Journal of Applied Mathematics and Stochastic Analysis
7, Number 3, 1994, 423-436.

LARGE DEVIATIONS FOR UNBOUNDED ADDITIVE FUNCTIONALS OF A MARKOV PROCESS WITH DISCRETE TIME (NONCOMPACT CASE)

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(Received February, 1993; revised June, 1993)

ABSTRACT

We combine the Donsker and Varadhan large deviation principle (l.d.p) for the occupation measure of a Markov process with certain results of Deuschel and Stroock, to obtain the l.d.p. for unbounded functionals. Our approach relies on the concept of exponential tightness and on the Puhalskii theorem. Three illustrative examples are considered.

Key words: Exponential Tightness, Large Deviations, Contraction Principle.

AMS (MOS) subject classifications: 60F10.

1. Introduction and Main Result

Consider an ergodic Markov process $\xi = (\xi_k)_{k \geq 0}$ having R as its state space, $\lambda_0(dx)$ as the distribution of the initial point ξ_0, and $\lambda = \lambda(dx)$ as the invariant measure. The transition probability $\pi(x, dy)$ is assumed to satisfy the Feller condition.

From the application point of view, it is interesting to get the large deviations for functionals of the type $\frac{1}{n} \sum_{k=0}^{n-1} g(\xi_k)$, for $n \geq 1$, with a continuous unbounded function $g = g(x)$. There exist different ways of solving this problem (see Gärtner [8], Dueschel and Stroock [3], Veretennikov [14], Acosta [1], Ellis [6], Orey and Pelikan [11], Freidlin and Wentzell [7]).

Here we combine the results of Deuschel and Stroock [3] and Donsker and Varadhan [5], and we use the representation

$$\frac{1}{n} \sum_{k=1}^{n} g(\xi_k) = \int_R g(x)\pi_n(dx) = (M_g(\pi_n)), \tag{1.1}$$

where $\pi_n(dx)$ is the empirical distribution

$$\pi_n(A) = \frac{1}{n} \sum_{k=1}^{n} I(\xi_k \in A).$$

$\pi_n(A)$ has been named the occupation measure by Donsker and Varadhan [4].

Assume the family $\{\pi_n, n \geq 1\}$ obeys the l.d.p. in the metric space (S, ρ) (S is the set of probability measures on R and ρ is the Levy-Prohorov metric) with a rate function $J(\mu)$, for $\mu \in S$. If $g = g(x)$ is a bounded continuous function, then $M_g(\nu) = \int g(x)\nu(dx)$. Here $\nu \in S$ defines a mapping continuous in the metric ρ; and the l.d.p. in (R, r) (r is the Euclidian metric) is implied by Varadhan's contraction principle [13] with a rate function

$$I_g(h) = \inf_{\mu \in \mathfrak{U}} J(\mu), \ \mathfrak{U} = \{\mu \in S: M_g(\mu) = y\} \text{ and } \inf\{\emptyset\} = \infty. \tag{1.2}$$

Deuschel and Stroock [3] have shown that under certain conditions this result remains valid for an unbounded function $g = g(x)$.

In this paper we give sufficient conditions for the sequence $\{\frac{1}{n}\sum_{k=0}^{n-1} g(\xi_k), n \geq 1\}$ to obey the l.d.p. in terms of $\lambda_0(dx)$, $\pi(x, dy)$, and $g(x)$. In view of (1.1) we need the l.d.p. for the family $\{\pi_n, n \geq 1\}$. In the noncompact case with a fixed initial point, $\xi_0 = x_0$, the l.d.p. has been proved by Donsker and Varadhan [5] under the following three assumptions.

(H^*) There is a nonnegative measurable function $v(x)$ such that $\sup_{|x| \leq N} v(x) < \infty$ for all $N > 0$. Furthermore, the function

$$w(x) = \ln \frac{v(x)}{\int_R e^{v(y)}\pi(x, dy)}$$

satisfies the following conditions:

$$\inf_{x \in R} w(x) = w_* > -\infty \text{ and } \lim_{l \to \infty} \inf_{|x| \geq l} [w(x) - w_*] = \infty.$$

(RM) There is a σ-finite reference measure $l = l(dx)$ such that

$$\pi(x, dy) = p(y \mid x) l(dx)$$

and

$$p(y \mid x) > 0, \ \forall x \in R \ \ l - a.s.$$

The rate function $J = J(\mu)$, for $\mu \in S$, is given by the formula

$$J(\mu) = \sup_{u \in \mathcal{N}} \int_R \ln \frac{e^{u(x)}}{\int_R e^{u(y)}\pi(x, dy)} \mu(dx) \tag{1.3}$$

($\mathcal{N}$ is a set of continuous finite-supported functions).

Since in our setting the initial point ξ_0 has the distribution λ_0, we add one more assumption.

(H_0) The function $v(x)$ from (H^*) is such that

$$\exists b > 0: \int_R e^{bv(x)}\lambda_0(dx) < \infty.$$

We show that the Donsker and Varadhan l.d.p. for $\{\pi_n, n \geq 1\}$ remains valid under these three assumptions with the same rate function (see Theorem 2 in the Appendix). The lower-bound part of this theorem is a simple generalization of the Donsker and Varadhan l.d.p. obtained by averaging with respect to λ_0. The proof of the upper-bound part is somewhat different. We show that (H^*) and (H_0) imply the exponential tightness of the family $\{\pi_n, n \geq 1\}$ and then use the Puhalskii theorem [12]. The same method is used in the proof of our main result concerning the l.d.p. for the family $\{M_g(\pi_n), n \geq 1\}$.

Theorem 1: *Suppose assumptions* (H^*), (RM), *and* (H_0) *hold. If a continuous function* $g = g(x)$ *is such that*

$$| g(x) | \leq L(1 + [w(x) - w_*]^{\beta}),\ \beta \in (0,1),\ L > 0,$$

where $w(x)$ *is the function from assumption* (H^*), *then the family* $\{M_g(\pi_n), n \geq 1\}$ *obeys the l.d.p. in* (R, r), *where the rate function is defined by* (1.2) *and* (1.3).

Remark: Assumption (RM) is used only in the lower bound part for the l.d.p. of $\{\pi_n, n \geq 1\}$. It has been weakened in Jain [10] and Wu [15]. Theorems 1 and 2 (see the Appendix) remain true if (RM) is replaced by any of the assumptions from [10] and [15].

The proof of Theorem 1 is given in Section 2. Elements of the proof of the theorem have been used in proving the l.d.p. for the family $\{\pi_n, n \geq 1\}$ (see the Appendix, Theorem 2). In Section 3, we consider three examples of Markov processes defined by nonlinear recursions to show how the assumptions of Theorem 1 can be checked.

2. Proof of Theorem 1

According to Deuschel and Stroock [3], Lemma 2.1.4, the following conditions are sufficient for the sequence $\{\frac{1}{n}\sum_{i=0}^{n-1} g(\xi_i), n \geq 1\}$ to obey the l.d.p. in (R, r):

(1) the sequence $\{\pi_n, n \geq 1\}$ obeys the l.d.p. in (S, ρ);

(2) there exists a sequence $\{g_k(x)\}_{k \geq 1}$ of continuous functions such that, for each fixed k, the function $g_k(x)$ is bounded, and

$$\lim_k \sup_{\{\mu \in S: J(\mu) \geq a\}} \left| \int_R [g(x) - g_k(x)]\mu(dx) \right| = 0,\ \forall a > 0,\ \text{while} \tag{2.1}$$

$$\lim_k \limsup_n \frac{1}{n} \log P\left(\left| \int_R [g(x) - g_k(x)]\pi_n(dx) \right| > \epsilon \right) = -\infty,\ \forall \epsilon > 0. \tag{2.2}$$

Take

$$g_k(x) = \begin{cases} g(x), & | g(x) | \leq k \\ k\ sign\ g(x), & | g(x) | > k. \end{cases} \tag{2.3}$$

Due to Theorem 2 (see the Appendix) it remains to be shown that, under assumptions of Theorem 1, each of the functions $g_k = g_k(x)$, for $k \geq 1$, satisfy conditions (2.1) and (2.2).

To this end we use the following.

Lemma 2.1: *Let function* $w = w(x)$ *be from* (H^*). *Then*

$$\int_R (w(x) - w_*)\mu(dx) \leq J(\mu) - w_*.$$

Proof: It goes without saying that $J(\mu)$ can be defined as (compare with (1.3))

$$J(\mu) = \sup_{u \in \mathfrak{U}} \int_R ln \frac{e^{u(x)}}{\int_R e^{u(y)}\pi(x, dy)}\mu(dx),$$

where $\mathfrak{U}$ is a set of measurable bounded functions. For $u \in \mathfrak{U}$ denote

$$G(u, \mu) = \int_R ln \frac{e^{u(x)}}{\int_R e^{u(y)}\pi(x, dy)}\mu(dx).$$

Let $v_n(x) = v(x) \wedge n$ where $v(x)$ is from (H^*). Make the following calculations:

$$G(v_n, \mu) = \int_R \left(v(x) \wedge n - log \int_R e^{v(y) \wedge n} \pi(x, dy) \right) \mu(dx)$$

$$= \int_R I(v(x) \geq n) \left(n - log \int_R e^{v(y) \wedge n} \pi(x, dy) \right) \mu(dx)$$

$$+ \int_R I(v(x) < n) \left(v(x) - log \int_R e^{v(y) \wedge n} \pi(x, dy) \right) \mu(dx)$$

$$\geq \int_R I(v(x) < n)(w(x) - w_*) \mu(dx) + w_* \int_R I(v(x) < n) \mu(dx).$$

Since $J(\mu) \geq G(v_n, \mu)$, we have the following estimate:

$$J(\mu) \geq \int_R I(v(x) < n)(w(x) - w_*) \mu(dx) + w_* \int_R I(v(x) < n) \mu(dx).$$

As $\sup_{|x| \leq N} v(x) < \infty$, we have $I(v(x) < n) \uparrow 1$ as $n \to \infty$; and by the Beppo-Levy Theorem the desired result holds. □

Now we shall establish (2.1). It follows from (2.3) that

$$|g(x) - g_k(x)| \leq |g(x)| I(|g(x)| > k). \tag{2.4}$$

Keeping in mind that $|g(x)| \leq L(1 + (w(x) - w_*)^{\beta})$ for $\beta < 1$, we get for $k > L$ that

$$\left| \int_R (g(x) - g_k(x)) \mu(dx) \right| \leq \int_R |g(x)| I(g(x) > k) \mu(dx)$$

$$\leq L \int_R I\left(w(x) - w_* \leq (k/L - 1)^{1/\beta} \right) \left(1 + (w(x) - w_*)^{\beta} \right) \mu(dx)$$

$$\leq L\left[(k/L - 1)^{-1/\beta} + (k/L - 1)^{1 - 1/\beta} \right] \int_R (w(x) - w_*) \mu(dx).$$

Due to Lemma 2.1, the last inequality implies (2.1).

It remains to be shown that (2.2) holds. This time we make use of Lemma 2.2.

Lemma 2.2: *For any $\mathcal{F}$-measurable sets, A_n for $n \geq 1$ and B_n for $n \geq 1$ and $i \geq 1$, such that $\lim_{i \to \infty} \limsup_n \frac{1}{n} \log P(B_{n,i}) = -\infty$, there holds the following equality:*

$$\lim_n \sup \frac{1}{n} \log P(A_n) = \lim_{i \to \infty} \sup \lim_n \sup \frac{1}{n} \log P(A_n, \Omega \backslash B_{n,i}).$$

The proof follows from the fact that $P(A_n) \geq P(A_n, \Omega \backslash B_{n,i})$ and $P(A_n) \leq 2[P(A_n, \Omega \backslash B_{n,i}) \vee P(B_{n,i})]$.

According to this lemma, (2.2) is valid if

$$\lim_i \limsup_n P(v(\xi_0) > in) = -\infty, \tag{2.5}$$

where $v(x)$ is from (H^*), and

$$\lim_{i\to\infty} \limsup_k \limsup_n \frac{1}{n} \log P\left(\int_R | g(x) | I(| g(x) | > k)\pi_n(dx) > \epsilon,\ v(\xi_0) \leq in \right) = -\infty. \tag{2.6}$$

(2.5) follows from (H_0) and the Chebychev inequality:

$$P(v(\xi_0) > in) \leq e^{-(in)b} \int_R e^{bv(x)}\lambda(dx). \tag{2.7}$$

To prove (2.6), define the random variable

$$Z_n = \prod_{k=0}^{n-1} \frac{e^{v(\xi_{k+1})}}{E(e^{v(\xi_{k+1})} | \xi_k)}, \tag{2.8}$$

with $v(x)$ from (H^*). By the Markovian property $E(e^{v(\xi_{k+1})} | \xi_k) = E(e^{v(\xi_{k+1})} | \xi_0, \ldots, \xi_k)$ $P-a.s.$ and so $EZ_n = 1$. Hence, the following inequality is obvious:

$$1 \geq EI\left(\int_R | g(x) | I(| g(x) | > k)\pi_n(dx) > \epsilon,\ v(\xi_0) \leq in \right) Z_n. \tag{2.9}$$

Also, it is easy to represent Z_n in the form:

$$Z_n = exp\left(\sum_{k=0}^{n-1} v(\xi_{k+1}) - \sum_{k=0}^{n-1} lnE(e^{v(\xi_{k+1})} | \xi_k) \right) = exp\left(v(\xi_n) - v(\xi_0) + \sum_{k=0}^{n-1} w(\xi_k) \right)$$

$$= exp\left(v(\xi_n) - v(\xi_0) + nw_* + n \int_R [w(x) - w_*]\pi_n(dx) \right).$$

For $k > 1$ define a function $f = f(k)$ as follows. $f(k) = inf(| x | : | g(x) | > k)$. Evidently, $\{ | g(x) | > k\} \subseteq \{ | x | \geq f(k)\}$ and $f(k)\uparrow\infty$ as $k\to\infty$.

Let us now evaluate Z_n from below on the set $\{\int_R | g(x) | I(| g(x) | > k)\pi_n(dx) > \epsilon,$ $v(\xi_0) \leq in\}$:

$$Z_n \geq exp\left(-in + nw_* + n \int_R [w(x) - w_*]\pi_n(dx) \right)$$

$$\geq exp\left(-in + nw_* + n \int_R [w(x) - w_*]^{1-\beta}\left(\frac{| g(x) |}{L} - 1 \right)\pi_n(dx) \right)$$

$$\geq exp\left(-in + nw_* + n \inf_{| x | \geq f(k)} [w(x) - w_*]^{1-\beta} \int_{| g(x) | > k} \left(\frac{| g(x) |}{L} - 1 \right)\pi_n(dx) \right)$$

$$\geq exp\left(-in + nw_* + n\frac{\epsilon(1-1/k)}{L\gamma^{2(1-\beta)}(f(k))}\right),$$

where

$$\gamma(y) = \frac{1}{\sqrt{\underset{|x|>y}{inf}[w(x)-w_*]}}. \quad (2.10)$$

It then follows from (2.9) that

$$\frac{1}{n}\, log\, P\left(\int_R |g(x)|I(|g(x)|>k)\pi_n(dx)>\epsilon,\, v(\xi_0)\leq in\right)$$
$$\leq i - w_* - \frac{\epsilon(1-1/k)}{L\gamma^{2(1-\beta)}(f(k))};$$

and, therefore, (2.6) holds due to (H^*). □

3. Examples of Nonlinear Recursion

Consider a Markov process $\xi = (\xi_k)_{-\infty<k<\infty}$ generated by a nonlinear recursion:

$$\xi_{k+1} = f(\xi_k) + h(\xi_k)\epsilon_{k+1}, \quad (3.1)$$

where $f(x)$ and $h(x)$ are continuous functions such that

$$\left|\frac{f(x)}{x}\right| \leq a < 1 \text{ and } 0 < |h(x)| \leq \alpha, \quad (3.2)$$

while $(\xi_k)_{-\infty<k<\infty}$ is a sequence of i.i.d. random variables. Under (3.2), the random value

$$\bar{\xi}_0 = \sum_{j=-\infty}^{0}\left[\prod_{l=j}^{-1}\frac{f(\xi_l)}{\xi_l}\right]h(\xi_{j-1})\epsilon_j \quad (3.3)$$

is well-defined since $|\bar{\xi}_0| \leq \sum_{j=-\infty}^{0} a^{-j}\alpha|\epsilon_j|$. Thus the process ξ defined in (3.1) has an invariant measure λ which is a distribution of the random variable $\bar{\xi}_0$.

Now, we consider a process ξ defined in (3.1) when $k\geq 0$.

1. Suppose that the distribution density w.r.t. the Lebesgue measure of ϵ_1 is Laplacian: $p_\epsilon(y) = \frac{1}{2}exp(-|y|)$ and $\lambda_0 = \lambda$. Then ξ is a stationary process with $\pi(x,dy) = \frac{1}{2}exp(-|\frac{y-f(x)}{h(x)}|)dy$ and so the assumption RM is met. (H^*) is met with $v(x) = c|x|$, for $\frac{a}{\alpha} < c < \frac{1}{\alpha}$:

$$w(x) = c|x| - ln\frac{1}{2}\int_R exp\left(c|y| - \frac{|y-f(x)|}{|h(x)|}\right)dy$$

$$\geq ln2 + ln\int_R exp\left(-|y|\left[\frac{1}{\alpha}-c\right]\right)dy + |x|\left[c-\frac{a}{\alpha}\right].$$

(H_0) is met since, for $0 < b < \frac{1}{\alpha}$, we have $Ee^{b\,|\,\xi_0\,|} \leq \prod_0^\infty \frac{1}{1 - b\alpha a^j} < \infty$.

By Theorem 1, the family $\{M_g(\pi_n), n \geq 1\}$ obeys the l.d.p. for any continuous function $g(x)$ with

$$|\,g(x)\,| \leq L(1 + |\,x\,|^{\beta}) \text{ for } L > 0 \text{ and } 0 < \beta < 1.$$

2. Suppose that the distribution density w.r.t. the Lebesgue measure of ϵ_1 is Gaussian: $p_\epsilon(y) = \frac{1}{\sqrt{2\pi}} exp(-\frac{y^2}{2})$ and $\lambda_0 = \lambda$. Then ξ is a stationary process with $\pi(x, dy) = \frac{1}{\sqrt{2\pi h^2(x)}} exp\left(-\frac{(y - f(x))^2}{2h^2(x)}\right)dy$ and so RM is met. (H^*) is met with $v(x) = cx^2$, for $0 < c < \frac{1 - a^2}{2\alpha^2}$:

$$w(x) = cx^2 - ln \frac{1}{\sqrt{2\pi h^2(x)}} \int_R exp\left(y^2 - \frac{(y - f(x))^2}{2h^2(x)}\right)dy$$

$$= cx^2 - \frac{cf^2(x)}{1 - 2ch^2(x)} + ln\sqrt{1 - 2ch^2(x)}$$

$$\geq cx^2 \frac{[(1 - a^2) - 2c\alpha^2]}{1 - 2c\alpha^2} + ln a.$$

(H_0) is satisfied since, for $0 < b < \frac{1}{2\alpha a}$,

$$E\ exp(b\xi_0^2) \leq \prod_{j,l=0}^{\infty} E\ exp(b\alpha a^{j+l}\,|\,\epsilon_j\,|\,|\,\epsilon_l\,|\,) \leq 4E \prod_{j,l=0}^{\infty} exp(b\alpha a^{j+l}\epsilon_j \epsilon_l)$$

$$\leq 4 \prod_{j=0}^{\infty} \frac{1}{\sqrt{1 - 2b\alpha a^{2j}}} \prod_{j \neq 1} \frac{1}{\sqrt{1 - b^2\alpha^2 a^{2(j+l)}}} < \infty.$$

By Theorem 1, the family $\{M_g(\pi_n), n \geq 1\}$ obeys the l.d.p. for any continuous function $g(x)$ with

$$|\,(x)\,| \leq L(1 + |\,x\,|^{2\beta}) \text{ for } L > 0 \text{ and } 0 < \beta < 1.$$

3. This time, consider a nonstationary process ξ given by (3.1) where every ϵ_k, for $k \geq 1$, is i.i.d. with the Cauchy distribution and where $\xi_0 = x_0$ is a constant. If $f(x)$ and $h(x)$ satisfy (3.2) and if, for some $\gamma < 1/2$,

$$\lim_{|\,x\,| \to \infty} \frac{|\,f(x)\,|}{|\,x\,|^{\gamma}} = 0,$$

then conditions (H^*) and (RM) are satisfied. ((H_0) is obviously satisfied).

Indeed,

$$\pi(x, dy) = \frac{dy}{\pi\left[1 + \left(\frac{y - f(x)}{h(x)}\right)^2\right]}$$

and so (RM) is met.

Take $v(x) = \alpha ln(1+x^2)$ with $\gamma < \alpha < 1/2$. Then

$$w(x) = \alpha ln(1+x^2) - ln \int_R \frac{(1+y^2)^\alpha}{1+\left(\frac{y-f(x)}{h(x)}\right)^2} \frac{dy}{\pi(1+y^2)^{1-\alpha}}$$

$$\geq \alpha ln(1+x^2) - ln(5f^2(x)+4h^2(x)+1) - C,$$

where $C = ln \int_R \frac{dy}{\pi(1+y^2)^{1-\alpha}} < \infty$. Therefore, (H^*) is satisfied because

$$\alpha ln(1+x^2) - ln(5f^2(x)+4h^2(x)+1) \geq ln\frac{(1+x^2)^\alpha}{1+9\,|\,x\,|^{2\gamma}} \uparrow\infty \text{ as } x\to\infty.$$

By Theorem 1, the family $\{M_g(\pi_n), n \geq 1\}$ obeys the l.d.p. for any continuous function $g(x)$ with

$$|\,g(x)\,| \leq L(1+[ln(1+|\,x\,|)]^\beta) \text{ for } L>0 \text{ and } 0<\beta<1.$$

Appendix

Theorem 2: *Let assumptions (H^*), (RM), and (H_0) be satisfied. Then the family $\{\pi_n, n\geq 1\}$ obeys the l.d.p. in (S,ρ) with the rate function $J=J(\mu)$, for $\mu \in S$, defined in* (1.3).

The proof of theorem 2 consists of several steps given below.

Lower bound in the l.d.p.

Lemma A.1: *Let RM be satisfied. Then for any open set $G \subset S$,*

$$\liminf_n P(\pi_n \in G) \geq -\inf_{\mu\in G} J(\mu). \quad (A.1)$$

Proof: Denote $J_G = \inf_{\mu \in G} J(\mu)$. Only the case $J_G < \infty$ needs to be checked. By Donsker and Varadhan [5], we have for any open set G that

$$\liminf_n \frac{1}{n} \log P(\pi_n \in G\,|\,\xi_0 = x) \geq -J_G$$

with $x \in R$.

By the Jensen inequality (for $c>0$),

$$\log(P(\pi_n \in G) + e^{-nc}) \geq \int_R \log(P(\pi_n \in G\,|\,\xi_0 = x) + e^{-nc})\lambda_0(dx).$$

Hence, by the Fatou lemma, we get that

$$\liminf_n \frac{1}{n}\log(P(\pi_n\in G)+e^{-nc})$$

$$\geq \int_R \liminf_n \frac{1}{n}\log(P(\pi_n \in G\,|\,\xi_0=x)+e^{-nc})\lambda_0(dx)$$

$$\geq \int_R \liminf_n \frac{1}{n} \log P(\pi_n \in G \mid \xi_0 = x)\lambda_0(dx) \geq -J_G.$$

Take $c = 2J_G$. Then

$$-J_G \leq \liminf_n \frac{1}{n} \log(P(\pi_n \in G) + e^{nc})$$
$$\leq \liminf_n \frac{1}{n} \log 2max[(P(\pi_n \in G), e^{-2nJ_G}]$$
$$= \liminf_n max\Big[\frac{1}{n} \log (P\pi_n \in G), -2J_G\Big].$$

The desired result follows from this inequality in an obvious way.

Exponential tightness

The family $\{\pi_n, n \geq 1\}$ is said to be *exponentially tight in* (S, ρ) if there exists a sequence $\{K_l, l \geq 1\}$ of compacts such that $K_l \subseteq K_{l+1}$ and

$$\limsup_l \limsup_n \frac{1}{n} \log P(\pi_n \in S \backslash K_l) = -\infty. \tag{A.2}$$

Take a positive decreasing function $\gamma = \gamma(y)$ for $y \geq 0$, with $\lim_{y \to \infty} \gamma(y) = 0$; and for any $j \geq 1$ and $\mu \in S$, define $L(j, \mu) = min\{l \geq j: \int_{|x| > l} \mu(dx) > \gamma(l)\}$ while $min\{\emptyset\} = \infty$. To check $(A.2)$ in an easy way, we use Lemma $A.2$.

Lemma *A*.2: *The family* $\{\pi_n, n \geq 1\}$ *is exponentially tight if*

$$\lim_{j \to \infty} \limsup_n \frac{1}{n} \log P(L(j, \pi_n) < \infty) = -\infty. \tag{A.3}$$

Proof: Let $K_j = \bigcap_{l \geq j} \{\mu \in S: \int_{|x| > l} \mu(dx) \leq \gamma(l)\}$. By the Prohov theorem [2], K_j is a relatively compact set and, since $\{x: |x| > l\}$ is open, the limit of any converging (in metric ρ) sequence from K_j belongs to K_j, i.e. K_j is compact in (S, ρ) and evidently $K_j \subseteq K_{j+1}$. The desired result follows from $(A.3)$ since $S \backslash K_j = \bigcup_{l \geq j} \{\mu \in S: \int_{|x| > l} \mu(dx) > \gamma(l)\} = \{\mu \in S: L(j, \mu) < \infty\}$.

The remainder of the proof of the exponential tightness is given in Lemma $A.3$.

Lemma *A*.3: *Let assumptions* (H^*) *and* (H_0) *be satisfied. Then the family* $\{\pi_n, n \geq 1\}$ *is exponentially tight in* (S, ρ).

Proof: Consider $(A.3)$ with $\gamma(y)$ defined in (2.10). Let us use Lemma 2.2. Introduce sets $A_n = \{L(j, \pi_n) < \infty\}$ and $B_{n,i} = \{v(\xi_0) > in\}$. From prior concepts discussed, $\lim_{i \to \infty} \limsup_n \frac{1}{n} \log P(B_{n,i}) = -\infty$. Then it remains to be shown that $\lim_{i \to \infty} \limsup_{j \to \infty} \limsup_n \frac{1}{n} \log P(A_n \cap \Omega \backslash B_{n,i}) = -\infty$.

Introduce Z_n as in (2.8). Then

$$1 \geq EI(A_n \cap \Omega \backslash B_{n,i})Z_n. \tag{A.4}$$

Taking into account $L(j, \pi_n) \geq j$ and so $\gamma(L(j, \pi_n)) \leq \gamma(j)$, evaluate $\ln Z_n$ from below on the set $\{L(j, \pi_n) < \infty, v(\xi) \leq in\}$. Since $v(x) \geq 0$ and $\gamma(j)$ is given by (2.10), we get

$$ln\ Z_n \geq -v(\xi_0) + nw_* + n \inf_{|x| > L(j, p_n)} [w(x) - w_*] \int_{|x| > L(j,\pi_n)} \pi_n(dx)$$

$$\geq -in + nw_* + \frac{n}{\gamma(L(j,\pi_n))} \geq -in + nw_* + \frac{n}{\gamma(j)}. \qquad (A.5)$$

It is easy to find from the last inequality and $(A.4)$ that

$$\frac{1}{n} \log P(A_n \cap \Omega \backslash B_{n,i}) \equiv (1/n) \log P(L(j,\pi_n) < \infty, v(\xi_0) \leq in)$$

$$\leq i - w_* - 1/\gamma(j),$$

and so the desired result holds. □

Corollary: $\lim_{i \to \infty} \lim_n \sup \frac{1}{n} \log \left(P \int_{|x| > i} \pi_n(x) > \beta \right) = -\infty$ *for* $\beta > 0$.

Upper bound in the l.d.p.

We first establish one auxiliary result.

Lemma A.4: *Let μ', $\mu'' \in S$ and $V \in \mathcal{N}$. Then for any $\epsilon > 0$, there exists a nonnegative function $h_c \in \mathcal{N}$, depending on ϵ and V, such that*

$$\left| \int_R V(x)[\mu'(dx) - \mu''(dx)] \right| \leq \epsilon + \rho(\mu', \mu'') \int_R h_\epsilon(x) dx$$

$$+ \int_R h_\epsilon(x)[F(x + \rho(\mu', \mu'')) - F(x - \rho(\mu', \mu''0)]dx,$$

where $F(x)$ denotes either $F'(x) = \int_{-\infty}^{x} \mu'(dy)$ or $F''(x) = \int_{-\infty}^{x} \mu''(dy)$.

Proof: If $V(x)$ is continuously differentiable, then integrating by parts we get $| \int_R V(x)[\mu'(dx) - \mu''(dx)] \leq \int_R \left| \frac{dV(x)}{dx} \right| | F'(x) - F''(x) | dx$. From the definition of the Levy-Prohorov metric, it follows that $a > 0$,

$$F'(x) - F''(x) \leq a + F''(x+a) - F'(x) \text{ and } F'(x) - F''(x) \geq -a + F''(x-a) - F'(x).$$

The desired result holds with $a = \rho(\mu', \mu'')$ and $h_\epsilon(x) = \left| \frac{dV(x)}{dx} \right|$.

In general, we approximate $V(x)$ by a continuously differentiable function V_ϵ from $\mathcal{N}$ such that $\sup_{x \in R} | V(x) - V_\epsilon(x) | \leq \frac{\epsilon}{2}$ and we use an estimate $| \int_R V(x)[\mu'(dx) - \mu''(dx)] | \leq \epsilon + | \int_R V_\epsilon(x)[\mu'(dx) - \mu''(dx)] |$. This gives the desired result with $h_\epsilon = \left| \frac{dV_\epsilon(x)}{dx} \right|$. □

The upper bound in the l.d.p. will be derived from the exponential tightness and the following.

Lemma A.5: *Let assumptions (H^*) and (H_0) be satisfied. Then*

$$\lim_{\delta \to 0} \sup \lim_n \sup \log P(\rho(\pi_n, \mu) \leq \delta) \leq -J(\mu) \text{ for } \mu \in S,$$

where $J(\mu)$ is defined in (1.3).

Proof: Denote $A_n = \{\rho(\pi_n, \mu) \le \delta\}$ and $B_{n,i} = \{ \int_{|x| > i} \pi_n(dx) > \beta\}$. By the Corollary to Lemma $A.3$, $\limsup_i \limsup_n \frac{1}{n} P(B_{n,i}) = -\infty$, and so

$$\lim_{\beta \to 0} \limsup_i \limsup_{\delta \to 0} \limsup_n \frac{1}{n} P(B_{n,i}) = -\infty.$$

It follows from Lemma 2.1 that the inequality,

$$\lim_{\beta \to 0} \limsup_i \limsup_{\delta \to 0} \limsup_n \frac{1}{n} \log P(A_n \cap \Omega \backslash B_{n,i}) \le -J(\mu) \text{ for } \mu \in S, \tag{A.6}$$

implies the desired result.

To prove $(A.6)$, let $Z_n = exp(\sum_{k=0}^{n-1} u(\xi_{k+1}) - \sum_{k=0}^{n-1} ln\ E(e^{u(\xi_{k+1})} \mid \xi_k))$, where $u \in \mathcal{N}$ and $EZ_n = 1$. An obvious inequality,

$$1 \ge EI(A_n \cap \Omega \backslash B_{n,i}) Z_n, \tag{A.7}$$

arises for our use. Denote

$$V(x) = ln \frac{e^{u(x)}}{\int_R e^{u(y)} p(y \mid x) \lambda(dy)} \tag{A.8}$$

and express Z_n in the form convenient for evaluation from below. We get

$$Z_n = exp\left(u(\xi_n) - u(\xi_0) + \sum_{k=0}^{n-1} V(\xi_k) \right) = exp\left(u(\xi_n) - u(\xi_0) + n \int_R V(x) \pi_n(dx) \right)$$

$$= exp\left(u(\xi_n) - u(\xi_0) + n \int_R V(x) \mu(dx) + n \int_R V(x) [\pi_n(dx) - \mu(dx)] \right)$$

$$\ge exp\left(-2u^* + n \int_R V(x) \mu(dx) - \left| n \int_R V(x) [\pi_n(dx) - \mu(dx)] \right| \right), \tag{A.9}$$

where $u^* = \sup_{x \in R} |u(x)|$. Estimate $|\int_R V(x)[\pi_n(dx) - \mu(dx)]|$ from above in terms of $\rho(\pi_n, \mu)$. Take an even function $g_i(x)$ such that

$$g_i(x) = \begin{cases} 1, & 0 \le x \le i-1 \\ i - x, & i-1 < x \le i \\ 0, & x > i, \end{cases}$$

and put $V_i(x) = V(x) g_i(x)$. Denote $V^* = \sup_{x \in R} |V(x)| (< \infty)$. Then

$$\left| \int_R V(x)[\pi_n(dx) - \mu(dx)] \right| \le V^* \left[\int_{|x| > i} \pi_n(dx) + \int_{|x| > i} \mu(dx) \right] + \left| \int_R V_i(x)[\pi_n(dx) - \mu(dx)] \right|.$$

The Feller property of $\pi(x,dy)$ implies that $V_i \in \mathcal{N}$. Therefore, by Lemma $A.4$, for any $\epsilon > 0$ there exists a positive, continuous, finite-supported function $h_\epsilon(x)$, depending on $V_i(x)$ and ϵ, such that

$$\left| \int_R V_i(x)[\pi_n(dx) - \mu(dx)] \right|$$

$$\leq \epsilon + \rho(\pi_n,\mu) \int_R h_\epsilon(x)dx + \int_R h_\epsilon(x)[F(x+\rho(\pi_n,\mu)) - F(x-\rho(\pi_n,\mu))]dx,$$

where $F(x) = \int_{-\infty}^{x} \mu(dy)$. Hence, on the set $\{A_n \cap \Omega \backslash B_{n,i}\}$,

$$Z_n \geq exp\Bigg\{ -2u^* + n\int_R V(x)\mu(dx) - V^*\beta - nV^*\int_{|x|>i} \mu(dx)$$

$$-n\left(\epsilon + \delta\int_R h_\epsilon(x)dx + \int_R h_\epsilon(x)[F(x+\delta) - F(x-\delta)]dx\right)\Bigg\};$$

and consequently

$$\lim_{\beta\to 0}\limsup_{i\to\infty}\limsup_{\delta\to 0}\limsup_n \frac{1}{n}\log P\left(\rho(\pi_n,\mu)\leq\delta, \int_{|x|>i}\pi_n(dx)\leq\beta\right)$$

$$\leq -\int_R V(x)\mu(dx)$$

is implied by $(A.7)$ and the arbitrariness of ϵ. Then by the arbitrariness of $u \in \mathcal{N}$ (see $(A.8)$ for the definition of $V(x)$), the desired upper bound holds.

Now, we can establish the upper bound for any closed $F \in S$.

Lemma $A.6$: *Let assumptions (H^*) and (H_0) be satisfied. Then for any closed $F \in S$,*

$$\limsup_n \frac{1}{n}\log P(\pi_n \in F) \leq -\inf_{\mu\in F} J(\mu),$$

where $J(\mu)$ is defined in (1.3).

Proof: Since (S,ρ) is a Polish space (see [9] for the proof) and since by Lemma $A.3$ the family $\{\pi_n, n\geq 1\}$ is exponentially tight, then by the Puhalskii theorem [12], any subsequence of the sequence $\{\pi_n, n\geq 1\}$ contains further a subsequence $\{\pi_{n'}\}$ satisfying the l.d.p. with a good rate function $J' = J'(\mu)$. Thus

$$\limsup_{\delta\to 0}\limsup_{n'}\frac{1}{n'}\log P(\rho(\pi_{n'},\mu)\leq\delta)$$

$$= \liminf_{\delta\to 0}\liminf_{n'}\frac{1}{n'}\log P(\rho(\pi_{n'},\mu)\leq\delta) = -J(\mu) \text{ for } \mu\in S. \tag{A.10}$$

Taking into account Lemma $A.5$, we get

$$-J(\mu) \geq \limsup_{\delta\to 0}\limsup_n \frac{1}{n}\log P(\rho(\pi_n,\mu)\leq\delta)$$

$$\geq \limsup_{\delta \to 0} \lim_{n'} \sup \frac{1}{n'} \log P(\rho(\pi_{n'}, \mu) \leq \delta) = -J'(\mu),$$

i.e.

$$J(\mu) \leq J'(\mu), \ \forall \mu \in S. \tag{A.11}$$

Let $F \in S$ be a closed set. Assume the subsequence $\{\pi_{n'}\}$ was chosen such that parallel with the l.d.p. for $\{\pi_{n'}\}$, $\lim\sup_n \frac{1}{n} \log P(\pi_n \in F) = \lim_{n'} \frac{1}{n'} \log P(\pi_{n'} \in F)$. Then,

$$\limsup_n \frac{1}{n} \log P(\pi_n \in F) \leq -\inf_{\mu \in F} J'(\mu) \leq -\inf_{\mu \in F} J(\mu), \tag{A.12}$$

where the last inequality follows from $(A.11)$. □

Now that we have the lower and upper bounds for the family $\{\pi_n, n \geq 1\}$ (see $(A.1)$ and $(A.12)$), to complete the proof of Theorem 2 we need to show that $J = J(\mu)$ is a good rate function. To this end we use the arguments similar to those in the proof of Lemma $A.6$.

By the Puhalskii theorem [12], any subsequence of the sequence $\{\pi_n, n \geq 1\}$ contains further a subsequence $\{\pi_{n'}\}$ satisfying the l.d.p. with a good rate function $J' = J'(\mu)$. Thus $(A.10)$ holds. Taking into account Lemma $A.1$, we get

$$-J(\mu) \leq \liminf_{\delta \to 0} \lim_n \inf \frac{1}{n} \log P(\rho(\pi_n, \mu) \leq \delta)$$

$$\leq \limsup_{\delta \to 0} \lim_{n'} \sup \frac{1}{n'} \log P(\rho(\pi_{n'}, \mu) \leq \delta) = -J'(\mu),$$

i.e.

$$J(\mu) \geq J'(\mu), \ \forall \mu \in S.$$

The last inequality together with $(A.11)$ implies that $J(\mu) = J'(\mu)$, i.e. $J(\mu)$ is a good rate function.

Theorem 2 is proved.

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