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OPTIMAL CONTROL OF NONLINEAR SECOND ORDER EVOLUTION EQUATIONS[1]

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ABSTRACT

In this paper we study the optimal control of systems governed by second order nonlinear evolution equations. We establish the existence of optimal solutions for Lagrange problem.

Key words: Evolution triple, compact embedding, monotone and hemicontinuous map, hyperbolic systems, Dirichlet form.

AMS (MOS) subject classifications:

1. INTRODUCTION

In this paper we establish the existence of optimal controls for a class of systems governed by second order nonlinear evolution equations. Our results extend some earlier work of Papageorgiou [8]. We introduce more general conditions, admitting strong nonlinearities. In fact, Papageorgiou's result follows from our general results.

2. BASIC ASSUMPTIONS

Let $T = [0, r]$ and Y a separable, reflexive Banach space. Let H be a separable Hilbert space and X a dense subspace of H, carrying the structure of a separable, reflexive, Banach space, which embeds in H continuously. Identifying H with its dual (pivot space), we have $X \hookrightarrow H \hookrightarrow X^*$, with all embeddings being continuous and dense. We will also assume that all the embeddings are compact. By $\|\cdot\|_X$ (resp. $|\cdot|_H$, $\|\cdot\|_{X^*}$) we will denote the norm of X

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(resp. of H, X^*). Also by $\langle \cdot, \cdot \rangle$, we will denote the duality brackets for the pair (X, X^*) and by $(\cdot, \cdot)$, the inner product of H. The two are compatible in the sense that $\langle \cdot, \cdot \rangle_{X \times H} = (\cdot, \cdot)$. Let $W_{p,q}(T) = \{x \in L^p(X): \dot{x} \in L^q(X^*)\}$. The derivative in this definition is understood in the sense of vector-valued distributions. This is a separable, reflexive Banach space with the norm $\| x \|_{W_{p,q}(T)} = (\| x \|^2_{L^p(X)} + \| \dot{x} \|^2_{L^q(X^*)})^{1/2}$. Recall that $W_{p,q}(T)$ embeds into $C(T, H)$ continuously (see Ahmed and Teo [1]). So very equivalence class in $W_{p,q}(T)$ has a unique representative in $C(T, H)$. Furthermore, since we have assumed that X embeds into H compactly, we have that $W_{p,q}(T)$ embeds into $L^p(H)$, compactly too. Finally, Nagy [6] proved that if X is a Hilbert space, then the injection $W_{p,q}(T) \hookrightarrow C(T, H)$ is compact. For further details on evolution triples and the Banach space $W_{p,q}(T)$, we refer to Zeidler [11], chapter 23.

3. EXISTENCE OF OPTIMAL CONTROLS

Let $T = [0, r]$, (X, H, X^*) an evolution triple, with $X \hookrightarrow H$ compactly (hence $H \hookrightarrow X^*$ compactly) and Y a separable, reflexive Banach space, modeling the control space. We consider the following Lagrange type optimal control problem:

$$\left\{ \begin{array}{l} J(x,u) = \int_0^r L(t, x(t), \dot{x}(t), u(t))dt \rightarrow inf = m \\ \text{subject to the following state and control constraints:} \\ \ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t, x(t))u(t), x(0) = x_0 \in X, \dot{x}(0) = x_1 \in H, u(t) \in U(t) a.e. \end{array} \right\} \qquad (P)$$

By an admissible "state-control" pair for (P), we understand a pair of a state trajectory $x(\cdot) \in C(T, X)$ and of a control function $u(\cdot) \in L^\infty(Y)$ so that $\dot{x}(\cdot) \in W_{p,q}(T)$ and both functions $x(\cdot), u(\cdot)$ satisfy the constraints of problem (P). Recall that $W_{p,q}(T)$ embeds into $C(T, H)$ continuously, and so the initial condition $\dot{x}(0) = x_1 \in H$ makes sense. An admissible "state-control" pair $\{x, u\}$, is said to be "optimal", if $J(x, u) = m$.

To establish the existence of an optimal pair for (P), we will need the following hypotheses on the data:

<u>$H(A)$:</u> $A: T \times X \rightarrow X^*$ is a map s.t.

(1) $t \rightarrow A(t, v)$ is measurable,

(2) $v \rightarrow A(t, v)$ is monotone (i.e. $\langle A(t,v) - A(t,v'), v - v' \rangle \geq 0$ for all $v, v' \in X$) and

hemicontinuous (i.e., $\lambda \to \langle A(t, v + \lambda y, z\rangle$ is continuous for all $v, y, z \in X$).

(3) $\langle A(t,v), v\rangle \geq c \| v \|_X^p - d\,|v|_H^2$ a.e. with $c > 0$, $d \geq 0$,

(4) $\| A(t,v) \|_{X^*} \leq a(t) + b \| v \|_X^{p-1}$ a.e. with $a(\cdot) \in L^q(T)$, $b > 0$ or $b \in L_+^\infty(T)$.

<u>$H(B)$:</u> $B \in \mathcal{L}(X, X^*)$ (i.e. B is continuous, linear), is symmetric (i.e. $\langle Bx, z\rangle = \langle x, Bz\rangle$ for all $x, z, \in X$) and $\langle Bx, x\rangle \geq c' \| x \|_X^2$, $c' > 0$ (i.e. $B(\cdot)$ is coercive).

<u>$H(f)$:</u> $f: T \times H \to \mathcal{L}(Y, H)$ is a map s.t.

(1) $t \to f(t,x)u$ is measurable for every $(x,u) \in H \times Y$,

(2) $x \to f(t,x)^* h$ is continuous for every $(t,h) \in T \times H$,

(3) $\| f(t,x) \|_{\mathcal{L}(Y,H)}^q \leq a_1(t) + b_1 |x|_H^2$ for $2 \leq p < \infty$ and $1 < q \leq 2$.

<u>$H(U)$:</u> $U: T \to P_{wkc}(Y)$ is a measurable multifunction so that $t \to |U(t)| = sup\{\| u \|_Y : u \in U(t)\} \equiv g(t)$, $g \in L_+^\infty$,

<u>$H(L)$:</u> $L: T \times H \times H \times Y \to \bar{R} = R \cup \{+\infty\}$ is an integrand so that

(1) $(t,x,y,u) \to L(t,x,y,u)$ is Borel measurable,

(2) $(x,y,u) \to L(t,x,y,u)$ is *l.s.c.*,

(3) $u \to L(t,x,y,u)$ is convex,

(4) $\varphi(t) - \widehat{M}(|x|_H + |y|_H + \| u \|_Y)) \leq L(t,x,y,u)$ a.e. with $\varphi(\cdot) \in L^1, \widehat{M} > 0$.

Finally since our cost-functional is $\bar{R}$-valued, we will need the following feasibility hypothesis.

<u>H_0:</u> there exists admissible "state-control" pair (x,u) so that $J(x,u) < \infty$. Denote by $\mathcal{U}_{ad} = \{u: u(t) \in U(t) \text{ a.e.}\}$ the admissible set of controls.

Lemma 1. *Under the assumptions $H(A)$, $H(B)$, $H(f)$ and $H(U)$, for each $x_0 \in X$, $x_1 \in H$, and $u \in \mathcal{U}_{ad}$ the evolution equation of problem (P) has unique solution x satisfying*

(*a*) $x \in L^\infty(X)$

(*b*) $\dot{x} \in L^\infty(H) \cap L^p(X)$

(*c*) $\ddot{x} \in L^q(X^*)$

(*d*) (*b*) *and* (*c*) $\Rightarrow \dot{x} \in W_{p,q}$

(*e*) $A(\cdot, \dot{x}(\cdot)) \in L^q(X^*)$.

The proof follows from standard application of Galerkin technique and the a priori estimates given in lemma 2, see [2, 11].

Before studying the problem of existence of optimal controls, we will start by deriving some a priori bounds for the admissible trajectories of (P).

Denote by S the set of solution trajectories of the evolution equation of problem (P) corresponding to the admissible set of controls as defined above.

Lemma 2. (A priori estimates). *Under the assumptions $H(A3)$, $H(f)$, $H(B)$ and $H(U)$, the set $Z \equiv \{\dot{x}, x \in S\}$ is a bounded subset of $W_{p,q}(T)$.*

Proof. Let x be any solutions trajectory of the evolution equation in problem (P), corresponding to an admissible control $u(\cdot) \in L^{\infty}(Y)$. By lemma 1, the following scalar multiplication is well defined,

$$\langle \ddot{x}(t), \dot{x}(t)\rangle + \langle A(t, \dot{x}(t)), \dot{x}(t)\rangle + \langle Bx(t), \dot{x}(t)\rangle = (f(t, x(t))u(t), \dot{x}(t) \text{ a.e.}$$

Since $\dot{x} \in W_{p,q}(T)$ it follows from proposition 23.23 (iv), p. 422 of Zeidler [11], that

$$\langle \ddot{x}(t)\dot{x}(t)\rangle = \frac{1}{2}\frac{d}{dt} |\dot{x}(t)|_H^2 \text{ a.e.}$$

Furthermore, because of hypothesis $H(A3)$, we have

$$c \| \dot{x}(t) \|_X^p - d |\dot{x}(t)|_H^2 \leq \langle A(t, \dot{x}(t)), \dot{x}(t)\rangle \text{ a.e.}$$

Also using the product rule and exploiting the symmetry of the operator $B \in \mathcal{L}(X, X^*)$ (see hypothesis $H(B)$), we obtain

$$\frac{d}{dt}\langle Bx(t), x(t)\rangle = \langle B\dot{x}(t), x(t)\rangle + \langle Bx(t), \dot{x}(t)\rangle = 2\langle B\dot{x}(t), x(t)\rangle \text{ a.e.}$$

So finally we can write that

$$\frac{1}{2}\frac{d}{dt} |\dot{x}(t)|_H^2 + c \| \dot{x}(t) \|_X^p + \frac{1}{2}\frac{d}{dt}\langle Bx(t), x(t)\rangle \leq d |\dot{x}(t)|_H^2 + (f(t, x(t))u(t), \dot{x}(t)) \text{ a.e.}$$

Integrating the above inequality, we have

$$\frac{1}{2} |\dot{x}(t)|_H^2 - \frac{1}{2} |x_1|_H^2 + c\int_0^t \| \dot{x}(s) \|_X^p ds + \frac{1}{2}\langle Bx(t), x(t)\rangle - \frac{1}{2}\langle Bx_0, x_0\rangle$$

$$\leq d\int_0^t |\dot{x}(s)|_H^2 ds + \int_0^t (f(s, x(s))u(s), \dot{x}(s))ds,$$

$$\Rightarrow |\dot{x}(t)|_H^2 + 2c\int_0^t \| \dot{x}(s) \|_X^p ds + c' \| x(t) \|_X^2 \tag{1}$$

$$\leq |x_1|_H^2 + \| B \|_{\mathcal{L}(X, X^*)} \| x_0 \|_X^2 + 2d\int_0^t |\dot{x}(s)|_H^2 ds + 2\int_0^t (f(s, x(s))u(s), \dot{x}(s))ds.$$

Note that by applying Cauchy's inequality,

$$a.b. \leq \frac{\epsilon^p}{p} |a|^p + \frac{\epsilon^{-q}}{q} |b|^q, \epsilon > 0, a, b \in R,$$

to the last integral on the right-hand side and using $H(f)$, $H(U)$ we obtain,

$$\int_0^t (f(s,x(s))u(s),\dot{x}(s))ds \leq \int_0^t |f(s,x(s))u(s)|_H \cdot |\dot{x}(s)|_H ds$$

$$\leq (\int_0^t |\dot{x}(s)|_H^p ds)^{\frac{1}{p}}(\int_0^t |f(s,x(s))u(s)|_H^q ds)^{1/q}$$

$$\leq \frac{\epsilon^p}{p}\int_0^t |\dot{x}(s)|_H^p ds + \frac{\epsilon^{-q}}{q}\int_0^t |f(s,x(s))u(s)|_H^q ds$$

$$\leq \beta\frac{\epsilon^p}{p}\int_0^t \|\dot{x}(s)\|_X^p ds + \frac{\epsilon^{-q}}{q}\|g\|_\infty^q \|a_1\|_{L^1} + b_1\frac{\epsilon^{-q}}{q}\|g\|_\infty^q \int_0^t |x(s)|_H^2 ds$$

where $\beta > 0$ is the embedding constant $X \hookrightarrow H$.

Hence

$$|\dot{x}(t)|_H^2 + 2(c - \beta\frac{\epsilon^p}{p})\int_0^t \|\dot{x}(s)\|_X^p ds + c'\|x(t)\|_X^2$$

$$\leq M + 2d\int_0^t |\dot{x}(s)|_H^2 ds + 2\frac{\epsilon^{-1}}{q}\|g\|_\infty^q \|a_1\|_{L^1} + 2b_1\frac{\epsilon^{-q}}{q}\|g\|_\infty^q \int_0^t |x(s)|_H^2 ds$$

with $M = |x_1|_H^2 + \|B\|_{\mathcal{L}(X,X^*)}\|x_0\|_X^2$ and consequently, for sufficiently small $\epsilon > 0$, so that ($c > \beta\frac{\epsilon^p}{p}$), we obtain

$$|\dot{x}(t)|_H^2 + c_1\int_0^t \|\dot{x}(s)\|_X^p ds + c'\|x(t)\|_X^2 \leq c_2$$

$$+ 2d\int_0^t |\dot{x}(s)|_H^2 ds + c_2 + c_3\int_0^t |x(s)|_H^2 ds \text{ a.e.,} \tag{2}$$

where c_1, c_2, c_3 are suitable positive constants. Observe that since $\dot{x} \in W_{p,q}(T)$, from theorem 22, p. 19 of Barbu [5], we have $x(s) = x_0 + \int_0^s \dot{x}(\tau)d\tau$ in X (hence in H too),

$$\Rightarrow |x(s)|_H^2 \leq 2|x_0|_H^2 + 2(\int_0^s |\dot{x}(\tau)|_H d\tau)^2 \leq 2|x_0|_H^2 + 2r\int_0^s |\dot{x}(\tau)|_H^2 d\tau.$$

Substituting this estimate in the inequality (2), we obtain

$$|\dot{x}(t)|_H^2 + c_1\int_0^t \|\dot{x}(s)\|_X^p ds + c'\|x(t)\|_X^2 \leq c_4 + c_5\int_0^t |\dot{x}(\tau)|_H^2 d\tau,$$

where c_4 and c_5 are positive constants depending on c_2, c_3, d and $|x_0|_H$.

Hence by Gronwall's inequality, there exists a constant $M_2 > 0$ so that for every admissible trajectory $x(\cdot) \in C(T,X)$ and all $t \in T$, we have

$$|\dot{x}(t)|_H \leq M_2 \tag{3}$$

But recall that $x(t) = x_0 + \int_0^t \dot{x}(s)ds$ in H, for all $t \in T$. So for every trajectory $x(\cdot)$ of (P) and every $t \in T$, we have

$$|x(t)|_H \leq |x_0|_H + \int_0^t |\dot{x}(s)|_H ds \leq |x_0|_H + M_2 r = M_3 \tag{4}$$

Using estimates (3) and (4) in inequality (2), we obtain:

$$|\dot{x}(t)|_H^2 + c_1 \int_0^t \|\dot{x}(s)\|_X^p ds + c' \|x(t)\|_X^2 \leq M_4$$

where M_4 is a positive constant depending on c_5, M_1 and M_2. Then from the last inequality, it follows that

$$\dot{x} \in L^\infty(H) \cap L^p(X), x \in L^\infty(X). \tag{5}$$

Finally let $z \in L^p(X)$, and by $((\cdot,\cdot))_0$ denote the duality brackets for the pair $(L^p(X), L^q(X^*))$ (i.e., if $v \in L^q(X^*), z \in L^p(X)$, then $((v,z))_0 = \int_0^r \langle v(t), z(t)\rangle dt$. Also, let $\widehat{A}: L^p(X) \rightarrow L^q(X^*)$ be the Nemitsky operator corresponding to $A(t,x)$; i.e. $\widehat{A}(y)(t) = A(t,y(t))$ a.e. and similarly for every $u \in S_U^\infty$, $(\widehat{f}(x)u)(t) = f(t,x(t))u(t)$. Clearly by assumption $H(f3)$ $\widehat{f}(x)u(\cdot) \in L^q(H)$. With those notation we can rewrite the evolution equation of problem (P) as an abstract equation in $L^q(X^*)$:

$$\ddot{x} + \widehat{A}(\dot{x}) + Bx = \widehat{f}(x)u.$$

Scalar multiplying this by $z \in L^p(X)$ we have

$$\begin{aligned} ((\ddot{x}, z))_0 &\leq |((\widehat{A}(\dot{x}), z))_0| + |((Bx, z))_0| + ((\widehat{f}(x)u, z))_0 \\ &\leq \Big[\|\widehat{A}(\dot{x})\|_{L^q(X^*)} + \|Bx\|_{L^q(X^*)} + \|\widehat{f}(x)u\|_{L^q(X^*)} \Big] \|z\|_{L^p(X)} \\ &\leq \Big[\|a\|_{L^q} + bM_5 + \|B\|_{\mathcal{L}(X,X^*)} M_6 + \beta' \|g\|_\infty \|\widetilde{a}_1\|_{L^q} + \widetilde{b}_1 M_2) \Big] \|z\|_{L^p(X)} \end{aligned}$$

where β' is the embedding constant $H \hookrightarrow X^*$, and the existence of M_5, M_6 follows from (5) and (6).

Since $z(\cdot) \in L^p(X)$ was arbitrary, we deduce that there exists $M_7 > 0$ so that for all arbitrary trajectories $x(\cdot)$ of (P), we have

$$\| \ddot{x} \|_{L^q(X^*)} \leq M_7. \tag{6}$$

Thus, the assertion of lemma 1 follows from (5) and (6).

Theorem 3.1: *If hypotheses $H(A)$, $H(B)$, $H(U)$, $H(L)$, H_0 hold and $x_0 \in X$, $x_1 \in H$, then problem (P) admits an optimal pair.*

Proof: From lemma 2 it follows that Z is bounded subset of the reflexive Banach space $W_{p,q}(T)$. So Z is relatively weakly compact subset of $W_{p,q}(T)$. Now let $\{(x_n, u_n)\}_{n \geq 1}$ be a minimizing sequence of admissible "state-control" pairs for the problem (P); i.e. $\lim_{n \to \infty} J(x_n, u_n) = Inf\{J(x,u),$ for *admissible "state-control" pair* $(x,u)\} \equiv m$. Since $\{x_n\}_{n \geq} \subseteq S$, by passing to a subsequence if necessary, we may assume that $\dot{x}_n \xrightarrow{w} y$ in $W_{p,q}(T)$. Hence one can easily see that $x \in C(T,X)$ and that $y = \dot{x}$ in the distribution sense. But recall that $W_{p,q}(T)$ embeds compactly into $L^p(H)$. So $\dot{x}_n \xrightarrow{s} y$ in $L^p(H)$ and clearly $x_n(t) \xrightarrow{s} x(t)$ in H uniformly on T. Furthermore, from hypothesis $H(U)$ and proposition 3.1 of [7] we have $S_U^\infty \equiv \{u \in L^\infty(Y): u(t) \in U(t) a.e.\}$ is w_*-compact in $L^\infty(Y)$. So we may assume that $u_n \xrightarrow{w_*} u$ in $L^\infty(Y)$. Then invoking theorem 2.1 of Balder [4], we conclude that $J(x,u)$ is strong-w_* *l.s.c.* i.e., $J(x,u) \leq \underline{lim} J(x_n, u_n) = m$, whenever $x_n \xrightarrow{s} x$ in $L^1(H)$ and $u_n \xrightarrow{w_*} u$ in $L^\infty(Y)$.

It suffices to show that (x,u) is an admissible "state-control" pair for (P). To this end, we have

$$((\ddot{x}_n, \dot{x}_n - \dot{x}))_0 + ((\widehat{A}(\dot{x}_n), \dot{x}_n - \dot{x})) + ((Bx_n, \dot{x}_n - \dot{x}))_0 = ((\widehat{f}(x_n)u_n, \dot{x}_n - \dot{x}))_0. \tag{7}$$

From the integration by parts formula for functions in $W_{p,q}(T)$ (see Zeidler [11], proposition 23.23 (iv), pp. 422-423), we have:

$$((\ddot{x}_n, \dot{x}_n - \dot{x}))_0 = \frac{1}{2} | \dot{x}_n(r) - \dot{x}(r) |_H^2 - \frac{1}{2} | \dot{x}_n(0) - x_1 |_H^2 + ((\ddot{x}, \dot{x}_n - \dot{x}))_0 \tag{8}$$

since $\dot{x}_n(0) = x_1$, the second term vanishes and $| \dot{x}_n(r) - \dot{x}(r) |_H \to 0$ as $n \to \infty$, $(\dot{x}_n \in C(T,H))$ and also since $\dot{x}_n \xrightarrow{w} y = \dot{x}$ in $L^p(X)$ we have $((\ddot{x}, \dot{x}_n - \dot{x}))_0 \to 0$ as $n \to \infty$. Then by passing to the limit as $n \to \infty$ in (8) we have

$$((\ddot{x}_n, \dot{x}_n - \dot{x}))_0 \to 0 \text{ as } n \to \infty. \tag{9}$$

Note that for every $h \in L^p(H)$, we have

$$\int_0^r (f(t, x_n(t)) u_n(t), h(t)) dt = \int_0^r (u_n(t), (f(t, x_n(t))^* h(t)) dt.$$

But since $x_n(t) \xrightarrow{s} x(t)$ in H, $(f(t, x_n(t)))^* h(t) \xrightarrow{s} (f(t, x(t)))^* h(t)$ in Y^* for almost all $t \in T$ (see hypothesis $H(f2)$). Also by $H(f3)$ we have

$$\| (f(t,x_n(t)))^* h(t) \|_{Y^*} \leq \| (f(t,x_n))^* \|_{\mathcal{L}(H,Y^*)} \cdot | h(t) |_H$$
$$\leq \tilde{a}(t) + (\tilde{b}\, M_3^{2/q}) | h(t) |_H.$$

So there exists $\eta(\cdot) \in L^1(Y^*)$ so that $\| (f(t,x_n(t)))^* h(t) \|_{Y^*} \leq \| \eta(t) \|_{Y^*}$ *a.e.* and therefore it follows from dominated convergence theorem that $(\hat{f}(x_n))^* h \xrightarrow{s} (\hat{f}(x))^* h$ in $L^1(Y^*)$.

Hence

$$\int_0^r (u_n(t),(f(t,x_n(t))^* h(t))dt \to \int_0^r (u(t),(f(t,x(t))^* h(t))dt$$
$$\Rightarrow \int_0^r (f(t,x_n(t))u_n(t),h(t))dt \to \int_0^r (f(t,x(t))u(t),h(t))dt$$
$$\Rightarrow \hat{f}(x_n)u_n \xrightarrow{w} \hat{f}(x)u \text{ in } L^q(H).$$

On the other hand, since $\dot{x}_n \xrightarrow{w} \dot{x}$ in $W_{p,p}(T)$ and since the embedding $W_{p,q}(T) \hookrightarrow L^p(H)$ is compact, we have that $\| \dot{x}_n - \dot{x} \|_{L^p(H)} \to 0$ and hence

$$((\hat{f}(x_n)u_n, \dot{x}_n - \dot{x}))_{L^q(H), L^p(H)} \to 0. \tag{10}$$

Exploiting the symmetry of the operator B, we have

$$\frac{d}{dt}\langle B(x_n(t)-x(t)), x_n(t)-x(t)\rangle = 2\langle B(x_n(t)-x(t)), \dot{x}_n(t)-\dot{x}(t)\rangle a.e.$$

Integrating the above equality, we get

$$\langle B(x_n(r)-x(r)), x_n(r)-x(r)\rangle = 2((B(x_n-x), \dot{x}_n-\dot{x}))_0$$
$$\Rightarrow c' \| x_n(r)-x(r) \|_X^2 + 2((Bx,\dot{x}_n-\dot{x}))_0 \leq 2((Bx_n,\dot{x}_n-\dot{x}))_0.$$

Note that since $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^p(X)$, we have $x_n(r) \xrightarrow{w} x(r)$ in X. Obviously

$$0 \geq \underline{lim} \| x_n(r)-x(r) \|_X^2,$$

and clearly $((Bx,\dot{x}_n-\dot{x}))_0 \to 0$. Thus we have

$$c'\underline{lim} \| x_n(r)-x(r) \|_X^2 + 2lim((Bx,\dot{x}_n-\dot{x}))_0 \leq 2\underline{lim}((Bx_n,\dot{x}_n-\dot{x}))_0$$
$$\Rightarrow 0 \leq \underline{lim}((Bx_n,\dot{x}_n-\dot{x}))_0. \tag{11}$$

Now passing to the limit as $n \to \infty$ in (7) and using (9), (10) and (11) above we get that

$$\overline{lim}((\hat{A}(\dot{x}_n),\dot{x}_n-\dot{x}))_0 \leq 0.$$

Also note that because of hypothesis $H(A4)$, $\{\widehat{A}(\dot{x}_n)\}_{n \geq 1} \subseteq L^q(X^*)$ is bounded and so by passing to a subsequence we may assume that $\widehat{A}(\dot{x}_n) \xrightarrow{w} v$ in $L^q(X^*)$. But $\widehat{A}(\cdot)$ is hemicontinuous, monotone (since $A(t, \cdot)$ is), hence it has property $(\underline{M})$ (see Zeidler [11], pp. 583-584 and Ahmed [2, 3]). thus $\widehat{A}(\dot{x}) = v$; i.e., $\widehat{A}(\dot{x}_n) \xrightarrow{w} \widehat{A}(\dot{x})$ in $L^q(X^*)$. Then for any $z \in L^p(X)$, we have:

$$((\ddot{x}_n, z))_0 + ((\widehat{A}(\dot{x}_n), z))_0 + ((Bx_n, z))_0 = ((\widehat{f}(x_n)u_n, z))_0$$

$$\rightarrow ((\ddot{x}, z))_0 + ((\widehat{A}(\dot{x}), z))_0 + ((Bx, z))_0 = ((\widehat{f}(x)u, z))_0 \text{ as } n \rightarrow \infty$$

$$\Rightarrow \ddot{x}(t) + A(t, \dot{x}(t)) + Bx(t) = f(t, x(t))u(t) \ a.e.\ x(0) = x_0 \in X, \dot{x}(0) = x_1 \in H, u \in S_U^\infty$$

$\Rightarrow (x, u)$ is an admissible "state-control pair for (P). So

$$J(x, u) = m$$

$\Rightarrow (x, u)$ is the desired optimal pair.

Q.E.D.

4. AN EXAMPLE

In this section we work out in detail an example of a nonlinear, hyperbolic optimal control problem.

So let $T = [0, r]$ and Ω a bounded domain in R^n, with smooth boundary $\Gamma = \partial\Omega$. We consider the following Lagrange control problem:

$$(P')\quad \left\{ \begin{array}{l} J(\phi, u) = \int_0^r \int_Z L(t, z, \phi(t, z), u(t, z)) dz dt \rightarrow inf = m' \\ \text{subject to } \{\phi, u\} \text{ satisfying the following constraints:} \\ \frac{\partial^2 \phi}{\partial t^2} - \Delta\phi = \sum_{ij=1}^N D_i(k_{ij}(t, | D\phi_t |^{p-2})) D_j \phi_t = f(t, z, \phi(t, z)) u(t, z) a.e. \text{ on } T \times \Omega \\ \phi |_{T \times \Gamma} = 0, \phi(0, z) = \phi_0(z_0), \phi_t(0, z) = \phi_1(z) \text{ and } \| U(t, \cdot) \|_{L^\infty(\Omega)} \leq \eta(t) a.e. \end{array} \right.$$

Here $D_i = \frac{\partial}{\partial z_i}$ $i = 1, 2, \ldots, N$, $D\phi = (D_1\phi, \ldots, D_{N\phi}) = gradient\ of\ \phi$, $D\phi D\psi = \sum_{i,j=1}^N D_i \phi D_j \psi$ and $| D\psi |^2 = \sum_{i=1}^N | D_i \psi |^2$. We will need the following hypotheses on the data of (P').

$\underline{H(k)}$: k is a matrix from $T \times R_+ \rightarrow \mathcal{L}^+(R^n)$ so that:

(1) $t \rightarrow k(t, \mu)$ is measurable,

(2) $\mu \rightarrow k(t, \mu)$ is continuous,

(3) $| k(t, \xi) |_{\mathcal{L}(R^n)} \leq \alpha + \beta | \xi |$ for all $(t, \xi) \in T \times R^n$ with $\beta > 0$ and $\alpha \geq 0$,

(4) $\langle k(t, | \xi |^{p-2})\xi - k(t, | \eta |^{p-2})\eta, \xi - \eta \rangle_{R^n} \geq 0$ for all $(t, \xi, \eta) \in T \times R^n \times R^n$,

(5) $\langle k(t, | \xi |^{p-2})\xi, \xi \rangle_{R^n} \geq \beta | \xi |^p_{R^n}$ for all $(t, \xi) \in T \times R^n$ and $\beta > 0$.

$\underline{H(f)_1}$: $f: T \times \Omega \times R \rightarrow R$ is a function satisfying

(1) $(t, z) \rightarrow f(t, z, x)$ is measurable,

(2) $x \rightarrow f(t, z, x)$ is continuous,

(3) $| f(t, z, x) | \leq a_1(t, z) + b_1(z) | x |$ *a.e.* with $a_1(\cdot, \cdot) \in L^2(T \times \Omega), b_1(\cdot) \in L^{\infty}(\Omega)$.

$\underline{H(\eta)}$: $\eta(\cdot) \in L^1_+$.

$\underline{H_0}$: $\phi_0 \in W_0^{1,p}(\Omega), \phi_1 \in L^2(\Omega)$ and $m' < \infty$.

$\underline{H(\hat{L})}$: $\hat{L}: T \times \Omega \times R \times R \times R \rightarrow \bar{R} = R \cup \{+\infty\}$ is an integrand s.t.

(1) $(t, z, x, y, u) \rightarrow \hat{L}(t, z, x, y, u)$ is measurable,

(2) $(x, y, u) \rightarrow \hat{L}(t, z, x, y, u)$ is *l.s.c.*,

(3) $u \rightarrow \hat{L}(t, z, x, y, u)$ is convex,

(4) $\varphi(t, z) - \hat{M}(x)(| x |_R + | y |_R + | u |_R) \leq L(t, z, x, y, u)$ *a.e.* with $\varphi(\cdot, \cdot) \in L^1(T \times \Omega)$, and $\hat{M}(\cdot) \in L^{\infty}_+(\Omega)$.

Consider the following Dirichlet forms:

$$a_1(t, \phi, \psi) = \int_{\Omega} \sum_{i,j=1}^{N} k_{i,j}(t, | D\phi |^{p-2}) D_i\phi D_j\psi dz = \int_{\Omega} \langle k(t, | D\phi |^{p-2}) D\phi, D\psi \rangle_{R^n} dz$$

and

$$a_2(\phi, \psi) = \int_{\Omega} \sum_{i,j=1}^{N} D_i\phi D_j\psi dz = \int_{\Omega} D\phi D\psi dz$$

for all $\phi, \psi \in W_0^{1,p}(\Omega)$. Using hypothesis $H(k3)$, we get

$$| a_1(t, \phi, \psi) | \leq (\zeta \| \phi \|_{W_0^{1,p}(\Omega)} + \beta \| \phi \|^{p-1}_{W_0^{1,p}(\Omega)}) \| \psi \|_{W_0^{1,p}(Z)}$$

where ζ is a positive constant dependent on the embedding constant $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,2}(\Omega)$ and α as defined in $H(k3)$.

So there exists an operator $A: T \times X \rightarrow X^*$ s.t.

$$\langle A(t, \phi), \psi \rangle = a_1(t, \phi, \psi).$$

Note that by Fubini's theorem, $t \rightarrow a_1(t, \phi, \psi)$ is measurable for all $\phi, \psi \in W_0^{1,p}(\Omega)$. Hence,

$t \to A(t,\phi)$ is weakly measurable from T into $W^{-1,q}(\Omega)$. But recall that $W^{-1,q}(\Omega)$ is a separable Hilbert space. Thus the Pettis' measurability theorem tells us that $t \to A(t,\phi)$ is measurable. Also let $\phi_n \xrightarrow{s} \phi$ in $W_0^{1,p}(\Omega)$. Then $D\phi_n \xrightarrow{s} D\phi$ in $L^p(\Omega, R^N)$ and since by hypothesis $H(k2)$, $k(t,\cdot)$ is continuous, we have $k(t, |D\phi_n(z)|^{p-2}) \to k(t, |D\phi(z)|^{p-2})$ *a.e.* $\Rightarrow \int_\Omega \langle k(t, |D\phi_n|^{p-2}) D\phi_n, D\psi\rangle_{R^n} dz \to \int_\Omega \langle k(t, |D\phi|^{p-2}) D\phi, D\psi\rangle_{R^n} dz \Rightarrow A(t,\phi_n) \xrightarrow{w} A(t,\phi)$ in $W^{-1,p}(\Omega) \Rightarrow A(t,\cdot)$ is demicontinuous, hence hemicontinuous (see Zeidler [11]). Also for every $\phi, \psi \in W_0^{1,p}(\Omega)$, we have

$$\langle A(t,\phi) - A(t,\psi), \phi - \psi\rangle = \int_\Omega \langle k(t, |D\phi|^{p-2}) D\phi - k(t, |D\psi|^{p-2}) D\psi, (D\phi - D\psi)\rangle_{R^n} dz.$$

Therefore, the monotonicity of $A(t,\cdot)$ follows from hypothesis $H(k4)$. Furthermore, from hypothesis $H(k5)$ we obtain

$$\langle A(t,\phi), \phi\rangle \geq \beta \|\phi\|^p_{W_0^{1,p}(\Omega)}, \text{ with } \beta > 0.$$

Thus we have satisfied hypothesis $H(A)$.

Next note that through the Cauchy-Schwartz inequality, we get

$$|a_2(\phi,\psi) \leq \mu(\Omega)^{\frac{p-2}{p}} \|\phi\|_{W_0^{1,p}(\Omega)} \|\psi\|_{W_0^{1,p}(\Omega)}.$$

Thus there exists $B \in \mathcal{L}(X, X^*)$ s.t.

$$a_2(\phi,\psi) = \langle B\phi, \psi\rangle$$

for all $\phi, \psi \in W_0^{1,p}(\Omega)$. Clearly B is symmetric and using Poincare's inequality, we obtain

$$\langle B\phi, \phi\rangle \geq c' \|\phi\|^2_{W_0^{1,p}(\Omega)}, c' > 0.$$

Thus we have satisfied hypothesis $H(B)$.

Let $\hat{f}: T \times L^2(\Omega) \to L^2(\Omega)$ be defined by

$$\hat{f}(t,\phi)(z) = f(t, z, \phi(z)).$$

In this case, $H = L^2(\Omega)$. Thus $\hat{f}(t,\phi)$ is the Nemitsky operator corresponding to f and so by Krasnosel'skii's theorem, it satisfies hypothesis $H(f)$.

For the control space we put $Y = L^\infty(\Omega)$ and $U(t) = \{u \in L^\infty(\Omega): \|u\|_\infty \leq \eta(t)\}$. Note that $GrU = \{(t,u) \in T \times L^\infty(\Omega): u(t) \in U(t) a.e.\}$. Observe that the function $(t,u) \to (\eta(t) - \|u\|_\infty)$ is measurable in t, continuous in u, thus jointly measurable. Hence $GrU \in B(T) \times B(L^\infty(\Omega))$ with $B(T)$ (resp. $B(L^\infty(\Omega))$)), being the Borel σ-field of T (resp. of

$L^\infty(\Omega))$. Then by theorem 4.2 of Wagner [10] $U(\cdot)$ is measurable, while from hypothesis $H(U)$, we deduce that $t \to |U(t)| \in L^\infty_+$. So we have satisfied hypothesis $H(U)$.

Also let $\widehat{\phi}_0 = \phi_0(\cdot) \in W_0^{1,p}(\Omega)$ and $\widehat{\phi}_1 = \phi_1(\cdot) \in L^2(\Omega)$ (see hypothesis H_0). Finally let $\widehat{L}: T \times L^2(\Omega) \times L^2(\Omega) \times L^\infty(\Omega) \to \bar{R}$ be defined by

$$L(t,\phi,\psi,u) = \int_\Omega \widehat{L}(t,z,\phi(z),\psi(z),u(z))dz, \phi,\psi \in L^2(\Omega), u \in L^\infty(\Omega).$$

Invoking theorem 1 of Pappas [9], we can find Caratheodory integrands $\widehat{L}_k$: $T \times \Omega \times R \times R \times R \to R$, $k \geq 1$ (i.e. $(t,z) \to \widehat{L}_k(t,z,\phi,\psi,u)$ is measurable, $(\phi,\psi,u) \to \widehat{L}_k(t,\phi,z,\psi,u)$ is continuous), so that $\widehat{L}_k \uparrow \widehat{L}$ and $\varphi(t,z) - M(z) \ \ |\phi|_R + |\psi|_R + |u|_R) \leq \widehat{L}_k(t,z,\phi,\psi,u) \leq k$ *a.e.* $k \geq 1$. Set $L_k(t,\phi,\psi,u) = \int_\Omega \widehat{L}(t,z,\phi(z),\psi(z),u(z))dz$. It is easy to check that $t \to L(t,\phi,\psi,u)$ is measurable, while $(\phi,\psi,u) \to L_k(t,\phi,\psi,u)$ is continuous, thus $L_k(\cdot,\cdot,\cdot,\cdot)$ is jointly measurable. Furthermore, from the monotone convergence theorem, we get $L_k \uparrow L$, hence $L(\cdot,\cdot,\cdot,\cdot)$ is measurable. Also from Balder [4], we know that $(\phi,\psi,z) \to L(t,\phi,\psi,z)$ is *l.s.c.*, while $L(t,\phi,\psi,\cdot)$ is clearly convex and $\widehat{\varphi}(t) - \widehat{M}(\|\phi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)} + \|u\|_\infty) \leq L(t,\phi,\psi,u)$, with $\widehat{\varphi}(t) = \|(\varphi(t,\cdot)\|_{L^2(\Omega)}$ and $\widehat{M} = \|M(\cdot)\|_\infty$. So we have satisfied hypothesis $H(L)$. In this case, $X = W_0^{1,p}(\Omega)$, $H = L^2(\Omega)$ and $X^* = W^{-1,q}(\Omega)$. We know that (X,H,X^*) is an evolution triple, with all embeddings being compact (Sobolev embedding theorem). Defining $x(t) = \phi(t,\cdot)$, it is easy to check that the example problem (P') is a special case of the abstract problem (P).

Theorem 3.1: *If hypotheses $H(k)$, $H(f)_1$, $H(\eta)$, H_0, $H(L)$ hold, then (P') admits an optimal pair $[x,u] \in C(T, W_0^{1,p}(\Omega)) \times L^\infty(T \times \Omega)$ so that*

$$\frac{\partial x}{\partial t} \in L^\infty(T, W_0^{1,p}(\Omega)) \cap C(T, L^2(\Omega)) \text{ and } \frac{\partial^2 x}{\partial t^2} \in L^q(T, W^{-1,q}(\Omega)).$$

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