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EXISTENCE PRINCIPLES FOR SECOND ORDER NONRESONANT BOUNDARY VALUE PROBLEMS

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ABSTRACT

We discuss the two point singular "nonresonant" boundary value problem $\frac{1}{p}(py')' = f(t,y,py')$ a.e. on $[0,1]$ with y satisfying Sturm Liouville, Neumann, Periodic or Bohr boundary conditions. Here f is an L^1-Carathéodory function and $p \in C[0,1] \cap C^1(0,1)$ with $p > 0$ on $(0,1)$.

Key words: Existence, Singular, Nonresonant, Boundary Value Problems, Sturm Liouville Problems.

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1. Introduction

In this paper, problems of the form

$$\frac{1}{p(t)}(p(t)y'(t))' = f(t, y(t), p(t)y'(t)) \text{ a.e. on } [0,1] \tag{1.1}$$

are discussed with y satisfying either

(i) (Sturm Liouville)

$$\begin{cases} -\alpha y(0) + \beta \lim_{t \to 0^+} p(t)y'(t) = c_0, \ \alpha \geq 0, \ \beta \geq 0, \ \alpha^2 + \beta^2 > 0 \\ a y(1) + b \lim_{t \to 1^-} p(t)y'(t) = c_1, \ a \geq 0, \ b \geq 0, a^2 + b^2 > 0 \\ max\{a, \alpha\} > 0 \end{cases} \tag{SL}$$

(ii) (Neumann)

$$\begin{cases} \lim_{t \to 0^+} p(t)y'(t) = c_0 \\ \lim_{t \to 1^-} p(t)y'(t) = c_1 \end{cases} \tag{N}$$

($\underline{iii}$) (Periodic)

$$\begin{cases} y(0) = y(1) \\ \lim_{t \to 0^+} p(t)y'(t) = \lim_{t \to 1^-} p(t)y'(t) \end{cases} \tag{P}$$

or

($\underline{iv}$) (Bohr)

$$\begin{cases} y(0) = c_0 \\ \int_0^1 \frac{ds}{p(s)} \lim_{t \to 1^-} p(t)y'(t) - y(1) = c_1. \end{cases} \tag{Br}$$

Remark: If a function $u \in C[0,1] \cap C^1(0,1)$ with $pu' \in C[0,1]$ satisfies boundary condition ($\underline{i}$), we write $u \in (SL)$. A similar remark applies for the other boundary condition. If u satisfies ($\underline{i}$) with $c_0 = c_1 = 0$, we write $u \in (SL)_0$, etc.

Throughout the paper, $p \in C[0,1] \cap C^1(0,1)$ together with $p > 0$ on $(0,1)$. Also $pf{:}[0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is an L^1-Carathéodory function. By this we mean:

($\underline{i}$) $t \to p(t)f(t,y,q)$ is measurable for all $(y,q) \in \mathbf{R}^2$,

($\underline{ii}$) $(y,q) \to p(t)f(t,y,q)$ is continuous for a.e. $t \in [0,1]$,

($\underline{iii}$) for any $r > 0$ there exists $h_r \in L^1[0,1]$ such that $|\,p(t)f(t,y,q)\,| \le h_r(t)$ for a.e. $t \in [0,1]$ and for all $|\,y\,| \le r$, $|\,q\,| \le r$.

The results in the literature [7, 10, 13-16] concern the nonresonant second order problem

$$\begin{cases} y'' + f(t,y) = 0 \text{ a.e. on } [0,1] \\ y \in (SL),\ (N) \text{ or } (P). \end{cases} \tag{1.2}$$

In particular, if $\frac{f(t,y)}{y}$ stays asymptotically between two consecutive eigenvalues or to the left of the spectrum of the differential operator then certain existence results can be established. The most advanced results to date seem to be [7], where quadratic forms associated with the eigenvalues and eigenfunctions are used to establish various existence criteria.

This paper deals with the more general problem (1.1). By using properties of the Green's function and by examining appropriate Sturm Liouville eigenvalue problems, we are able to establish various existence results. The paper will be divided into three sections. In section 2, fixed point methods, in particular a nonlinear alternative of Leray-Schauder type, will be used to establish existence principles for (1.1) with the various boundary conditions. We remark here that the existence principles are constructed with the nonresonant problem in mind. Section 3 establishes various existence theorems and section 4 discusses the Sturm Liouville eigenvalue problem.

In the remainder of the introduction we gather together some facts on second order differential equations which will be used throughout this paper. For notational purposes, let w be a weight function. By $L^1_w[0,1]$ we mean the space of functions u such that $\int_0^1 w(t)\,|\,u(t)\,|\,dt < \infty$. $L^2_w[0,1]$ denotes the space of functions u such that $\int_0^1 w(t)\,|\,u(t)\,|^2 dt < \infty$; also for $u, v \in L^2_w[0,1]$ define $\langle u,v \rangle = \int_0^1 w(t)u(t)\overline{v(t)}dt$. Let $AC[0,1]$ be the space of functions which are absolutely continuous on $[0,1]$.

Theorem 1.1: *Suppose*

$$p \in C[0,1] \cap C^1(0,1) \text{ with } p > 0 \text{ on } (0,1) \text{ and } \int_0^1 \frac{ds}{p(s)} < \infty \tag{1.3}$$

and

$$r, g \in L^1_p[0,1] \tag{1.4}$$

are satisfied. Then

$$\begin{cases} \frac{1}{p}(py')' + r(t)y = g(t) \text{ a.e. on } [0,1] \\ y(0) = a_0, \lim_{t \to 0^+} p(t)y'(t) = b_0 \end{cases} \tag{1.5}$$

has exactly one solution $y \in C[0,1] \cap C^1(0,1)$ *with* $py' \in AC[0,1]$. *(By a solution to (1.5), we mean a function* $y \in C[0,1] \cap C^1(0,1)$, $py' \in AC[0,1]$ *which satisfies the differential equation a.e. on* $[0,1]$ *and the stated initial condition).*

Proof: Let $C[0,1]$ denote the Banach space of continuous functions on $[0,1]$ with norm

$$|u|_K = \sup_{t \in [0,1]} |e^{-KR(t)}u(t)| \text{ where } K = \int_0^1 \frac{ds}{p(s)} \text{ and } R(t) = \int_0^t p(s)r(s)ds.$$

Solving (1.5) is equivalent to finding a $y \in C[0,1]$ which satisfies

$$y(t) = a_0 + b_0 \int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)} \int_0^s p(s)[-r(x)y(x) + g(x)]dxds.$$

Define the operator $N: C[0,1] \to C[0,1]$ by

$$Ny(t) = a_0 + b_0 \int_0^t \frac{ds}{p(s)} + \int_0^t \frac{1}{p(s)} \int_0^s p(x)[-r(x)y(x) + g(x)]dxds.$$

Now N is a contraction since

$$|Nu - Nv|_K \le |u - v|_K \max_{t \in [0,1]} |e^{-KR(t)} \int_0^t \frac{1}{p(s)} \int_0^s p(x)r(x)e^{KR(x)}dxds|$$

$$= \frac{|u-v|_K}{K} \max_{t \in [0,1]} |e^{-KR(t)} \int_0^t \frac{1}{p(s)}[e^{KR(s)} - 1]ds| \le |u - v|_K [1 - e^{-KR(1)}].$$

The Banach contraction principle now establishes the result. □

Let u_1 be the unique solution to

$$\begin{cases} \frac{1}{p}(py')' + r(t)y = 0 \text{ a.e. on } [0,1] \\ y(0) = 1, \lim_{t \to 0^+} p(t)y'(t) = 0 \end{cases}$$

and u_2 the unique solution to

$$\begin{cases} \frac{1}{p}(py')' + r(t)y = 0 \text{ a.e. on } [0,1] \\ y(0) = 0, \lim_{t \to 0^+} p(t)y'(t) = 1. \end{cases}$$

Now u_1 and u_2 are linearly independent and their Wronskian $W(t)$, at t, satisfies $p(t)W'(t)+p'(t)W(t)=0$ so $p(t)W(t)=\text{constant}\neq 0$, $t\in[0,1]$. The general solution (method of variation of parameters) of

$$\frac{1}{p}(py')'+r(t)y=g(t) \text{ a.e. on } [0,1]$$

is

$$y(t)=d_0u_1(t)+d_1u_2(t)+\int_0^t \frac{[u_2(t)u_1(s)-u_1(t)u_2(s)]}{W(s)}g(s)ds \tag{1.6}$$

where d_0 and d_1 are constants. The standard construction of the Green's function, see [17-18] for example, yields

Theorem 1.2: *Let B denote either (SL), (N), (P) or (Br) and B_0 either $(SL)_0$, $(N)_0$, (P) or $(Br)_0$. Suppose (1.3) and (1.4) are satisfies. If*

$$\begin{cases} \frac{1}{p}(py')'+r(t)y=0 \text{ a.e. on } [0,1] \\ y\in B_0 \end{cases}$$

has only the trivial solution, then

$$\begin{cases} \frac{1}{p}(py')'+r(t)y=0 \text{ a.e. on } [0,1] \\ y\in B \end{cases} \tag{1.7}$$

has exactly one solution y, given by (1.6), where d_0 and d_1 are uniquely determined from the boundary condition. In fact,

$$y(t)=A_0y_1(t)+A_1y_2(t)+\int_0^1 G(t,s)g(s)ds \tag{1.8}$$

where $G(t,s)$ is the Green's function and A_0 and A_1 are uniquely determined by the boundary conditions. Of course,

$$G(t,s)=\begin{cases} \dfrac{y_1(s)y_2(t)}{W(s)}, & 0<s\leq t \\ \dfrac{y_1(t)y_2(s)}{W(s)}, & t\leq s<1 \end{cases}$$

where y_1 and y_2 are the two "usual" linearly independent solutions i.e., choose $y_1\neq 0$, $y_2\neq 0$ so that y_1, y_2 satisfy $\frac{1}{p}(py')'+r(t)y=0$ a.e. on $[0,1]$ with y_1 satisfying the first boundary condition of B_0 and y_2 satisfying the second boundary condition of B_0.

Of course, analogue versions of theorems 1.1 and 1.2 hold for the more general problem

$$\begin{cases} \frac{1}{p}(py')'+r(t)y+\kappa(t)p(t)y'(t)=g(t) \text{ a.e. on } [0,1] \\ y\in (SL),\ (N),\ (P) \text{ or } (Br) \end{cases} \tag{1.9}$$

where κ satisfies

$$\kappa \in L^1_p[0,1]. \tag{1.10}$$

Theorem 1.3: *If* (1.3), (1.4) *and* (1.10) *are satisfied and if*

$$\begin{cases} \frac{1}{p}(py')' + r(t)y + \kappa(t)p(t)y'(t) = 0 \ a.e. \ on \ [0,1] \\ y \in (SL)_0, \ (N)_0, \ (P) \ or \ (Br)_0 \end{cases}$$

has only the trivial solution, then (1.9) *has exactly one solution given by* (1.8) (*where* $G(t,s)$ *is the appropriate Green's function*).

In practice, one usually examines (1.7) and not the more general problem (1.9). This is due to the fact that numerical schemes [3] are available for Sturm Liouville eigenvalue problems (see section 4). However from a theoretical point of view, it is of interest to establish the most general result.

2. Existence Principles

We use a fixed point approach to establish our existence principles. In particular, we use a nonlinear alternative of Leray-Schauder type [9] which is an immediate consequence of the topological transversality theorem [8] of Granas. For completeness, we state the result. By a map being **compact** we mean it is continuous with relatively compact range. A map is **completely continuous** if it is continuous and the image of every bounded set in the domain is contained in a compact set of the range.

Theorem 2.1: (*Nonlinear Alternative*) *Assume* U *is a relatively open subset of a convex set* K *in a Banach space* E. *Let* $N: \bar{U} \to K$ *be a compact map with* $p \in U$. *Then either*

(*i*) N *has a fixed point in* $\bar{U}$; *or*

(*ii*) *there is a point* $u \in \partial U$ *and* $\lambda \in (0,1)$ *such that* $u = \lambda N u + (1-\lambda)p$.

Consider first the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y = f(t, y, py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N). \end{cases} \tag{2.1}$$

By a solution to (2.1) we mean a function $y \in C[0,1] \cap C^1(0,1)$, $py' \in AC[0,1]$ which satisfies the differential equation in (2.1) a.e. on $[0,1]$ and the stated boundary conditions.

Theorem 2.2: *Let* $pf:[0,1] \times \mathbf{R}^2 \to \mathbf{R}$ *be an* L^1*-Carathéodory function and assume* p *satisfies* (1.3) *and* τ *satisfies*

$$\tau \in L^1_p[0,1]. \tag{2.2}$$

In addition, suppose

$$\begin{cases} \frac{1}{p}(py')' + \tau y = 0 \ a.e. \ on \ [0,1] \\ y \in (SL)_0 \ or \ (N)_0 \end{cases} \tag{2.3}$$

has only the trivial solution. Now suppose there is a constant M_0, *independent of* λ, *with*

$$\| y \|_1 = max\{\sup_{[0,1]} | y(t) |, \sup_{(0,1)} | p(t)y'(t) | \} \le M_0$$

for any solution y to

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y = \lambda f(t, y, py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N) \end{cases} \tag{2.4$_\lambda$}$$

for each $\lambda \in (0,1)$. *Then* (2.1) *has at least one solution.*

Proof: Let y_1 and y_2 be two linearly independent solutions (see section 1) of $(py')' + \tau py = 0$ a.e. on $[0,1]$ with $y_1, y_2 \in C[0,1]$ and $py_1', py_2' \in AC[0,1]$.

Remark: In the analysis that follows, (N) will be thought of as (SL) with $\alpha = a = 0$, $\beta = b = 1$.

Choose y_2 so that $-\alpha y_2(0) + \beta \lim_{t\to 0^+} p(t)y_2'(t) \neq 0$. If this is not possible, then the two linearly independent solutions are such that $-\alpha y_1(0) + \beta \lim_{t\to 0^+} p(t)y_1'(t) = -\alpha y_2(0) + \beta \lim_{t\to 0^+} p(t)y_2'(t) = 0$. Let

$$u(x) = [ay_2(1) + b\lim_{t\to 1^-} p(t)y_2'(t)]y_1(x) - [ay_1(1) + b\lim_{t\to 1^-} p(t)y_1'(t)]y_2(x)$$

so u satisfies $(pu')' + \tau pu = 0$ a.e. on $[0,1]$ with $-\alpha u(0) + \beta \lim_{t\to 0^+} p(t)u'(t) = 0$ and $au(1) + b\lim_{t\to 1^-} p(t)u'(t) = 0$. Consequently, $u \equiv 0$, a contradiction since y_1 and y_2 are linearly independent. Solving $(2.4)_\lambda$ is equivalent to finding a $y \in C[0,1]$ with $py' \in C[0,1]$ which satisfies

$$y(t) = A_\lambda y_1(t) + B_\lambda y_2(t) + \lambda \int_0^t \frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]}{W(s)} f(s, y(s), py')ds \tag{2.5}$$

where $W(s)$ is the Wronskian of y_1 and y_2 at s and

$$B_\lambda = \frac{c_0 - A_\lambda Q_3}{Q_1} \text{ and } A_\lambda = \frac{c_0 Q_2 - c_1 Q_1 + \lambda Q_5}{Q_3 Q_2 - Q_4 Q_1}.$$

Here $Q_1 = -\alpha y_2(0) + \beta \lim_{t\to 0^+} p(t)y_2'(t)$, $Q_2 = ay_2(1) + b\lim_{t\to 1^-} p(t)y_2'(t)$, $Q_3 = -\alpha y_1(0) + \beta \lim_{t\to 0^+} p(t)y_1'(t)$ and $Q_4 = ay_1(1) + b\lim_{t\to 1^-} p(t)y_1'(t)$ with

$$Q_5 = aQ_1 \int_0^1 \frac{[y_1(s)y_2(1) - y_1(1)y_2(s)]}{W(s)} f(s, y(s), p(s)y'(s))ds$$

$$+ bQ_1 \int_0^1 \frac{[y_1(s)\lim_{t\to 1^-} p(t)y_2'(t) - y_2(s)\lim_{t\to 1^-} p(t)y_1'(t)]}{W(s)} f(s, y(s), p(s)y'(s))ds.$$

Remarks: ($\underline{i}$) Note $Q_3Q_2 - Q_4Q_1 \neq 0$. To see this, let $u(x) = Q_1 y_1(x) - Q_3 y_2(x)$. Notice $(pu')' + \tau pu = 0$ a.e. on $[0,1]$ and $-\alpha u(0) + \beta \lim_{t\to 0^+} p(t)u'(t) = 0$. If $Q_3Q_2 - Q_4Q_1 = 0$ then $au(1) + b\lim_{t\to 1^-} p(t)u'(t) = 0$. Consequently, $u \equiv 0$, i.e., $y_1(x) = \frac{Q_3}{Q_1} y_2(x)$, a contradiction. ($\underline{ii}$) Since $pW' + p'W = 0$ then $pW = $ constant.

We can rewrite (2.5) as

$$y(t) = \lambda\left(Cy_1(t) + Dy_2(t) + \int_0^t \frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]}{W(s)} f(s, y(s), py')ds\right)$$

$$+ (1-\lambda)[Ey_1(t) + Fy_2(t)]$$

where

$$F = \frac{c_0 - EQ_3}{Q_1},\ E = \frac{c_0Q_2 - c_1Q_1}{Q_3Q_2 - Q_4Q_1},\ C = \frac{Q_5 + c_0Q_2 - c_1Q_1}{Q_3Q_2 - Q_4Q_1} \text{ and } D = \frac{c_0 - CQ_3}{Q_1}.$$

Define the operator $N: K^1_{\mathfrak{B}} \to K^1_{\mathfrak{B}}$ by setting

$$Ny(t) = Cy_1(t) + Dy_2(t) + \int_0^t \frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]}{W(s)} f(s, y(s), py')ds.$$

Here $K^1_{\mathfrak{B}} = \{u \in C[0,1], pu' \in C[0,1]: u \in (SL) \text{ or } (N)\}$. Then $(2.4)_\lambda$ is equivalent to the fixed problem

$$y = \lambda Ny + (1-\lambda)p \tag{2.6}$$

where $p = Ey_1(t) + Fy_2(t)$. We **claim** that $N: K^1_{\mathfrak{B}} \to K^1_{\mathfrak{B}}$ is continuous and completely continuous. Let $u_n \to u$ in $K^1_{\mathfrak{B}}$, i.e., $u_n \to u$ and $pu'_n \to pu'$ uniformly on $[0,1]$. Thus there exists $r > 0$ with $|u_n(t)| \le r$, $|p(t)u'_n(t)| \le r$, $|u(t)| \le r$, $|p(t)u'(t)| \le r$ for $t \in [0,1]$. By the above uniform convergence we have $p(t)f(t, u_n(t), p(t)u'_n(t)) \to p(t)f(t, u(t), p(t)u'(t))$ pointwise a.e. on $[0,1]$. Also there exists an integrable function h_r with

$$|p(t)f(t, u_n(t), p(t)u'_n(t))| \le h_r(t) \text{ a.e. } t \in [0,1]. \tag{2.7}$$

Now

$$Nu_n(t) = Cy_1(t) + Dy_2(t) + \int_0^t \frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]}{W(s)} f(s, u_n(s), pu'_n ds$$

together with

$$p(t)(Nu_n)'(t) = Cp(t)y'_1(t)$$

$$+ Dp(t)y'_2(t) + \int_0^t \frac{[y_1(s)p(t)y'_2(t) - p(t)y'_1(t)y_2(s)]}{W(s)} f(s, u_n(s), pu'_n)ds$$

and the Lebesgue dominated convergence theorem implies that $Nu_n \to Nu$ and $p(Nu_n)' \to p(Nu)'$ pointwise for each $t \in [0,1]$. In fact, the convergence is uniform because of (2.7). Consequently, $Nu_n \to Nu$ in $K^1_{\mathfrak{B}}$ so N is continuous. To see that N is completely continuous, we use the Arzela-Ascoli theorem. To see this, let $\Omega \subseteq K^1_{\mathfrak{B}}$ be bounded, i.e., there exists a constant $M > 0$ with $\|y\|_1 \le M$ for each $y \in \Omega$. Also there exist constants C^* and D^* (which may depend on M) such that $|C| \le C^*$ and $|D| \le D^*$ for all $y \in \Omega$. The boundedness of $N\Omega$ is immediate and to see the equicontinuity on $[0,1]$ consider $y \in \Omega$ and t, $z \in [0,1]$. Then

$$|Ny(t) - Ny(z)| \le C^* |y_1(t) - y_1(z)| + D^* |y_2(t) - y_2(z)|$$

$$+ |y_2(t)| \left| \int_t^z \frac{y_1(s)}{W(s)} f(s, y(s), py')ds \right|$$

$$+ \mid y_2(t) - y_2(z) \mid \mid \int_0^z \frac{y_1(s)}{W(s)} f(s, y(s), py')ds \mid$$

$$+ \mid y_1(t) \mid \mid \int_t^z \frac{y_2(s)}{W(s)} f(s, y(s), py')ds \mid$$

$$+ \mid y_1(t) - y_1(z) \mid \mid \int_0^z \frac{y_2(s)}{W(s)} f(s, y(s), py')ds \mid$$

and

$$\mid p(t)(Ny)'(t) - p(z)(Ny)'(z) \mid \le C^* \mid p(t)y_1'(t) - p(z)y_1'(z) \mid$$

$$+ D^* \mid p(t)y_2'(t) - p(z)y_2'(z) \mid$$

$$+ \mid p(t)y_2'(t) \mid \mid \int_t^z \frac{y_1(s)}{W(s)} f(s, y(s), py')ds \mid$$

$$+ \mid p(t)y_2'(t) - p(z)y_2'(z) \mid \mid \int_0^z \frac{y_1(s)}{W(s)} f(s, y, py')ds \mid$$

$$+ \mid p(t)y_1'(t) \mid \mid \int_t^z \frac{y_2(s)}{W(s)} f(s, y(s), py')ds \mid$$

$$+ \mid p(t)y_1'(t) - p(z)y_1'(z) \mid \mid \int_0^z \frac{y_2(s)}{W(s)} f(s, y, py')ds \mid$$

so the equicontinuity of $N\Omega$ follows from the above inequalities. Thus $N: K^1_{\mathfrak{B}} \to K^1_{\mathfrak{B}}$ is completely continuous. Set

$$U = \{u \in K^1_{\mathfrak{B}}: \| u \|_1 < M_0 + 1\},\ K = K^1_{\mathfrak{B}} \text{ and } E = \{u \in C[0,1] \text{ with } pu' \in C[0,1]\}.$$

Then theorem 2.1 implies that N has a fixed point, i.e. (2.1) has a solution $y \in C[0,1]$ with $py' \in C[0,1]$. The fact that $py' \in AC[0,1]$ follows from (2.5) with $\lambda = 1$. □

We next consider the problem

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y = f(t, y, py') \text{ a.e. on } [0,1] \\ y \in (P). \end{cases} \tag{2.8}$$

Theorem 2.3: *Let $pf: [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be an L^1-Carathéodory function and assume (1.3) and (2.2) hold. In addition, suppose*

$$\begin{cases} \frac{1}{p}(py')' + \tau y = 0 \textit{ a.e. on } [0,1] \\ y \in (P) \end{cases}$$

has only the trivial solution. Also suppose there is a constant M_0, independent of λ, with $\| y \|_1 \leq M_0$ for any solution y to

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y = \lambda f(t,y,py') \text{ a.e. on } [0,1] \\ y \in (P) \end{cases} \tag{2.9$_\lambda$}$$

for each $\lambda \in (0,1)$. Then (2.8) has at least one solution.

Proof: Let y_1 and y_2 be two linearly independent solutions of $(py')' + \tau py = 0$ with $y_1, y_2 \in C[0,1]$ and $py_1', py_2' \in AC[0,1]$. Choose y_2 with $y_2(0) - y_2(1) \neq 0$.

If this is not possible, then the two linearly independent solutions are such that $y_2(0) - y_2(1) = y_1(0) - y_1(1) = 0$. Let

$$u(x) = [\lim_{t \to 0^+} p(t)y_2'(t) - \lim_{t \to 1^-} p(t)y_2'(t)]y_1(x) - [\lim_{t \to 0^+} p(t)y_1'(t) - \lim_{t \to 1^-} p(t)y_1'(t)]y_2(x)$$

so u satisfies $(pu')' + \tau pu = 0$ a.e. on $[0,1]$ with $u(0) = u(1)$ and $\lim_{t \to 0^+} p(t)u'(t) = \lim_{t \to 1^-} p(t)u'(t)$. Consequently, $u \equiv 0$, a contradiction since y_1 and y_2 are linearly independent.

Solving $(2.9)_\lambda$ is equivalent to finding a $y \in C[0,1]$ with $py' \in C[0,1]$ which satisfies (2.5) where

$$B_\lambda = \frac{A_\lambda[y_1(1) - y_1(0)] + \lambda I_2}{y_2(0) - y_2(1)}$$

and

$$A_\lambda = \frac{\lambda[I_2 + I_3]}{[y_2(0) - y_2(1)]I_0 - [y_1(1) - y_1(0)]I_1}.$$

Here $I_0 = \lim_{t \to 0^+} p(t)y_1'(t) - \lim_{t \to 1^-} p(t)y_1'(t), I_1 = \lim_{t \to 0^+} p(t)y_2'(t) - \lim_{t \to 1^-} p(t)y_2'(t)$ with

$$I_2 = \int_0^1 \frac{[y_1(s)y_2(1) - y_1(1)y_2(s)]}{W(s)} f(s, y(s), p(s)y'(s))ds$$

and

$$I_3 = [y_2(0) - y_2(1)] \int_0^1 \frac{[y_1(s)\lim_{t \to 1^-} p(t)y_2'(t) - y_2(s)\lim_{t \to 1^-} p(t)y_1'(t)]}{W(s)} f(s, y(s), py')ds.$$

Remark: Notice $[y_2(0) - y_2(1)]I_0 - [y_1(1) - y_1(0)]I_1 \neq 0$ for if not, then

$$u(x) = y_1(x) + \frac{[y_1(1) - y_1(0)]}{[y_2(0) - y_2(1)]} y_2(x)$$

satisfies $(pu')' + \tau pu = 0$ a.e. on $[0,1]$ with $u(0) = u(1)$ and $\lim_{t \to 0^+} p(t)u'(t) = \lim_{t \to 1^-} p(t)u'(t)$. Then $u \equiv 0$, a contradiction.

Essentially, the same reasoning as in theorem 2.2 establishes the result. □

Next consider the problem

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y = f(t, y, py') \text{ a.e. on } [0,1] \\ y \in (Br). \end{cases} \tag{2.10}$$

Theorem 2.4: *Let $pf: [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ be a Carathéodory function and assume (1.3) and (2.2) hold. In addition, suppose*

$$\begin{cases} \frac{1}{p}(py')' + \tau y = 0 \text{ a.e. on } [0,1] \\ y \in (Br)_0 \end{cases}$$

has only the trivial solution. Also, suppose there is a constant M_0, independent of λ, with $\| y \|_1 \leq M_0$ for any solution y to

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y = \lambda f(t, y, py') \text{ a.e. on } [0,1] \\ y \in (Br) \end{cases} \tag{2.11$_\lambda$}$$

for each $\lambda \in (0,1)$. Then (2.10) has at least one solution.

Proof: Let y_1 and y_2 be two linearly independent solutions of $(py')' + \tau py = 0$ with $y_1, y_2 \in C[0,1]$ and $py_1', py_2' \in AC[0,1]$. Choose y_2 with $y_2(0) \neq 0$. Solving $(2.11)_\lambda$ is equivalent to finding a $y \in C[0,1]$ with $py' \in C[0,1]$ which satisfies (2.5) where

$$B_\lambda = \frac{c_0 - A_\lambda y_1(0)}{y_2(0)}$$

and

$$C = \int_0^1 \frac{ds}{p(s)} \lim_{t \to 1^-} p(t)y_1'(t) - \left(\frac{y_1(0)}{y_2(0)}\right) \int_0^1 \frac{ds}{p(s)} \lim_{t \to 1^-} p(t)y_2'(t) - y_1(1) + \left(\frac{y_1(0)}{y_2(0)}\right) y_2(1)$$

with

$$A_\lambda = \frac{1}{C}(c_1 + \int_0^1 \frac{ds}{p(s)}[\lambda \int_0^1 \frac{[y_1(s)\lim_{t \to 1^-} p(t)y_2'(t) - y_2(s)\lim_{t \to 1^-} p(t)y_1'(t)]}{W(s)} f(s, y(s), py')ds$$

$$- \frac{c_0 \lim_{t \to 1^-} p(t)y_2'(t)}{y_2(0)}] + \frac{c_0 y_2(0)}{y_2(0)} + \lambda \int_0^1 \frac{[y_1(s)y_2(1) - y_2(s)y_1(1)]}{W(s)} f(s, y(s), p(s)y'(s))ds).$$

Remark: Notice $C \neq 0$. To see this, let

$$u(x) = y_1(x) - \left(\frac{y_1(0)}{y_2(0)}\right) y_2(x)$$

so $C = \int_0^1 \frac{ds}{p(s)} \lim_{t \to 1^-} p(t)u'(t) - u(1)$. Now $(pu')' + \tau pu = 0$ a.e. on $[0,1]$ with $u(0) = 0$. If $C = 0$ then, $\int_0^1 \frac{ds}{p(s)} \lim_{t \to 1^-} p(t)u'(t) - u(1) = 0$. Consequently, $u \equiv 0$, a contradiction.

Essentially the same reasoning as in theorem 2.2 establishes the result. $\square$

Of course, more general forms of theorems 2.2, 2.3 and 2.4 are immediately available for us for the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y + \sigma(t)py' = f(t, y, py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N) \text{ or } (P) \text{ or } (Br). \end{cases} \tag{2.12}$$

Theorem 2.5: *Let* $pf: [0,1] \times \mathbf{R} \to \mathbf{R}$ *be an* L^1*-Carathéodory function and assume* (1.3) *and* (2.2) *hold with* σ *satisfying*

$$\sigma \in L^1_p[0,1]. \tag{2.13}$$

In addition, suppose

$$\begin{cases} \frac{1}{p}(py')' + \tau y + \sigma p y' = 0 \text{ a.e. on } [0,1] \\ y \in (SL)_0 \text{ or } (N)_0 \text{ or } (P) \text{ or } (Br)_0 \end{cases}$$

has only the trivial solution. Also suppose there is a constant M_0, *independent of* λ, *with* $\| y \|_1 \le M_0$ *for any solution* y *to*

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y + \sigma(t)py' = \lambda f(t,y,py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N) \text{ or } (P) \text{ or } (Br) \end{cases} \tag{2.14$_\lambda$}$$

for each $\lambda \in (0,1)$. *Then* (2.12) *has at least one solution.*

Proof: Essentially the same reasoning as in theorems 2.2, 2.3 and 2.4 establishes the result. □

3. Existence Theory

We begin by establishing an existence result for the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y = f(t,y,py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N) \text{ or } (P) \text{ or } (Br). \end{cases} \tag{3.1}$$

Theorem 3.1: *Let* $pf: [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ *be an* L^1*-Carathéodory function and assume* (1.3) *and* (2.2) *hold. In addition, suppose*

$$\begin{cases} \frac{1}{p}(py')' + \tau y = 0 \text{ a.e. on } [0,1] \\ y \in (SL)_0 \text{ or } (N)_0 \text{ or } (P) \text{ or } (Br)_0 \end{cases} \tag{3.2}$$

has only the trivial solution. Let $f(t,u,v) = nv + g(t,u,v)$ *and assume*

$$n \in L^1_p[0,1] \tag{3.3}$$

$$\begin{cases} pg \text{ is an } L^1\text{-Carathéodory function and} \\ |g(t,u,v)| \le \phi_1(t) + \phi_2(t)|u|^\gamma + \phi_3(t)|v|^\theta \\ \text{for a.e. } t \in [0,1], \text{ for constants } \gamma, \theta \text{ with } 0 \le \gamma,\ \theta < 1 \\ \text{and functions } \phi_i \in L^1_p[0,1],\ i = 1,2,3 \end{cases} \tag{3.4}$$

and

$$\left\{\begin{array}{c} \sup_{t\in[0,1]} \int_0^1 \mid p(t)G_t(t,s)n(s) \mid ds < 1. \\ \textit{Here } G(t,s) \textit{ is the Green's function associated with} \\ (py') + p\tau y = 0 \textit{ a.e. on } [0,1] \textit{ with } y \in (SL) \textit{ or } (N) \textit{ or } (P) \textit{ or } (Br) \end{array}\right. \tag{3.5}$$

hold. Then (3.1) *has at least one solution.*

Remark: Since $pW' + p'W = 0$ then $\sup_{t\in[0,1]} \mid p(t)G_t(t,s) \mid \le E_0 p(s)$ for some constant E_0.

Proof: Let y be a solution to

$$\left\{\begin{array}{l} \frac{1}{p}(py')' + \tau(t)y = \lambda f(t,y,py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N) \text{ or } (P) \text{ or } (Br) \end{array}\right. \tag{3.6$_\lambda$}$$

for $0 < \lambda < 1$. Then

$$y(t) = y_3(t) + \int_0^1 G(t,s)f(s,y(s),p(s)y'(s))ds, \quad t \in [0,1] \tag{3.7}$$

where y_3 is the unique solution of $(py')' + p\tau y = 0$ a.e. on $[0,1]$ with $y \in (SL)$ or (N) or (P) or (Br) and $G(t,s)$ is as described in (3.5). Also notice

$$p(t)y'(t) = p(t)y_3'(t) + \int_0^1 p(t)G_t(t,s)f(s,y(s),p(s)y'(s))ds. \tag{3.8}$$

Now (3.4) together with (3.7) yields

$$\begin{aligned} \mid y \mid_0 \equiv \sup_{[0,1]} \mid y(t) \mid &\le \sup_{[0,1]} \mid y_3(t) \mid \\ &+ \mid py' \mid_0 \sup_{t\in[0,1]} \int_0^t \mid G(t,s)n(s) \mid ds + \sup_{t\in[0,1]} \int_0^t \mid G(t,s)\phi_1(s) \mid ds \\ &+ \mid y \mid_0^\gamma \sup_{t\in[0,1]} \int_0^t \mid G(t,s)\phi_2(s) \mid ds + \mid py' \mid_0^\theta \sup_{t\in[0,1]} \int_0^t \mid G(t,s)\phi_3(s) \mid ds \end{aligned}$$

so there exist constants A_0, A_1, A_2 and A_3 with

$$\mid y \mid_0 \le A_0 + A_1 \mid py' \mid_0 + A_2 \mid y \mid_0^\gamma + A_3 \mid py' \mid_0^\theta. \tag{3.9}$$

Also there exists a constant $A_4 > 0$ with

$$A_2 x^\gamma \le \frac{1}{2}x + A_4 \text{ for all } x > 0.$$

Putting this into (3.9) yields

$$\mid y \mid_0 \le 2(A_0 + A_4) + 2A_1 \mid py' \mid_0 + 2A_3 \mid py' \mid_0^\theta. \tag{3.10}$$

Also (3.8) implies that there are constants A_5, A_6, and A_7 with

$$|py'|_0 \leq A_5 + |py'|_0 \left(\sup_{t \in [0,1]} \int_0^t |p(t)G_t(t,s)n(s)|\, ds \right) + A_6 |y|_0^{\gamma} + A_7 |py'|_0^{\theta}. \tag{3.11}$$

Put (3.10) into (3.11) to obtain

$$|py'|_0 \leq A_5 + |py'|_0 \left(\sup_{t \in [0,1]} \int_0^t |p(t)G_t(t,s)n(s)|\, ds \right)$$

$$+ A_7 |py'|_0^{\theta} + A_6(2(A_0 + A_4) + 2A_1 |py'|_0 + 2A_3 |py'|_0^{\theta})^{\gamma}$$

and so there exist constants A_8, A_9, A_{10}, and A_{11} with

$$\left(1 - \sup_{t \in [0,1]} \int_0^t |p(t)G_t(t,s)n(s)|\, ds \right) |py'|_0$$

$$\leq A_8 + A_9 |py'|_0^{\gamma} + A_{10} |py'|_0^{\gamma\theta} + A_{11} |py'|_0^{\theta}.$$

Consequently there exists a constant M_0^*, independent of λ, with $|py'|_0 \leq M_0^*$. This together with (3.10) yields the existence of a constant M_0^{**} with $|y|_0 \leq M_0^{**}$. Let $M_0 = max\{M_0^*, M_0^{**}\}$ and this together with either theorems 2.2, 2.3 or 2.4 establishes the result. □

Consider the Sturm Liouville eigenvalue problem

$$\begin{cases} Lu = \lambda u \text{ a.e. on } [0,1] \\ u \in (SL)_0 \text{ or } (N)_0 \text{ or } (P) \text{ or } (Br)_0 \end{cases} \tag{3.12}$$

where $Lu = -\frac{1}{pq(t)}[(pu')' + r(t)pu]$, with p satisfying (1.3) and

$$r, q \in L_p^1[0,1] \text{ with } q > 0 \text{ a.e. on } [0,1]. \tag{3.13}$$

Then L has a countably infinite number of real eigenvalues (see section 4) and it is possible to estimate these eigenvalues numerically [3].

Theorem 3.1 immediately yields an existence result for

$$\begin{cases} \frac{1}{p}(py')' + r(t)y + \mu q(t)y = f(t,y,py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N) \text{ or } (P) \text{ or } (Br) \end{cases} \tag{3.14}$$

where μ is **not** an eigenvalue of (3.12).

Theorem 3.2: Let $pf:[0,1] \times \mathbf{R}^2 \rightarrow \mathbf{R}$ *be an* L^1*-Carathéodory function and assume* (1.3) *and* (3.13) *hold. Let* $f(t,u,v) = nv + g(t,u,v)$ *and assume* (3.3), (3.4) *and* (3.5), *with* $\tau(t) = r(t) + \mu q(t)$, *are satisfied. Then* (3.14) *has at least one solution.*

Proof: Let $\tau(t) = r(t) + \mu q(t)$ in theorem 3.1. $\square$

Next in this section we obtain an existence result for the boundary value problem

$$\begin{cases} \frac{1}{p}(py')' + \tau(t)y + \sigma(t)py' = g(t, y, py') \text{ a.e. on } [0,1] \\ y \in (SL) \text{ or } (N) \text{ or } (P) \text{ or } (Br). \end{cases} \tag{3.15}$$

Theorem 3.3: *Let* $pg: [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ *be an* L^1*-Carathéordory function and assume* (1.3), (2.2), (2.13) *and* (3.4) *hold. In addition, suppose*

$$\begin{cases} \frac{1}{p}(py')' + \tau y + \sigma py' = 0 \text{ a.e. on } [0,1] \\ y \in (SL)_0 \text{ or } (N)_0 \text{ or } (P) \text{ or } (Br)_0 \end{cases}$$

has only the trivial solution. Then (3.15) *has at least one solution.*

Proof: Essentially the same argument as in theorem 3.1 (except easier) yields the result. $\square$

Remark: We remark here that theorem 3.2 seems to be the most applicable result in this paper since it is possible to estimate numerically [3] the eigenvalues of (3.12).

Finally we obtain a more subtle existence result for

$$\begin{cases} \frac{1}{p}(py')' + \tau y = g(t, y, py')y + h(t, y, py') \equiv f(t, y, py') \text{ a.e. on } [0,1] \\ y(0) = y(1) = 0. \end{cases} \tag{3.16}$$

Theorem 3.4: *Let* $pg, ph: [0,1] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ *be* L^1*-Carathéodory functions and assume* (1.3) *and* (2.2) *hold. In addition, suppose* (2.3), *with* $\beta = b = 0$, *has only the trivial solution. Also assume*

$$\begin{cases} |h(t,u,v)| \leq \phi_1(t) + \phi_2(t)|u|^{\gamma} + \phi_3(t)|v|^{\theta} \\ \text{for a.e. } t \in [0,1] \text{ with } 0 \leq \gamma,\ \theta < 1 \end{cases} \tag{3.17}$$

$$\begin{cases} \text{there exist } \tau_1, \tau_2 \in L^1_p[0,1] \text{ with } \tau_1(t) \leq g(t,u,v) \leq \tau_2(t) \text{ for a.e.} \\ t \in [0,1]; \text{ here } \tau_1 \leq 0 \text{ for a.e. } t \in [0,1] \text{ and } \tau_2 \geq 0 \text{ for a.e. } t \in [0,1] \end{cases} \tag{3.18}$$

$$\phi_1, \phi_2 \in L^1_p[0,1] \text{ with } \int_0^1 [p(t)]^{\frac{2+\theta}{2-\theta}} [\phi_3(t)]^{\frac{2}{2-\theta}} dt < \infty \tag{3.19}$$

and

$$\begin{cases} W^{1,2}_p[0,1] = \Omega \oplus \Gamma \text{ where } \Omega \subseteq K^* \text{ is finite dimensional and for every} \\ 0 \neq y = u + v \in K^* \text{ with } u \in \Omega, v \in \Gamma \text{ we have } R(y) > 0;\ \Gamma = \Omega^{\perp} \end{cases} \tag{3.20}$$

hold; here

$$R(y) = \int_0^1 [p(v')^2 - (\tau - \tau_1)pv^2]dt - \int_0^1 [p(u')^2 - (\tau - \tau_2)pu^2]dt$$

and

$$K^* = \{w:[0,1]\rightarrow \boldsymbol{R}\text{: } w \in AC[0,1] \text{ with } w' \in L_p^2[0,1] \text{ and } w(0) = w(1)\}.$$

Then (3.16) *has at least one solution.*

Remark: (*i*) In (3.20) we have $y = u + v$ with $u \in \Omega$, $v \in \Gamma$ so $\int_0^1 puv\,dt + \int_0^1 pu'v'dt = 0$.

(*ii*) For notational purposes, let $\| u \|_p = (\int_0^1 p \mid u \mid^2 dt)^{\frac{1}{2}}$.

(*iii*) Recall by $W_p^{1,2}[0,1]$ we mean the space of functions $u \in AC[0,1]$ with $u' \in L_p^2[0,1]$ and with norm

$$\| u \|_* = \left(\int_0^1 p \mid u \mid^2 dt + \int_0^1 p \mid u' \mid^2 dt \right)^{\frac{1}{2}}.$$

Proof: First recall Lemma 2.8 in [7] implies there exists $\epsilon > 0$ with

$$R(y) \geq \epsilon\left(\| y \|_p^2 + \| y' \|_p^2 \right) \tag{3.21}$$

for any $y \in K^*$; here $y = u + v$ with $u \in \Omega$ and $v \in \Gamma$. Let $y(= u + v)$ be a solution to

$$\begin{cases} \frac{1}{p}(py')' + \tau y = \lambda f(t, y, py') \text{ a.e. on } [0,1] \\ y(0) = y(1) = 0 \end{cases} \tag{3.22}$$

for some $0 < \lambda < 1$. Then

$$-\int_0^1 (v-u)[(py')' + p\tau y]dt = -\lambda \int_0^1 p(v-u)yg(t,y,py')dt$$
$$-\lambda \int_0^1 p(v-u)h(t,y,py')dt$$

and so integration by parts yields

$$\begin{aligned} &\int_0^1 [p(v')^2 + pv^2(-\tau + \lambda g(t,y,py'))]dt \\ &- \int_0^1 [p(u')^2 + pu^2(-\tau + \lambda g(t,y,py'))]dt \\ &\leq \int_0^1 p \mid v - u \mid \mid h(t,y,py') \mid dt. \end{aligned} \tag{3.23}$$

Also

$$pv^2[-\tau + \lambda g(t,y,py')] = pv^2[-(\tau - \tau_1) + \lambda g(t,y,py') - \tau_1]$$

$$\geq pv^2[-(\tau-\tau_1)+(\lambda-1)\tau_1] \geq -p(\tau-\tau_1)v^2 \text{ a.e. on } [0,1].$$

Similarly

$$pu^2[-\tau+\lambda g(t,y,py')] \leq -p(\tau-\tau_2)u^2 \text{a.e. on } [0,1].$$

Putting this into (3.23) yields

$$R(y) \leq \int_0^1 p\,|\,v-u\,|\,|\,h(t,y,py')\,|\,dt.$$

This together with (3.21) implies that there is an $\epsilon > 0$ with

$$\epsilon\Big(\|y\|_p^2 + \|y'\|_p^2\Big) \leq \int_0^1 p\,|\,v-u\,|\,|\,h(t,y,py')\,|\,dt. \tag{3.24}$$

We also have $\|v-u\|_p^2 + \|v'-u'\|_p^2 = \|y\|_p^2 + \|y'\|_p^2$. Now Sobolev's inequality (since we have the imbedding $W_p^{1,2}[0,1] \rightarrow C[0,1]$) implies

$$\int_0^1 p\phi_1\,|\,v-u\,|\,dt \leq |\,v-u\,|_0 \int_0^1 p\phi_1 dt \leq F_1(\|v-u\|_p^2 + \|v'-u'\|_p^2)^{\frac{1}{2}}$$

for some constant F_1. Thus

$$\int_0^1 p\phi_1\,|\,v-u\,|\,dt \leq F_1(\|y\|_p + \|y'\|_p). \tag{3.25}$$

Also

$$\int_0^1 p\phi_2\,|\,v-u\,|\,|\,y\,|^{\gamma}dt \leq |\,v-u\,|_0\,|\,y\,|_0^{\gamma} \int_0^1 p\phi_2 dt$$

$$\leq F_2\Big(\|v-u\|_p^2 + \|v'-u'\|_p^2\Big)^{\frac{1}{2}}\Big(\|y\|_p^2 + \|y'\|_p^2\Big)^{\frac{\gamma}{2}}$$

for some constant F_2. Thus there exists a constant F_3 with

$$\int_0^1 p\phi_2\,|\,v-u\,|\,|\,y\,|^{\gamma}dt \leq F_3\Big(\|y\|_p^{\gamma+1} + \|y'\|_p^{\gamma+1}\Big). \tag{3.26}$$

Finally Hölder's inequality implies that there is a constant F_4 with

$$\int_0^1 p\phi_3\,|\,v-u\,|\,|\,py'\,|^{\theta}dt \leq |\,v-u\,|_0 \|y'\|_p^{\theta}\left(\int_0^1 [p(t)]^{\frac{2+\theta}{2-\theta}}[\phi_3(t)]^{\frac{2}{2-\theta}}dt\right)^{\frac{2-\theta}{2}}$$

$$\leq F_4\|y'\|_p^{\theta}\Big(\|v-u\|_p^2 + \|v'-u'\|_p^2\Big)^{\frac{1}{2}}.$$

Thus there exists a constant F_5 with

$$\int_0^1 p\phi_3 \mid v-u \mid \mid py' \mid^\theta dt \leq F_5\Big(\| y \|_p^{\theta+1} + \| y' \|_p^{\theta+1}\Big). \tag{3.27}$$

Put (3.25), (3.26) and (3.27) into (3.24) and since $\theta, \gamma < 1$ there exists a constant F_6 with

$$\tfrac{\epsilon}{2}\Big(\| y \|_p^2 + \| y' \|_p^2\Big) \leq F_6. \tag{3.28}$$

Thus for $t \in [0,1]$,

$$\mid y(t) \mid \leq \int_0^1 \mid y'(s) \mid ds \leq \| y' \|_p \left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}}$$

and this together with (3.28) implies that there exists a constant F_7, independent of λ, with

$$\mid y \mid_0 = \sup_{[0,1]} \mid y(t) \mid \leq F_7,\ \| y \|_p \leq F_7,\ \| y' \|_p \leq F_7 \tag{3.29}$$

for any solution y to $(3.22)_\lambda$.

This bound together with $(3.22)_\lambda$ implies there exist a constant F_8 with

$$\int_0^1 \mid (py')' \mid dt \leq F_8 + \int_0^1 p\phi_3 \mid py' \mid^\theta dt.$$

Hölder's inequality implies there exist a constant F_9 with

$$\int_0^1 \mid (py')' \mid dt \leq F_8 + F_9 \| y' \|_p^\theta \leq F_8 + F_9 F_7^\theta \equiv F_{10}.$$

Also there exist $t_0 \in (0,1)$ with $y'(t_0) = 0$ so

$$\mid p(t)y'(t) \mid \leq \Big| \int_{t_0}^t \mid (py')' \mid dt \Big| \leq F_{10}.$$

Thus

$$\sup_{(0,1)} \mid p(t)y'(t) \mid \leq F_{10}. \tag{3.30}$$

Theorem 2.2 together with (3.29) and (3.30) completes the proof. □

4. Appendix-Eigenvalues

We now use the ideas of section 2 and results on compact self adjoint operators to give a unified treatment of the Sturm Liouville eigenvalue problem.

In particular, consider

$$\begin{cases} Lu = \lambda u \text{ a.e. on } [0,1] \\ u \in (SL)_0 \text{ or } (N)_0 \text{ or } (P) \text{ or } (Br)_0 \end{cases} \tag{4.1}$$

where $Lu = -\frac{1}{pq(t)}[(pu')' + r(t)pu]$ and assume (1.3) and (3.13) hold. We first show that there exists $\lambda^* \in \mathbf{R}$ such that λ^* is **not** an eigenvalue of (4.1).

Remark: If $r \equiv 0$ and $y \in (SL)_0$, then $\lambda^* = 0$ will work whereas if $r \equiv 0$ and $y \in (N)_0$ or (P) or $(Br)_0$, then $\lambda^* = -1$ will work.

Let

$$D(L) = \{w \in C[0,1]: w, pw' \in AC[0,1] \text{ with } w \in (SL)_0 \text{ or } (N)_0 \text{ or } (P) \text{ or } (Br)_0\}$$

and notice that

$$L: D(L)\Big(\subseteq C[0,1] \subseteq L^2_{pq}[0,1]\Big) \rightarrow L^1_{pq}[0,1].$$

Theorem 4.1: *Let u_n be a sequence in $D(L)$. Suppose $u \in C[0,1]$ and $y \in L^1_{pq}[0,1]$ be such that $u_n \rightarrow u$ in $C[0,1]$ and $Lu_n \rightarrow y$ in $L^1_{pq}[0,1]$. Thus $u \in D(L)$ and $Lu = y$ a.e. on $[0,1]$.*

Proof: Let $\| \cdot \|_{L^1}$, $\| \cdot \|_{L^1_{pq}}$ and $| \cdot |_0$ denote the usual norms in $L^1[0,1]$, $L^1_{pq}[0,1]$ and $C[0,1]$ respectively. For $n = 1, 2, \ldots$ there exist constants C_1 and C_2 independent of n with

$$\| (pu_n')' + pru_n \|_{L^1} = \| Lu_n \|_{L^1_{pq}} \le C_1 \text{ and } \| (pu_n')' \|_{L^1} \le C_2.$$

This together with the boundary condition implies that there exists a constant C_3 independent of n with

$$| pu_n' |_0 \le C_3.$$

Consequently Sobolev's imbedding theorem [1] guarantees the existence of a subsequence S of integers with $pu_n' \rightarrow pu'$ in $C[0,1]$ as $n \rightarrow \infty$ in S. For $x \in (0,1)$ and $n \in S$ we have

$$p(x)u_n'(x) = \lim_{t \rightarrow 0^+} p(t)u_n'(t) + \int_0^x \Big(p(s)u_n'(s)\Big)'ds$$

and this together with the fact that $(pu_n')' + pru_n \rightarrow -pqy$ in $L^1[0,1]$ implies

$$p(x)u_n'(x) = \lim_{t \rightarrow 0^+} p(t)u_n'(t) + \int_0^x p(s)[-q(s)y(s) - r(s)u(s)]ds.$$

Thus $u \in D(L)$ and $(pu')' + pru = -pqy$ a.e. on $[0,1]$. □

Theorem 4.2: *The eigenvalues, λ, of the eigenvalue problem (4.1) are real and the eigenfunctions corresponding to the distinct eigenvalues of (4.1) are orthogonal in $L^2_{pq}[0,1]$. In addition, the eigenvalues form at most a countable set with no finite limit point.*

Proof: Suppose $\lambda_0 \in \mathbf{R}$ is a limit point of the set of eigenvalues of (4.1). Then there exists a distinct sequence $\{\lambda_n\}$, $\lambda_n \neq \lambda_0$, of eigenvalues of (4.1) with $\lambda_n \rightarrow \lambda_0$. Let ϕ_n denote the eigenfunction for (4.1) corresponding to λ_n and with $| \phi_n |_0 = 1$ for all n. Now ϕ_n satisfies the boundary condition and of course $L\phi_n = \lambda_n \phi_n$ a.e. on $[0,1]$. Thus

$$\lambda_n \int_0^1 pq | \phi_n |^2 dt = A_0 + \int_0^1 p | \phi_n' |^2 dt - \int_0^1 pr | \phi_n |^2 dt$$

where

$$A_0 = \begin{cases} \frac{a}{b} \mid \phi_n(1) \mid^2 + \frac{\alpha}{\beta} \mid \phi_n(0) \mid^2, \ \phi_n \in (SL)_0 \\ \\ 0, \ \phi_n \in (N)_0 \text{ or } (P) \\ \\ \dfrac{- \mid \phi_n(1) \mid^2}{\int_0^1 \frac{ds}{p(s)}}, \ \phi_n \in (Br)_0. \end{cases}$$

This together with $\mid \phi_n \mid_0 = 1$ implies that there is a constant C_4 independent of n with

$$\int_0^1 p \mid \phi_n' \mid^2 dt \leq C_4.$$

Consequently, $\{\phi_n\}$ is bounded and equicontinuous on $[0,1]$ so there exists a $\phi \in C[0,1]$ and a subsequence S of integers with $\phi_n \to \phi$ in $C[0,1]$ as $n \to \infty$ in S. Let $n \in S$ and notice $L\phi_n = \lambda_n \phi_n$ a.e. on $[0,1]$ implies $L\phi_n \to \lambda_0 \phi$ in $L^1_{pq}[0,1]$. Then theorem 4.1 implies $\phi \in D(L)$ and $L\phi = \lambda_0 \phi$ a.e. on $[0,1]$. Notice also $\mid \phi \mid_0 = 1$ and so λ_0 is an eigenvalue of L. Now

$$\langle \phi_n, \phi \rangle = \int_0^1 pq\phi_n \bar{\phi}\, dt = 0 \text{ for all } n \in S$$

so $\int_0^1 pq \mid \phi \mid^2 dt = 0$. Consequently, $\phi(x) = 0$ a.e. on $[0,1]$, a contradiction. □

Now theorem 4.2 implies that there exists $\lambda^* \in \mathbf{R}$ such that λ^* is **not** an eigenvalue of (4.1). Assume without loss of generality for the remainder of this section that 0 is **not** an eigenvalue of (4.1).

Let y_1 and y_2 be two linearly independent solutions of $(py')' + rpy = 0$ a.e. on $[0,1]$ with $y_1, y_2 \in C[0,1]$ and $py_1', py_2' \in AC[0,1]$.

Remark: If $(SL)_0$ or $(N)_0$ is considered, choose y_2 as in theorem 2.2. For (P) choose y_2 as in theorem 2.3 whereas for $(Br)_0$ choose y_2 as in theorem 2.4.

Now for any $h \in L^1_{pq}[0,1]$ the boundary value problem

$$\begin{cases} Lu = h \text{ a.e. on } [0,1] \\ u \in (SL)_0 \text{ or } (N)_0 \text{ or } (P) \text{ or } (Br)_0 \end{cases}$$

has a unique solution

$$L^{-1}h(t) = u(t) = A_h y_1(t) + B_h y_2(t) + \int_0^t \frac{[y_1(s)y_2(t) - y_1(t)y_2(s)]}{W(s)} q(s)h(s)ds$$

where A_h and B_h may be constructed as in theorems 2.2, 2.3 or 2.4; see [14, 16]. It follows immediately that

$$L^{-1}: L^1_{pq}[0,1] \to D(L) \subseteq C[0,1] \subseteq L^2_{pq}[0,1].$$

The Arzela-Ascoli theorem (see [14, 16] or the ideas in theorem 2.2) implies that L^{-1} is **completely continuous.** Next, define the imbedding $j: L^2_{pq}[0,1] \to L^1_{pq}[0,1]$ by $ju = u$. Note j is continuous since Hölder's inequality yields

$$\int_0^1 pq\,|u|\,dt \le \left(\int_0^1 pq\,|u|^2 dt\right)^{\frac{1}{2}} \left(\int_0^1 pq dt\right)^{\frac{1}{2}}.$$

Consequently,

$$L^{-1}j: L^2_{pq}[0,1] \to D(L) \subseteq L^2_{pq}[0,1]$$

is completely continuous. In addition, [14, 16], for $u, v \in L^2_{pq}[0,1]$ it is easy to check that

$$\langle L^{-1}ju, v\rangle = \langle u, L^{-1}jv\rangle.$$

The spectral theorem for compact self adjoint operators [18] implies that L has an countably infinite number of real eigenvalues λ_i with corresponding eigenfunctions $u_i \in D(L)$. Of course

$$\lambda_i = \frac{A_0 + \int_0^1 p\,|u_i'|^2 dt - \int_0^1 pr\,|u_i|^2 dt}{\int_0^1 pq\,|u_i|^2 dt}$$

where

$$A_0 = \begin{cases} \frac{a}{b}\,|u_i(1)|^2 + \frac{\alpha}{\beta}\,|u_i(0)|^2, & u_i \in (SL)_0 \\ 0, & u_i \in (N)_0 \text{ or } (P) \\ \dfrac{-\,|u_i(1)|^2}{\int_0^1 \frac{ds}{p(s)}}, & u_i \in (Br)_0. \end{cases}$$

Remark: Notice that $\lambda_i \ge \dfrac{-\int_0^1 pr\,|u_i|^2 dt}{\int_0^1 pq\,|u_i|^2 dt}$. This is clear in all cases except maybe when $u_i \in (Br)_0$. However if $u_i \in (Br)_0$, then

$$u_i(1) = \int_0^1 \sqrt{p(s)}u_i'(s)\frac{1}{\sqrt{p(s)}}ds \le \left(\int_0^1 p(s)\,|u_i'(s)|^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}}$$

and the result follows.

Now the eigenfunctions u_i may be chosen so that they form a orthonormal set. We may also arrange the eigenvalues so that

$$\lambda_0 < \lambda_1 < \lambda_2 < \ldots .$$

In addition, [18, pp. 282, 373] implies that the set of eigenfunctions u_i form a basis for $L^2_{pq}[0,1]$ and if $h \in L^2_{pq}[0,1]$ then h has a Fourier series representation and h satisfies Parseval's equality, i.e.,

$$h = \sum_{i=0}^{\infty} \langle h, u_i \rangle u_i \text{ and } \int_0^1 pq \, | \, h \, |^2 dt = \sum_{i=0}^{\infty} | \langle h, u_i \rangle |^2.$$

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