

NUMERICAL SOLUTIONS OF NONSTANDARD FIRST ORDER INITIAL VALUE PROBLEMS¹

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ABSTRACT

Differential equations of the form $y' = f(t, y, y')$, where f is not necessarily linear in its arguments, represent certain physical phenomena and are known for quite some time. The well known Clairut's and Chrystal's equations fall into this category. Earlier, we established the existence of a (unique) solution of the nonstandard initial value problem (*NSTD IVP*) $y' = f(t, y, y')$, $y(t_0) = y_0$ under certain natural hypotheses on f . In this paper we present some first order convergent numerical methods for finding the approximate solutions of the *NSTD IVPs*.

Key words: Nonstandard initial value problem, existence, unique solution, real valued, continuous, differentiable, numerical scheme, inequality, partition, approximate, algorithm, influence, round-off error, estimate, truncation.

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1. INTRODUCTION

Differential equations of the form $y' = f(t, y, y')$, where f is not necessarily linear in its arguments, represent certain physical phenomena and are known for quite some time. The well known Clairut's and Chrystal's equations fall into this category [2]. A few authors, notably E.L. Ince [3], H.T. Davis [2], et. al. have given some methods for finding solutions of equations of the above type. In fact these methods are best described as follows.

If there exists (t_0, y_0) such that the equation $y' = f(t, y, y')$ can be solved for y' as a single valued function of (t, y) in a neighborhood of (t_0, y_0) , say $y' = g(t, y)$, then the solution of the initial value problem (*IVP*) $y' = g(t, y)$, $y(t_0) = y_0$, if it exists, is also a solution of the original equation $y' = f(t, y, y')$ (and satisfies the initial condition $y(t_0) = y_0$). Or, if there exists (t_0, y_0) such that the equation $y' = f(t, y, y')$ can be solved for y' as a multivalued function of (t, y) in a neighborhood of

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(t_0, y_0) , then a (nonunique) solution of the IVP $y' = f(t, y, y'), y(t_0) = y_0$ is given by certain (not necessarily convergent) infinite series.

In our earlier paper [7], we have established the existence of a (unique) solution of the nonstandard IVP $y' = f(t, y, y'), y(t_0) = y_0$ under certain natural hypotheses on f , and have shown the continuous dependence of the solution on initial conditions, and parameters. The existence result that we established in [7] is of theoretical nature and it does not provide methods of finding the solutions explicitly either analytically or numerically. Our aim in this paper is to present numerical methods for finding the approximate solutions of the nonstandard IVPs.

Before proceeding to numerical methods, we introduce few notations and the nonstandard IVP. Let R denote the real line, and let R^n denote the n -dimensional real space where 'n' is a positive integer. Let $(t_0, y_0) \in R^2$ and let D be a convex subset of R^3 defined by $D = \{(t, y, z) \in R^3 \mid |t - t_0| \leq a, |y - y_0| < b, |z| \leq c\}$ where a, b and c are some positive constants. Let f be a continuous real valued function defined on D .

Consider the nonstandard IVP (NSTD IVP)

$$y' = f(t, y, y'), y(t_0) = y_0 \quad (1)$$

where y' denotes the derivative of y .

Definition: By a solution y of the NSTD IVP (1), we mean a continuously differentiable real valued function $y(t)$ defined for $t \in I$, where I is some interval of the real line containing the point t_0 such that

- i) $y(t_0) = y_0$
- ii) the triplet $(t, y(t), y'(t)) \in D$ for all $t \in I$, and
- iii) $y'(t) = f(t, y(t), y'(t))$ holds for all $t \in I$.

In Section 2, we shall prove the existence of a unique higher order regular solution of the IVP(1). In Section 3, we shall present linearly convergent numerical schemes for constructing the solution of IVP(1). In Section 4, we shall illustrate our methods by solving some NSTD IVPs.

2. EXISTENCE OF A UNIQUE (HIGHER ORDER REGULAR) SOLUTION OF THE IVP (1)

Theorem 1: Suppose that the real valued function f defined on D satisfies the following conditions:

- I) f is 'p' times continuous differentiable with respect to $(t, y, z) \in D$ where $p \geq 1$ is an integer.
- II) $|f(t, y, z)| \leq c$ for all $(t, y, z) \in D$, and

III) $|\frac{\partial f(t, y, z)}{\partial y}| \leq k_1$, and $|\frac{\partial f(t, y, z)}{\partial z}| \leq k_2$ for all $(t, y, z) \in D$ where $k_1 > 0$ and $0 \leq k_2 < 1$ are constants.

Then the NSTD IVP (1) has a unique solution y existing on the interval $[t_0 - \alpha, t_0 + \alpha]$, where α is any number such that $0 < \alpha < \min(\frac{1-k_2}{k_1}, \frac{b}{c}, a)$. Moreover, the solution y is ' $p+1$ ' times differentiable on $[t_0 - \alpha, t_0 + \alpha]$.

Proof: To prove the existence of a unique solution, it is enough to verify conditions (i) – (iii) of Theorem 1 [7]. Clearly conditions (I) and (II) imply the conditions (i) and (ii) of Theorem 1 [7]. To verify condition (iii), let (t, y_1, z_1) and $(t, y_2, z_2) \in D$. Then by the mean value Theorem [1], we have that

$$\begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)| &\leq |\frac{\partial f(t, \hat{y}, \hat{z})}{\partial y}| |y_1 - y_2| + |\frac{\partial f(t, \hat{y}, \hat{z})}{\partial z}| |z_1 - z_2| \\ &\leq k_1 |y_1 - y_2| + k_2 |z_1 - z_2| \text{ (by (III)),} \end{aligned}$$

where (t, \hat{y}, \hat{z}) is a point on the line segment joining (t, y_1, z_1) and (t, y_2, z_2) . Thus f satisfies conditions (i) – (iii) of Theorem 1 [7], and hence the IVP (1) has a unique solution y existing on the interval $[t_0 - \alpha, t_0 + \alpha]$, for every α such that $0 < \alpha < \min(\frac{1-k_2}{k_1}, \frac{b}{c}, a)$. To show that y is ' $p+1$ ' times continuously differentiable, we proceed as follows.

From the definition of a solution, it follows that y' is a continuous function on $[t_0 - \alpha, t_0 + \alpha]$ and we now claim that y'' exists on $[t_0 - \alpha, t_0 + \alpha]$ and is continuous function. For, let $t \in [t_0 - \alpha, t_0 + \alpha]$ and $h \neq 0$ be sufficiently small. Consider the difference quotient

$$\frac{y'(t+h) - y'(t)}{h} = \frac{f(t+h, y(t+h), y'(t+h)) - f(t, y(t), y'(t))}{h}.$$

By the mean value theorem, there exists a point (τ, η, ς) on the line segment joining $(t, y(t), y'(t))$ and $(t+h, y(t+h), y'(t+h))$ such that

$$\frac{y'(t+h) - y'(t)}{h} = \frac{\partial f(\tau, \eta, \varsigma)}{\partial t} + \frac{\partial f(\tau, \eta, \varsigma)}{\partial y} \left(\frac{y(t+h) - y(t)}{h} \right) + \frac{\partial f(\tau, \eta, \varsigma)}{\partial z} \left(\frac{y'(t+h) - y'(t)}{h} \right). \quad (2)$$

But we have

$$\left| 1 - \frac{\partial f(\tau, \eta, \varsigma)}{\partial z} \right| \geq 1 - \left| \frac{\partial f(\tau, \eta, \varsigma)}{\partial z} \right| \geq 1 - k_2 > 0 \text{ (by (III)).} \quad (3)$$

Therefore, from relation (2), we get that

$$\frac{y'(t+h) - y'(t)}{h} = \left\{ \frac{\partial f(\tau, \eta, \varsigma)}{\partial t} + \frac{\partial f(\tau, \eta, \varsigma)}{\partial y} \left(\frac{y(t+h) - y(t)}{h} \right) \right\} / \left[1 - \frac{\partial f(\tau, \eta, \varsigma)}{\partial z} \right]$$

Now, taking the limit as $h \rightarrow 0$, we get that

$$\lim_{h \rightarrow 0} \frac{y'(t+h) - y'(t)}{h} = \frac{\frac{\partial f(t, y(t), y'(t))}{\partial t} + \frac{\partial f(t, y(t), y'(t))}{\partial y} y'(t)}{1 - \frac{\partial f(t, y(t), y'(t))}{\partial z}} \quad (4)$$

We observe that the right hand side of (4) is a well defined quantity and is a continuous function of t . Hence

$$y''(t) = \frac{\frac{\partial f(t, y(t), y'(t))}{\partial t} + \frac{\partial f(t, y(t), y'(t))}{\partial y} y'(t)}{1 - \frac{\partial f(t, y(t), y'(t))}{\partial z}} \text{ exists and is continuous.} \quad (5)$$

Now, from condition (I) and inequality (3), it follows that the right hand side of (5) is a ' $p-1$ ' times continuously differentiable function of ' t ' on $[t_0 - \alpha, t_0 + \alpha]$, and hence the solution y is ' $p+1$ ' times continuously differentiable on $[t_0 - \alpha, t_0 + \alpha]$.

3. NUMERICAL SCHEMES FOR THE SOLUTION OF IVP (1)

We assume that the conditions of Theorem 1 be satisfied and $p = 1$. For each positive integer ' n ', we divide the interval $[t_0, t_0 + \alpha]$ into ' n ' subintervals of equal length, i.e., we take a partition of the form $t_0 < t_1 < \dots < t_{n-1} < t_n = t_0 + \alpha$, where $t_i = t_0 + ih$, $h = \alpha/n$, $i = 0, 1, \dots, n$. For $t \in (t_0, t_0 + \alpha)$ and h sufficiently small such that $t + h \in [t_0, t_0 + \alpha]$, by Taylor's theorem, the solution y of the IVP (1) satisfies the relation

$$y(t+h) = y(t) + hy'(t) + \frac{h^2}{2}y''(t + \theta h), 0 < \theta < 1. \quad (6)$$

From (5) and (3), and condition (I), we easily get that $|y''(t)| \leq N$, for all $t \in [t_0 - \alpha, t_0 + \alpha]$ where $N > 0$ is a constant.

Let η_i denote the approximate value of y at $t_i, i = 0, 1, \dots, n$, and let ϵ denote tolerance limit.

Scheme I (Contraction-Euler Method):

We define

- a) $\eta_0 = y_0$
for $i = 0, 1, 2, \dots, n-1$
 $\eta'_{i0} = 0; j = 0$
- b) repeat
 $j = j + 1$
 $\eta'_{ij} = f(t_i, \eta_i, \eta'_{i,j-1})$ (7)
until $k_2^j < |h|$ and $|\eta'_{ij} - \eta'_{i,j-1}| \leq \epsilon$ are satisfied;
- c) $\eta_{i+1} = \eta_i + h\eta'_{ij}$
 $t_{i+1} = t_i + h.$

This algorithm computes the approximate value of $y(t_i)$ at $t_i = t_0 + ih, i = 1, 2, \dots, n$.

To show the validity of the above algorithm, let m be the smallest non-negative integer j which satisfies the conditions $k_2^j < |h|$ and $|\eta'_{ij} - \eta'_{i,j-1}| \leq \epsilon$, for a fixed 'i'. We have

$$\begin{aligned} |\eta_0 - y_0| &= 0 \leq b \\ |\eta_1 - y_0| &= |\eta_1 - \eta_0| = h |\eta'_{0m}| \leq \frac{\alpha}{n} c \leq b \\ |\eta_2 - y_1| &\leq |\eta_1 - y_0| + h |\eta'_{1m}| \leq \frac{c\alpha}{n} + \frac{c\alpha}{n} \leq b \\ &\vdots \\ &\vdots \\ |\eta_{i+1} - y_0| &\leq |\eta_i - y_0| + h |\eta'_{im}| \leq \frac{c\alpha i}{n} + \frac{c\alpha}{n} \leq b \\ &\vdots \\ &\vdots \\ |\eta_n - y_0| &\leq |\eta_{n-1} - y_0| + h |\eta'_{(n-1)m}| \leq \frac{(n-1)c\alpha}{n} + \frac{c\alpha}{n} = c\alpha \leq b, \end{aligned}$$

since α is such that $0 < \alpha < \min((1 - k_2)/k_1, b/c, a)$. Clearly, we have $|\eta'_{im}| \leq c$ for all $i = 0, 1, \dots, n$.

Lemma 1: [6] *If the numbers ξ_i satisfy the estimates of the form $|\xi_{i+1}| \leq (1 + \delta)|\xi_i| + B$, where $\delta > 0$, $B > 0$, $i = 1, \dots, q, \dots$, then $|\xi_q| \leq e^{q\delta} |\xi_0| + (\frac{e^{q\delta} - 1}{\delta})B$, $q = 1, 2, 3, \dots$*

Theorem 2 (Error estimate and Convergence of the Scheme I): *Let $e_i = \eta_i - y(t_i)$, $i = 0, 1, \dots, n$. Then $|e_i| \leq \hat{N}h(e^{\alpha k_1/(1 - k_2)} - 1)$, for $i = 0, 1, \dots, n$, where $\hat{N} > 0$ is a constant.*

Proof: Clearly, we have $e_0 = \eta_0 - y_0 = y_0 - y_0$. For brevity, let $y_i = y(t_i)$ and consider

$$\begin{aligned} e_{i+1} &= \eta_{i+1} - y_{i+1} \\ &= \eta_i + h\eta'_{im} - (y_i + hy'(t_i) + \frac{h^2}{2}y''(t_i + \theta h)), 0 < \theta < 1, \\ &\quad \text{(by (6) and (7)),} \\ &= (\eta_i - y_i) + h(\eta'_{im} - y'(t_i)) - \frac{h^2}{2}y''(t_i + \theta h) \end{aligned}$$

Therefore, we have that

$$|e_{i+1}| \leq |e_i| + \frac{Nh^2}{2} + h|\eta'_{im} - y'(t_i)|, \tag{8}$$

where N is the bound for y'' .

Also, we have that $\eta'_{im} = f(t_i, \eta_i, \eta'_{i,m-1})$ and f , for a fixed (t_i, η_i) , is a contraction on the closed set $\{z \in R \mid |z| \leq c\}$ with the contraction constant ' k_2 '. Therefore, by the contraction mapping theorem [4], there exists a unique value η'_i such that

$$\eta'_i = f(t_i, \eta_i, \eta'_i), \quad |\eta'_i| \leq c, \tag{9}$$

and

$$|\eta'_{im} - \eta'_i| \leq \frac{k_2^m}{1 - k_2} c. \tag{10}$$

Moreover we have that

$$\begin{aligned} |\eta'_i - y'(t_i)| &= |f(t_i, \eta_i, \eta'_i) - f(t_i, y_i, y'(t_i))| \\ &\leq k_1 |\eta_i - y_i| + k_2 |\eta'_i - y'(t_i)|, \end{aligned}$$

which implies that

$$|\eta'_i - y'(t_i)| \leq \left(\frac{k_1}{1-k_2}\right) |\eta_i - y_i| = \left(\frac{k_1}{1-k_2}\right) |e_i|. \quad (11)$$

Thus in view of (10) and (11), from (8) we get that

$$\begin{aligned} |e_{i+1}| &\leq |e_i| + \frac{Nh^2}{2} + h(|\eta'_{im} - \eta'_i| + |\eta'_i - y'(t_i)|) \\ &\leq |e_i| + \frac{Nh^2}{2} + h\left(\frac{k_1}{1-k_2}|e_i| + \frac{k_2^m c}{1-k_2}\right) \\ &= \left(1 + \frac{k_1 h}{1-k_2}\right) |e_i| + \frac{Nh^2}{2} + \frac{hk_2^m c}{1-k_2}, \quad i = 0, 1, \dots, n-1. \end{aligned} \quad (12)$$

Hence by lemma 1, we get that

$$\begin{aligned} |e_q| &\leq e^{\frac{qhk_1}{1-k_2}} e_0 + \frac{e^{\frac{qhk_1}{1-k_2}} - 1}{\frac{hk_1}{1-k_2}} \left[\left(\frac{Nh^2}{2} + \frac{hk_2^m c}{1-k_2}\right)\right] \leq \left(\frac{e^{\frac{\alpha k_1}{1-k_2}} - 1}{\frac{k_1}{1-k_2}}\right) \left(\frac{Nh}{2} + \frac{k_2^m c}{1-k_2}\right) \quad (\text{since } e_0 = 0) \\ &= \hat{N}h(e^{\alpha k_1/(1-k_2)} - 1), \quad \text{where } \hat{N} = \left(\frac{N}{2} + \frac{c}{1-k_2}\right) \left(\frac{1-k_2}{k_1}\right) \quad (\text{since } k_2^m < |h|), \\ &\quad \text{for } q = 1, 2, \dots, n. \end{aligned}$$

Influence of the round-off errors: Let $y_i = y(t_i)$ be the value of the exact solution y of the IVP (1), η_i be the approximate solution produced by the scheme I , and $\hat{\eta}_i$ be the approximate value of η_i actually obtained in the digit floating point arithmetic, at t_i for $i = 0, 1, \dots, n$. Let $R_i = \hat{\eta}_i - \eta_i$ be the round-off error at the i -th step. Then we have

$$\eta_{i+1} = \eta_i + h\eta'_{im} - R_{i+1}, \quad (13)$$

and

$$y_{i+1} = y_i + hy'(t_i) + \frac{h^2}{2}y''(t_i + \theta h). \quad (14)$$

Letting $e_i = y_i - \eta_i$, and subtracting (13) from (14), then in view of (12), we get

$$|e_{i+1}| \leq \left(1 + \frac{hk_1}{1-k_2}\right) |e_i| + \frac{Nh^2}{2} + \frac{hk_2^m c}{1-k_2} + R_{i+1}. \quad (15)$$

Denoting $R = \max_{i=0,1,\dots,n} R_i$ and since $k_2^m < |h|$, we get from (15) that $|e_{i+1}| \leq \left(1 + \frac{hk_1}{1-k_2}\right) |e_i| + \left(\frac{N}{2} + \frac{c}{1-k_2}\right)h^2 + R$, $i = 0, 1, \dots, n-1$, and hence by lemma 1, we get that

$$\begin{aligned} |e_q| &\leq \left(\left(\frac{N}{2} + \frac{c}{1-k_2}\right)h^2 + R\right) \left(\frac{e^{\frac{qhk_1}{1-k_2}} - 1}{\frac{hk_1}{1-k_2}}\right) \leq \left(N_1 h + \frac{R}{h}\right) \left(e^{\frac{\alpha k_1}{1-k_2}} - 1\right) \left(\frac{1-k_2}{k_1}\right), \\ &\quad q = 1, 2, \dots, n, \quad \text{where } N_1 = \frac{N}{2} + \frac{c}{1-k_2}. \end{aligned} \quad (16)$$

In the above inequality, $N_1 h$ represents the truncation-contraction error term, and R/h represents the round-off error term. We observe that as $h \rightarrow 0$, the truncation-contraction error term goes to zero where as the round-off error term becomes infinity. On the other hand, as $h \rightarrow +\infty$, the round-off error term goes to zero where as the truncation-contraction error term becomes infinitely large. Here, on account of the round-off errors, the total error begins to increase once again if h is reduced beyond a certain critical value and that value is $h = (R/N_1)^{\frac{1}{2}}$.

Scheme II (Euler-Contraction Method):

We define

- a) $\eta_0 = y_0$
 for $i = 0, 1, \dots, n-1$,
 $\eta_{i+1,0} = \eta_i, j = 0$
- b) repeat
 $j = j + 1$
 $\eta_{i+1,j} = \eta_i + hf(t_i, \eta_i, \frac{\eta_{i+1,j-1} - \eta_i}{h})$ (17)
 until $k_2^j < |h|$ and $|\eta_{i+1,j} - \eta_{i+1,j-1}| \leq \epsilon$ are satisfied
- c) $\eta_i = \eta_{i+1,j}$
 $t_{i+1} = t_i + h$.

As in the scheme I, the following theorem can be easily proved.

Theorem 3 (Error estimate and Convergence of the Scheme II): Let $e_i = \eta_i - y(t_i)$, $i = 0, 1, 2, \dots, n$. Then $|e_i| \leq \hat{N}h(e^{\alpha k_1/(1-k_2)} - 1)$, $i = 0, 1, 2, \dots, n$, where \hat{N} is the same constant as in theorem 2.

From theorems 2 and 3, it is clear that the numerical schemes I and II are equivalent in the sense that in both the cases the approximations converge to the true solution at the same rate. Also, it can be shown that the influence of the round-off errors in both the cases are same.

For the next two schemes to be developed below, we assume, in addition to the hypotheses of theorem 1, the following conditions:

- IV) $\frac{\partial^2 f(t, y, z)}{\partial z^2}$ exists and continuous for $(t, y, z) \in D$, and hence $|\frac{\partial^2 f(t, y, z)}{\partial z^2}| \leq k_3$ for all $(t, y, z) \in D$ where $k_3 > 0$ is a constant,
- V) $|\left[1 - \frac{\partial f(t, y, 0)}{\partial z}\right]^{-1}| \leq k_4$ for all $(t, y, 0) \in D$, where $k_4 > 0$ is a constant such that $k_3 k_4 c < 2/3$, and

VI) $|f(t, y, 0)| \leq c_1$ for all $(t, y, 0) \in D$, where $c_1 > 0$ is a constant such that $c_1 k_4 < (1 - \frac{3}{2} k_3 k_4 c)$.

Scheme III (Newton-Kantorovich-Euler Method):

We define

a) $\eta_0 = y_0$
for $i = 0, 1, 2, \dots, n-1$,
 $\eta'_{i0} = 0; j = 0$

b) repeat

$$j = j + 1$$

$$\eta'_{ij} = \eta'_{ij-1} - \frac{\eta'_{ij-1} - f(t_i, \eta_i, \eta'_{ij-1})}{1 - \frac{\partial f(t_i, \eta_i, \eta'_{ij-1})}{\partial z}} \quad (18)$$

until $(k_3 k_4 c / 6(1 - k_3 k_4 c))^{2^j - 1} < h$ and $|\eta'_{ij} - \eta'_{ij-1}| \leq \epsilon$ (19)
are satisfied;

c) $\eta_{i+1} = \eta_i + h\eta'_{ij}$
 $t_{i+1} = t_i + h.$

We note that the approximations η'_{im} are the Newton's approximations and by conditions (IV)–(VI), they are well defined (see section 1.4 of [5]). The values η_i are the Euler's approximations and are well defined as in scheme I.

Theorem 4 (Error estimate and Convergence of the Scheme III): Let $e_i = \eta_i - y(t_i)$, $i = 0, 1, 2, \dots, n$. Then $|e_i| \leq Nh(e^{\alpha k_1 / (1 - k_2)} - 1)$ where $N > 0$ is a constant.

Proof: Let m denote the least non-negative integer such that the conditions $(k_3 k_4 c / 6(1 - k_3 k_4 c))^{2^m - 1} < |h|$ and $|\eta'_{im} - \eta'_{im-1}| \leq \epsilon$ are true, for a fixed i . Clearly, we have $e_0 = \eta_0 - y_0 = y_0 - y_0 = 0$. For brevity, let $y_i = y(t_i)$ and consider

$$e_{i+1} = \eta_{i+1} - y_{i+1}$$

$$= \eta_i + h\eta'_{im} - (y_i + hy'(t_i) + \frac{h^2}{2}y''(t_i + \theta h)), 0 < \theta < 1$$

(by (6) and (18)),

which implies that

$$|e_{i+1}| \leq |e_i| + \frac{Nh^2}{2} + h|\eta'_{im} - y'(t_i)| \quad (20)$$

where N is the bound for $|y''|$.

Under the hypotheses of theorem 1 and conditions (IV)–(VI), for each fixed (t_i, η_i) , $f(t_i, \eta_i, z)$ as a function of 'z' satisfies the conditions of theorem 1.4.5 [5] in the closed ball $\{z \in R \mid |z| \leq c\}$ and hence by theorem 1.4.5 and problem 1.4.8 of [5], there exists a unique value

' η'_i ' such that

$$\eta'_i = f(t_i, \eta_i, \eta'_i), \quad |\eta'_i| \leq c \tag{21}$$

and

$$|\eta'_{im} - \eta'_i| \leq (\lambda/3)^{2^{m-1}}(c/\lambda), \tag{22}$$

where $\lambda = k_3 k_4 c / 2(1 - k_3 k_4 c)$ (< 3).

Now proceeding as in the proof of theorem 2, we get that

$$\begin{aligned} |e_q| &\leq (e^{\frac{\alpha k_1}{1-k_2}} - 1) \left(\frac{1-k_2}{k_1}\right) \left(\frac{Nh}{2} + \left(\frac{\lambda}{3}\right)^{2^{m-1}} \frac{c}{\lambda}\right) \\ &\leq Nh(e^{\frac{\alpha k_1}{1-k_2}} - 1), \text{ where } N = \left(\frac{N}{2} + \frac{c}{\lambda}\right) \left(\frac{1-k_2}{k_1}\right) \text{ (by (19)).} \end{aligned}$$

Influence of the round off errors: If we take the round-off errors into account, then as in the case of Scheme I, we get an estimate of the form $|e_q| \leq [(\frac{N}{2} + \frac{c}{\lambda})h + \frac{R}{h}](\frac{1-k_2}{k_1})(e^{\frac{\alpha k_1}{1-k_2}} - 1)$, $q = 1, 2, \dots, n$, and hence the critical value for h is given by $h = (R/(\frac{N}{2} + \frac{c}{\lambda}))^{\frac{1}{2}}$.

Scheme IV (Euler-Newton-Kantorovich Method):

We define

$$\begin{aligned} a) \quad &\eta_0 = y_0 \\ &\text{for } i = 0, 1, 2, \dots, n-1, \\ &\eta_{i+1,0} = \eta_i; \quad j = 0 \\ b) \quad &\text{repeat} \\ &j = j + 1 \\ &\eta_{i+1,j} = \eta_{i+1,j-1} - \frac{\eta_{i+1,j-1} - (\eta_i + hf(t_i, \eta_i, \frac{\eta_{i+1,j-1} - \eta_i}{h}))}{1 - \frac{\partial f}{\partial z}(t_i, \eta_i, \frac{\eta_{i+1,j-1} - \eta_i}{h})} \end{aligned} \tag{23}$$

$$\text{until } (k_3 k_4 c |h| / 6(1 - k_3 k_4 c |h|))^{2^{j-1}} < |h| \text{ and} \tag{24}$$

$$|\eta_{i+1,j} - \eta_{i+1,j-1}| \leq \epsilon \text{ are satisfied}$$

$$\begin{aligned} c) \quad &\eta_{i+1} = \eta_{i+1,j} \\ &t_{i+1} = t_i + h. \end{aligned}$$

Let m denote the least non-negative integer such that the conditions $(k_3 k_4 c |h| / 6(1 - k_3 k_4 c |h|))^{2^{m-1}} < |h|$ and $|\eta_{i+1,m} - \eta_{i+1,m-1}| \leq \epsilon$ are satisfied for a fixed i .

As in Scheme III, the approximations η_{im} are the Newton's approximations and by conditions (IV) – (VI), they are well defined, and the following theorem can be proved similarly.

Theorem 5 (Error estimate and Convergence of the Scheme IV): Let $e_i = \eta_i - y(t_i)$, $i = 0, 1, 2, \dots, n$. Then $|e_i| \leq Nh(e^{\alpha k_1/(1-k_2)} - 1)$, $i = 0, 1, 2, \dots, n$, where the constant N is same as in Theorem 4.

From Theorems 4 and 5, we note that the numerical Schemes III and IV are equivalent in the sense that in both the cases the approximations converge to the true solution at the same rate. Also it can be shown that the influence of the round-off errors in both the schemes are same.

The above schemes give approximate values of the solution in the interval $[t_0, t_0 + \alpha]$. By changing h to $-h$, the schemes can also be used to find approximate values of the solution in the interval $[t_0 - \alpha, t_0]$. For this case, in the above discussions h needs to be replaced by $|h|$ whenever h appears in an inequality.

4. COMPUTATION OF THE SOLUTIONS OF SOME NSTD IVPs

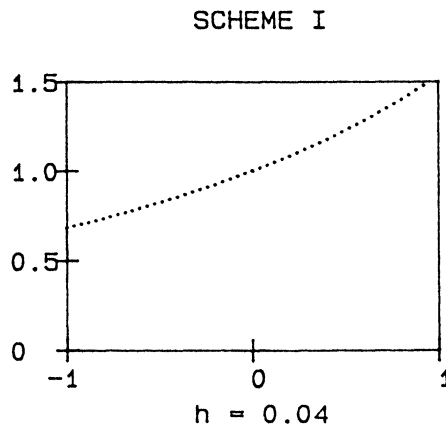
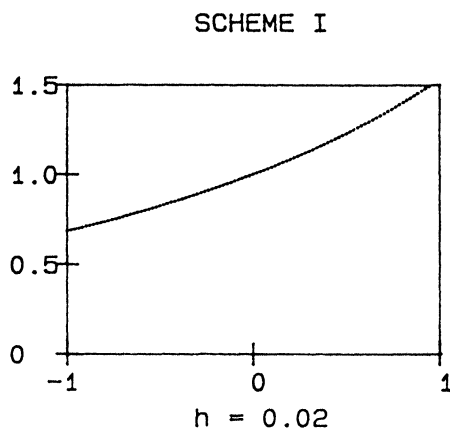
Example 1: Consider the NSTD IVP

$$y' = \frac{1}{7}e^{t-1} + \frac{2}{7}y + \frac{1}{14}y^2 \cos y', y(0) = 1. \quad (25)$$

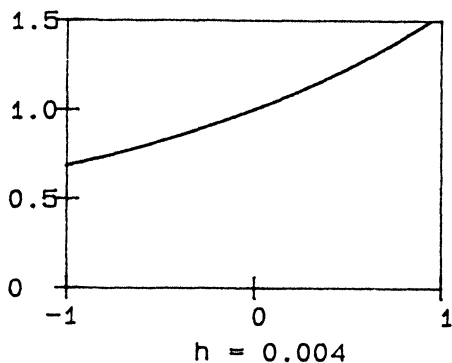
Taking the domain $D = \{(t, y, z) \in R^3 \mid |t| \leq 1, |y-1| \leq 1, |z| \leq 1\}$, it can be shown using theorem 1, as in section 3 of [7], that the IVP (25) has a unique solution existing on the interval $[-1, 1]$.

a) Computation of the solution using Scheme I:

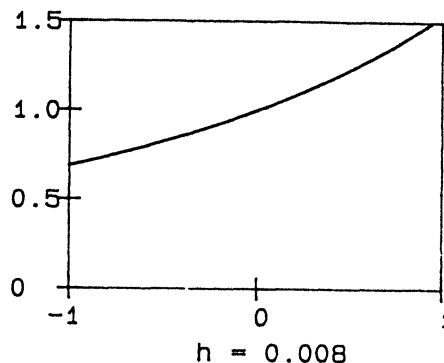
The solution of (25) is computed using Scheme I and the graphs of the (numerical) solution for various step sizes h are given below.



SCHEME I



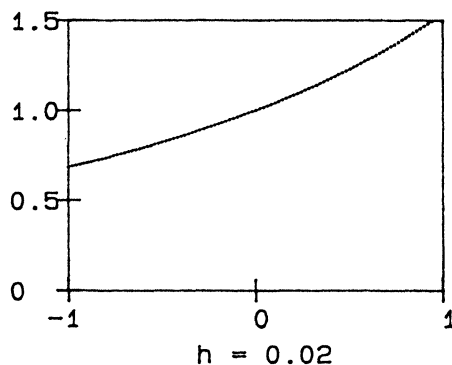
SCHEME I



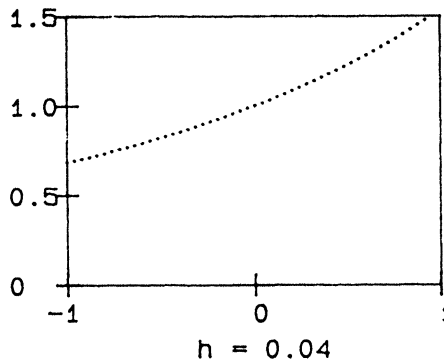
b) Computation of the solution using Scheme II:

The solution of (25) is computed using Scheme II and the graphs of the (numerical) solution for various step sizes h are given below.

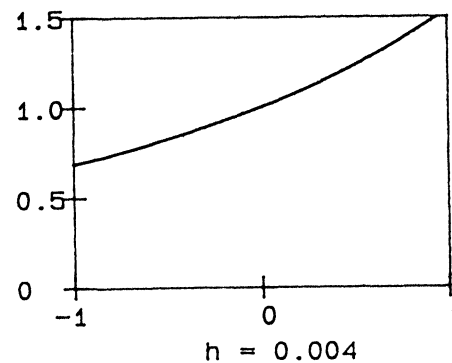
SCHEME II



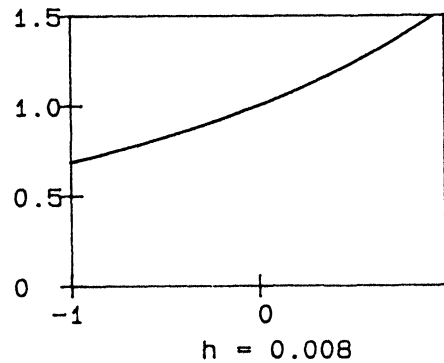
SCHEME II



SCHEME II



SCHEME II



Example 2: Consider the *NSTD IVP*

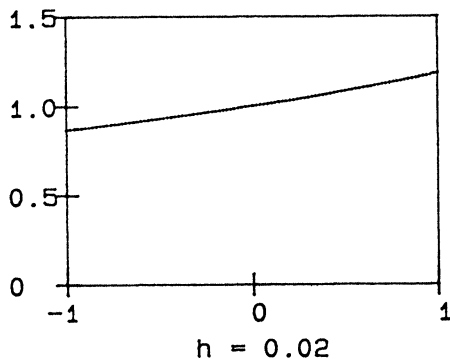
$$y' = \frac{1}{28}e^{t-1} + \frac{1}{14}y + \frac{1}{14}y^2 \cos y', \quad y(0) = 1. \quad (26)$$

Taking the same domain D as in Example 1, it can be shown, using Theorem 1, that the *IVP* (26) has a unique solution existing on the interval $[-1, 1]$.

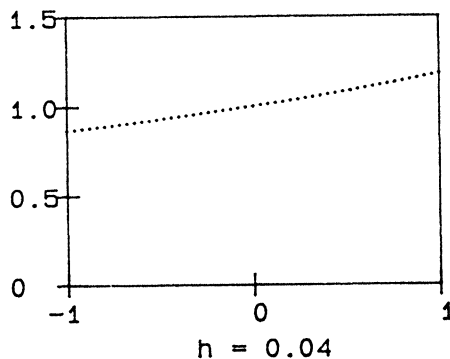
c) Computation of the solution using Scheme III:

The solution of (26) is computed using Scheme III and the graphs of the (numerical) solutions for various step sizes h are given below.

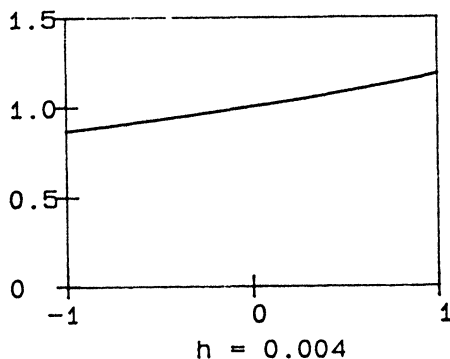
SCHEME III



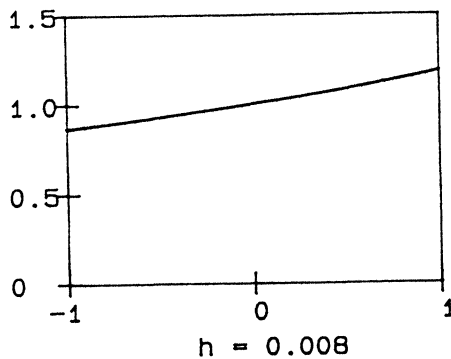
SCHEME III



SCHEME III

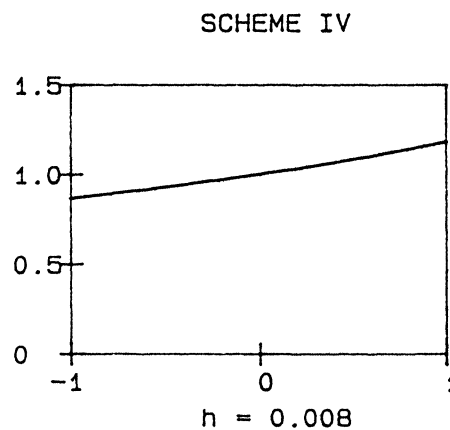
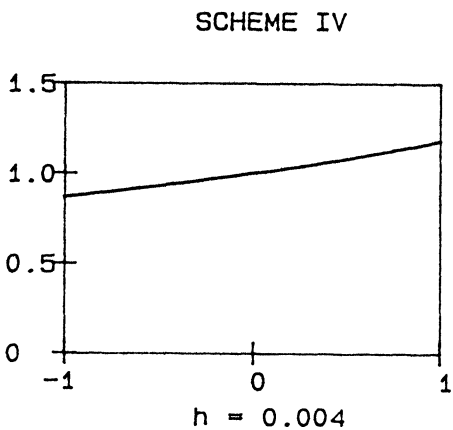
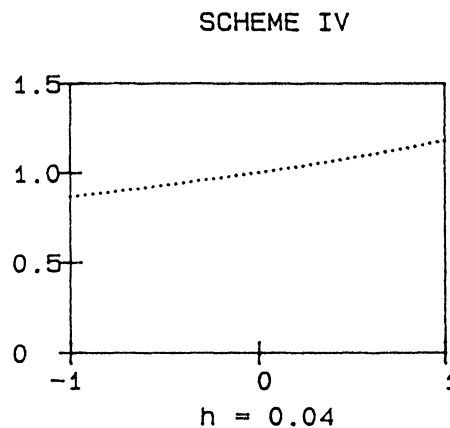
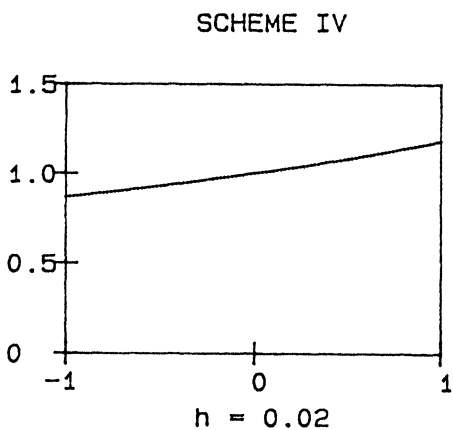


SCHEME III



d) Computation of the solution using Scheme IV:

The solution of (26) is computed using Scheme IV and the graphs of the (numerical) solution for various step sizes are given below.



ACKNOWLEDGEMENT

The authors dedicate the work to the Chancellor of the Institute Bhagawan Sri Sathya Sai Baba.

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