e^{-t}$ then

$$Q_\infty = \sup_{n \in N^+}\left(\sup_{t \in [0,n]}\left\{[1-e^{-t}] - \tfrac{t}{n}[1-e^{-n}]\right\}\right) \le \sup_{n \in N^+}[1-e^{-n}] = 1 < \infty.$$

Next we discuss a general boundary value problem on the semi-infinite interval, namely,

$$\begin{cases} y'' + \mu q(t) g(t,y) = 0, 0 < t < \infty \\ y(0) = a \ge 0 \\ \lim_{t\to\infty} y'(t) = 0. \end{cases} \tag{3.20}$$

Theorem 3.3: *Suppose* (3.14), (3.15) *and* (3.16) *hold and in addition, assume*

$$\int_0^\infty q(x)dx < \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\int_0^n sq(s)ds = 0 \tag{3.21}$$

and

$$\begin{cases} g:[0,\infty)\times[a,\infty)\to \boldsymbol{R} \text{ is continuous, } g(t,a) \ge 0 \text{ for} \\ t \in (0,\infty) \text{ and there exists a continuous nondecreasing function} \\ f:[a,\infty)\to[0,\infty) \text{ such that } f(u) > 0 \text{ for } u > a \\ \text{and } |g(t,u)| \le f(u) \text{ on } (0,\infty)\times(a,\infty) \end{cases} \tag{3.22}$$

are satisfied. Choose $b \geq a$ and fix it. Let μ_∞ satisfy (3.18). If $0 \leq \mu \leq \mu_\infty$, then (3.20) has a nonnegative solution $y \in C^1[0,\infty) \cap C^2(0,\infty)$.

Proof: Fix $\mu \leq \mu_\infty$. As in Theorem 3.2 we have that $(3.6)_1^n$ has a solution $y_n \in C^1[0,n] \cap C^2(0,n)$ with $a \leq y_n(t) \leq M_\infty$ for $t \in [0,n]$; here M_∞ is given as in (3.19). Also since

$$y_n'(t) = \frac{b}{n} + \mu\left(\int_t^n q(s)g(s,y_n(s))ds - \frac{1}{n}\int_0^n sq(s)g(s,y_n(s))ds \right)$$

we have that

$$|y_n'(t)| \leq \frac{b}{n} + \mu f(M_\infty) \left(\int_t^n q(s)ds + \frac{1}{n}\int_0^n sq(s)ds \right)$$

$$\leq \frac{b}{n} + \mu f(M_\infty) \left(\int_t^\infty q(s)ds + \frac{1}{n}\int_0^n sq(s)ds \right) \equiv c_n(t).$$

Thus for $t \in [0,n]$ we have

$$|y_n'(t)| \leq c_n(t). \tag{3.23}$$

Remarks: (i) Notice since (3.21) is true then $\lim_{n\to\infty}\frac{1}{n}\int_0^n sq(s)ds = 0$ and consequently

$$\lim_{n\to\infty} c_n(t) = \mu f(M_\infty) \int_t^\infty q(s)ds \text{ for } t \in [0,n].$$

(ii) Also (3.21) implies that that there exists a constant c_∞ with $|y_n'(t)| \leq c_\infty$ for $t \in [0,n]$.

Finally, as in Theorem 3.1, we have

$$|y_n''(t)| \leq \mu R_0 q(t) \text{ for } t \in [0,n] \tag{3.24}$$

where

$$R_0 = \sup_{[0,\infty)\times[a,M_\infty]} |g(t,u)|.$$

Define

$$u_n(x) = \begin{cases} y_n(x), x \in [0,n] \\ b, x \in (n,\infty). \end{cases}$$

Using the Arzela-Ascoli theorem [8] we obtain for $k = 1,2,\ldots$ a subsequence $N_k \subseteq \{k+1, k+2,\ldots\}$ with $N_k \subseteq N_{k-1}$ and a function $z_k \in C^1[0,k]$ with $u_n^{(j)} \to z_k^{(j)}$, $j = 0,1$ uniformly on $[0,k]$ as $n\to\infty$ through N_k.

Now define a function $y:[0,\infty)\to[a,\infty)$ by $y(x) = z_k(x)$ on $[0,k]$. Notice $y \in C^1[0,\infty)$ and $a \leq y(t) \leq M_\infty$ for $t \in [0,\infty)$ and $|y'(t)| \leq c_\infty$ for $t \in [0,\infty)$. In fact

$$|y'(t)| \leq \lim_{n\to\infty} c_n(t) = \mu f(M_\infty) \int_t^\infty q(s)ds \text{ for } t \geq 0. \tag{3.25}$$

As in Theorem 3.1 we have that y is a solution of (3.1). Also (3.25) implies $|y'(\infty)| = 0$ so $y'(\infty) = 0$. □

Similarly we have

Theorem 3.4: *Choose $b \geq a$ and fix it. Suppose* (3.3) *and* (3.21) *hold and in addition*

$$\begin{cases} g: [0,\infty) \times [a,\infty) \to [0,\infty) \text{ is continuous and there exists} \\ \text{a continuous nondecreasing function } f: [a,\infty) \to [0,\infty) \text{ such that} \\ f(u) > 0 \text{ for } u > a \text{ and } g(x,u) \leq f(u) \text{ on } (0,\infty) \times (a,\infty) \end{cases} \quad (3.26)$$

is satisfied. Let μ_∞ satisfy (3.5). *If $0 \leq \mu < \mu_\infty$ then* (3.20) *has a nonnegative solution $y \in C^1[0,\infty) \cap C^2(0,\infty)$.*

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