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A NEW SIMULATION ESTIMATOR OF SYSTEM RELIABILITY

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ABSTRACT

A basic identity is proven and applied to obtain new simulation estimators concerning (a) system reliability, (b) a multi-valued system. We show that the variance of this new estimator is often of the order α^2 when the usual raw estimator has variance of the order α and α is small. We also indicate how this estimator can be combined with standard variance reduction techniques of antithetic variables, stratified sampling and importance sampling.

Key words: Simulation, System Reliability, Variance Reduction, Multi-valued System.

AMS (MOS) subject classifications: Primary 65Cxx, Secondary 60K10, 62G05, 62N05, 90B25.

1. Introduction

Consider the problem of using simulation to estimate α, the probability that an m component binary system is failed. Making use of a basic identity concerning sums of non-independent Bernoulli random variables (presented in Section 1) we present a new estimator of this probability. It is shown that this new estimator is always bounded by the Boole inequality upper bound on α. It is also shown how our approach can be combined with the standard variance reduction techniques of antithetic variables, stratified sampling, and importance sampling. The final section applies the estimator to a multi-valued system.

2. A Basic Identity

Let $X_1, \ldots, X_n$ be Bernoulli random variables such that

$$P\{X_i = i\} = \lambda_i = 1 - P\{X_i = 0\},\ i = 1, \ldots, n.$$

Let $W = \sum_{i=1}^{n} a_i X_i$ where the a_i are positive constants. Set $\lambda = \sum_i a_i \lambda_i = E[W]$. Also let R and I be random variables defined on the same probability space as the X_i. Suppose that I is independent of the random vector $\boldsymbol{X}$, R and is such that

$$P\{I=i\} = a_i / \sum_i a_i, \; i = 1, \ldots, n.$$

The following identity will be basic to our work.

The Basic Identity:

(a) $P\{I=i \mid X_I = 1\} = \lambda_i a_i / \lambda$
(b) $E[WR] = \lambda E[R \mid X_I = 1]$
(c) $P\{W > 0\} = \lambda E[\frac{1}{W} \mid X_I = 1]$.

Proof: The proof of (a) is immediate. Part (b) is shown as follows:

$$\begin{aligned} E[WR] &= \sum_i a_i E[X_i R] \\ &= \sum_i a_i E[X_i R \mid X_i = 1] \lambda_i \\ &= \sum_i a_i \lambda_i E[R \mid X_i = 1] \end{aligned}$$

Also,

$$\begin{aligned} E[R \mid X_I = 1] &= \sum_i E[R \mid X_I = 1, \, I = i] a_i \lambda_i / \lambda \\ &= \sum_i E[R \mid X_i = 1] a_i \lambda_i / \lambda. \end{aligned}$$

Combining these two equalities yields part (b).

Part (c) follows directly from part (b) by letting

$$R = \begin{cases} 0 & \text{if } W = 0 \\ 1/W & \text{if } W > 0. \end{cases}$$

Remark: Part (b) of the basic identity is a generalization of an identity due to Charles Stein [7], who used it in his work on establishing bounds for Poisson approximations. Stein presented the identity when the a_i equal 1 and R is a function of W. We will need the added generality in certain of our applications. The proof presented above differs from the one given by Stein.

3. Some Applications in Simulation

3. 1. System Reliability

Consider an m component system in which each component is either working or failed. Let Y_i equal 1 if component i is failed and let it equal 0 if i is working, $i = 1, \ldots, m$. Assume that the Y_i are independent and $P\{Y_i = 1\} = p_i$, $i = 1, \ldots, m$. Also suppose that the system is itself either working or failed and its state is determined as a monotone function of the vector $\boldsymbol{Y}$.

Let $C_1, \ldots, C_n$ denote the minimal cut sets of this system. That is, if all of the components in C_i are failed then so is the system and such a statement is not true for any proper subset of C_i, $i = 1, \ldots, n$ (see Barlow and Proschan [1]). Let

$$X_i = \prod_{j \in C_i} Y_j, \; i = 1, \ldots, n.$$

That is, X_i is equal to 1 if C_i is down (that is, if all components in C_i are down). Set

$$W = \sum_i X_i$$

$\lambda_i = E[X_i] = \prod_{j \in C_i} p_j$, and let $\lambda = \sum_i \lambda_i$. Also let denote the number of minimal cut sets that are down. The probability that the system is failed is

$$\alpha = P\{\text{system is failed}\} = P\{W > 0\}.$$

In cases of highly reliable systems α will be quite small. However, it is important to be able to obtain a precise estimate of it since whereas a probability of system failure of 10^{-6} might be acceptable one of 10^{-3} might not be.

Suppose we are planning to approximate α by continually simulating the vector $\boldsymbol{Y}$. Whereas the raw simulation estimator from a simulation run yielding the vector $\boldsymbol{Y}$ is 1 if the system is failed under $\boldsymbol{Y}$ and 0 otherwise, we will use the basic identity (with a_i equal 1) to present a new estimator.

To use the basic identity, simulate the value of J, a random variable that is equal to i with probability λ_i/λ, $i = 1, \ldots, n$. (It follows from (a) of the Basic Identity that the random variable J has the same distribution as the conditional distribution of I given that $X_I = 1$). Now, set Y_i equal to 1 for $i \in C_J$, and simulate all of the other Y_i, $i \notin C_J$. Let W^* denote the number of minimal cut sets that are down; (note that $W^* \geq 1$). From the basic identity it follows that the estimator λ/W^* will be an unbiased estimator of α. Since $W^* \geq 1$, it follows that

$$0 \leq \lambda/W^* \leq \lambda$$

and we can conclude from the above in conjunction with the fact that $E[\lambda/W^*] = \alpha$, that

$$\begin{aligned}\text{Var}(\lambda/W^*) &= E[(\lambda/W^*)^2] - \alpha^2 \\ &\leq \lambda E[\lambda/W^*] - \alpha^2 \\ &= \alpha(\lambda - \alpha).\end{aligned}$$

Since the variance of the raw estimator is $\alpha(\lambda - \alpha)$ the above implies that the new estimator has a smaller variance when $\lambda < 1$, as will usually be the case when α is small. Indeed, in many applications we would expect that λ and α should be roughly of the same magnitude and so the above would yield that the variance of the new estimator is no greater than the order of α^2.

Remark: From a computational point of view the new estimator seems comparable to the old one. Whereas we must now count the number of minimal cut sets that are down and also simulate J we have the savings of not having to simulate the values of the components in C_J.

We now show how to use the new estimator in conjunction with the standard variance reduction techniques of antithetic variables, stratified sampling and importance sampling.

3.1.1 Use of Antithetic Variables

The variance of the estimator can be further reduced by using the method of antithetic variables. Once the value of J has been generated and the resulting Y_i, $i \notin C_J$, generated then we should compute the value of W^* first using these Y_i and secondly using the values $Y_i^a = 1 - Y_i$. The estimator of $E[1/W \mid X_I = 1]$ from that run should be the average of these two values of $1/W^*$.

3.1.2 Use of Stratified Sampling

The estimator presented above can be improved, in the sense of having its variance reduced, by using a stratified sampling approach. That is, rather than starting by simulating J we can arbitrarily set J equal to i in the fixed fraction λ_i/λ of simulation runs, $i = 1, ..., n$. This will result in an estimator having the same mean as λ/W^* but a smaller variance (see Section 8.4 of [6]). Of course, this is only feasible when n is of moderate size.

3.1.3 Importance Sampling and the New Estimator

Importance sampling often produces an estimator of α whose variance is significantly smaller than that of the raw estimator (see Jun-Ross [3]). Rather than simulating the vector $\boldsymbol{Y}$ according to the independent probabilities p_i, $i = 1, ..., n$ it simulates them according to a different set of independent probabilities q_i, $i = 1, ..., n$ and then uses the estimator

$$Imp = I_{\{W > 0\}} p(\boldsymbol{Y})/q(\boldsymbol{Y}),$$

where $I_{\{W > 0\}}$ is the indicator variable equal to 1 when the system is failed, and $p(\boldsymbol{Y})$ and $q(\boldsymbol{Y})$ are the probabilities of the simulated vector $\boldsymbol{Y}$ under the p_i and q_i respectively. That is,

$$p(\boldsymbol{Y}) = \prod p_i^{Y_i}(1-p_i)^{1-Y_i},\ q(\boldsymbol{Y}) = \prod q_i^{Y_i}(1-q_i)^{1-Y_i},$$

and

$$E_q[I_{\{W > 0\}} p(Y)/q(Y)] = \alpha,$$

where E_q is used to signify that the distribution of $\boldsymbol{Y}$ is given by the probabilities q_i, $i = 1, ..., n$.

Let $\lambda_i(q)$ denote the probability, under the q's, that C_i is down, and let $\lambda_q = \sum_i \lambda_i(q)$. We now show how to obtain a new estimator by using the basic identity in conjunction with the above.

By letting

$$R = \begin{cases} 0 & \text{if } W = 0 \\ I_{\{W > 0\}} p(\boldsymbol{Y})/[Wq(\boldsymbol{Y})] & \text{if } W > 0 \end{cases}$$

we obtain, from the basic identity, that

$$E_q[I_{\{W > 0\}} p(\boldsymbol{Y})/q(\boldsymbol{Y})] = \lambda_q E_q[I_{\{W > 0\}} p(\boldsymbol{Y})/[Wq(\boldsymbol{Y})] \mid X_I = 1]$$

$$= \lambda_q E_q[\frac{p(\boldsymbol{Y})}{Wq(\boldsymbol{Y})} \mid X_I = 1].$$

Therefore, we can obtain a new unbiased estimator of α by following these steps:

(*a*) Simulate the value of J, which is equal to i with probability $\lambda_i(q)/\lambda_q$, $i = 1, ..., n$.
(*b*) Set Y_i equal to 1 for $i \in C_J$.
(*c*) Simulate all of the other Y_i, $i \notin C_J$ according to the q_i.
(*d*) Let W^* denote the resulting number of minimal cut sets that are down.

The estimator of α is then $\lambda_q \dfrac{p(\boldsymbol{Y})}{W^* q(\boldsymbol{Y})}$ where $\boldsymbol{Y}$ is the vector obtained in parts (*a*) through (*d*). Call this estimator Est.

Let us now compare Var(Imp) and Var (Est). Since these estimators have the same mean, we need consider their second moments. Now

$$E[Imp^2] = E_q[I_{\{W > 0\}} p^2(\boldsymbol{Y})/q^2(\boldsymbol{Y})]$$

$$= E_q[\frac{p^2(\boldsymbol{Y})}{q^2(\boldsymbol{Y})} \mid W > 0]\alpha_q$$

where $\alpha_q = P_q\{W > 0\}$.

Now let

$$R = \begin{cases} \dfrac{p^2(\boldsymbol{Y})}{W^2 q^2(\boldsymbol{Y})}, & \text{if } W > 0 \\ 0, & \text{if } W = 0. \end{cases}$$

Now,

$$E[Imp^2] = \lambda_q^2 E_q[\frac{p^2(\boldsymbol{Y})}{W^2 q^2(\boldsymbol{Y})} \mid X_I = 1]$$
$$= \lambda_q^2 E_q[R \mid X_I = 1]$$
$$= \lambda_q E_q[RW] \text{ by the basic identity}$$
$$= \lambda_q E_q[WR \mid W > 0]\alpha_q$$
$$= \lambda_q \alpha_q E_q[\frac{p^2(\boldsymbol{Y})}{W q^2(\boldsymbol{Y})} \mid W > 0].$$

A comparison of the formulae for $E[Imp^2]$ and $E[Est^2]$ shows, since $W > 0$ is equivalent to $W \geq 1$, that a sufficient condition for the former to be the larger is that

$$\lambda_q \leq 1.$$

That is, if the expected number of minimal cut sets that are down, when simulating with the q_i, is less than 1 then the estimator Est has a smaller variance than does the importance sampling estimator. Also, this is not a necessary condition since $E[Est^2]/E[Imp^2] < \lambda_q$, and indeed might be quite a bit less since W^*, which has the conditional distribution of W given $W \geq 1$, might tend to be quite a bit larger than 1.

3.2 A Multi-Valued System

Suppose we have a set of m independent components, with component i functioning with probability p_i. There is a set of n experiments we want to perform. However, in order to perform experiment i all of the components in the set C_i must function. Suppose the return from experiment i is the positive amount a_i. If we let X_i be 1 if all the components in C_i are functioning and let it be 0 otherwise then the total return from the n experiments is

$$W = \sum_{i=1}^{n} a_i X_i.$$

Since W is a monotone function of the vector $\boldsymbol{Y}$, we can regard the above as a multi-valued system whose value is W.

Suppose we want to use simulation to estimate

$$\beta = P\{W > x\}.$$

The raw simulation estimator is to simulate $\boldsymbol{Y}$ and then determine the value of W. However, we can use the basic identity to obtain a different estimator. First, note that it follows from the basic identity, by letting

$$R = \begin{cases} 0 & \text{if } W \leq x \\ 1/W & \text{if } W > x, \end{cases}$$

that

$$\begin{aligned} P\{W > x\} &= \lambda E[R \mid X_I = 1] \\ &= \lambda E[I_{\{W > x\}}/W \mid X_I = 1], \end{aligned}$$

where $\lambda = \sum_i a_i \lambda_i$, $\lambda_i = E[X_i]$. Hence, we can estimate β by simulating the value of J, a random variable equal to i with probability $a_i\lambda_i/\lambda$. We then set Y_i equal to 1 if $i \in C_J$ and we simulate its value in $i \notin C_J$. Finally, we let W^* denote the resulting value of $\sum_i a_i X_i$. The estimator is

$$Estimator = \begin{cases} 0 & \text{if } W^* \leq x \\ \lambda/W^* & \text{if } W^* > x. \end{cases}$$

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