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ERROR ESTIMATES FOR THE SEMIDISCRETE FINITE ELEMENT APPROXIMATION OF LINEAR NONLOCAL PARABOLIC EQUATIONS[1]

DENNIS E. JACKSON

Florida Institute of Technology
Department of Applied Mathematics
150 W. University Blvd.
Melbourne, FL 32901 USA

ABSTRACT

Existence and uniqueness are proved for nonlocal (in time) for solutions of linear parabolic partial differential equations. Instead of an initial condition, there is a relation connecting the initial value to values of the solution at other times. L^2 error estimates are obtained for the semidiscrete approximation of the problem using finite elements in the space variables.

Key words: Nonlocal parabolic equations, semidiscrete finite element approximations, error estimates.

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1. INTRODUCTION

Let Ω be a bounded open subset of R^n with a smooth boundary Γ. The following nonlocal problem will be considered:

$$\left.\begin{aligned} &u_t + Au = f(x,t) \text{ on } \Omega \times (0,T), \\ &u\,|_{\Gamma} = 0, \\ &u(x,0) + g(t_1,\ldots,t_N,u) = \psi(x), \end{aligned}\right| \tag{1.1}$$

where $0 < t_1 < t_2 < \ldots < t_N \leq T$, $\psi(x) \in L^2(\Omega), f(x,t) \in L^\infty([0,T];L^2(\Omega))$ and $g(t_1,\ldots,t_N,\cdot)$ maps $C^0([0,T];L^2(\Omega))$ into $L^2(\Omega)$. Also assume A is a strongly elliptic operator defined by

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u}{\partial x_j}) + \sum_{i=1}^{n} a_i(x)\frac{\partial u}{\partial x_i} + a_0(x)u \tag{1.2}$$

with $a_{ij}(x), a_i(x) \in C^\infty(\overline{\Omega})$, with

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$$a(u,u) \geq \sigma \| u \|_1^2 - \lambda_0 \| u \|^2, \quad u \in H_0^1(\Omega), \tag{1.3}$$

where $\sigma > 0, \lambda_0 \in R$, $\| u \|^2 = \| u \|^2_{L^2(\Omega)} = (u,u)$,

$$a(u,v) = -\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x)\frac{\partial u}{\partial x_i}\ \frac{\partial v}{\partial x_j}\, dx + \sum_{i=1}^{n} \int_{\Omega} a_i(x)\frac{\partial u}{\partial x_i} v dx + \int_{\Omega} a_0(x) uv dx \tag{1.4}$$

and $H^s(\Omega)$ and $H_0^s(\Omega)$ are the usual Sobolev spaces with norms $\| \ \|_s$. See Adams [1] or Lions [10] for definitions.

Under the above conditions, A with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ generates an analytic semigroup $S(t) = e^{-At}$ such that for $a = \sigma - \lambda_0$

$$\| S(t)f \| \leq Me^{-at} \| f \|, \tag{1.5}$$

where $M \geq 1$ depends continuously on σ and λ_0 in (1.3). See Pazy [5].

The function $u \in C^0([0,T]; L^2(\Omega))$ is said to be a mild solution of (1.1) if

$$u(t) = S(t)\psi(x) - S(t)g(t_1, \quad , t_N, u) + \int_0^t S(t-\tau)f(x,\tau)d\tau. \tag{1.6}$$

We will assume for $u, v \in C^0([0,T]; L^2(\Omega))$ of the form $u, v = w$, where

$$w(t) = S(t)w(0) + \int_0^t S(t-\tau)f(x,\tau)d\tau, \tag{1.7}$$

we have the Lipschitz condition

$$\| g(t_1,\dots,t_N,u) - g(t_1,\dots,t_N,v) \| \leq \sum_{i=1}^{N} m_i \| u(t_i) - v(t_i) \|. \tag{1.8}$$

The following are some examples of $g(t_1,\dots,t_N,u)$: If $h_i(x) \in C^\infty(\overline{\Omega})$, let

$$g(t_1,\dots,t_N,u) = \sum_{i=1}^{N} h_i(x)u(t_i). \tag{1.9}$$

The m_i in (1.8) are $m_i = \max_{x \in \Omega} | h_i(x) |$.

Another useful example is

$$g(t_1,\dots,t_N,u) = \sum_{i=1}^{N} \frac{1}{k_i} \int_{t_i}^{t_i+k_i} h_i(x,\tau)u(\tau)d\tau, \tag{1.10}$$

where $k_i > 0$ and $h_i(x,t) \in C^\infty(\overline{\Omega} \times [0,T])$. If u, v are as in (1.7) and $t_i \leq \tau \leq t_i + k_i$, then

$$\| u(\tau) - v(\tau) \| = \| S(\tau - t_i)(u(t_i) - v(t_i)) \| \leq Me^{-a(\tau - t_i)} \| u(t_i) - v(t_i) \| .$$

Thus the m_i in (1.8) are

$$m_i = \frac{M}{ak_i}(1 - e^{-ak_i}) \cdot (\max_{(x,t) \in \bar{\Omega} \times [t_i, t_i + k_i]} | h_i(x,t) |).$$

Nonlocal parabolic problems have been studied by several authors. See Byszewski [2-5], Chabrowski [6], Hess [7], Kerefov [8], and Vabishchewich [13].

2. EXISTENCE AND UNIQUENESS FOR NONLOCAL PROBLEMS

In this section we will prove under the conditions of section 1, (1.6) has a unique solution.

Let $W = C^0([0,T]; L^2(\Omega))$ with norm

$$\| u \|_W = \sup_{0 \leq t \leq T} e^{at} \| u(t) \| ,$$

where a satisfies (1.5). We have the following:

Theorem 2.1: *Assume* (1.5), (1.8) *hold,* $\psi(x) \in L^2(\Omega)$, *and* $\sum_{i=1}^{N} m_i e^{-at_i} < \frac{1}{M^2}$ *for* m_i *in* (1.8) *and* a, M *in* (1.5). *Then there is a unique* u *in* W *such that* $u(t)$ *satisfies* (1.6).

Proof: Let $\Phi: W \rightarrow W$ be defined by

$$\Phi v(t) = S(t)\psi(x) - S(t)g(t_1, \ldots, t_N, S(t)v(0) + \int_0^t S(t-\tau)f(x,\tau)d\tau) \tag{2.1}$$
$$+ \int_0^t S(t-\tau)f(x,\tau)d\tau$$

for $v \in W$.

We will show Φ is a contraction mapping on W. Let $u, v \in W$. Then

$$e^{at} \| \Phi u(t) - \Phi v(t) \|$$

$$\leq e^{at} M e^{-at} \sum_{i=1}^{N} m_i \| S(t_i)(u(0) - v(0)) \|$$

$$\leq M \sum_{i=1}^{N} m_i M e^{-at_i} \| u(0) - v(0) \|$$

$$\leq M^2 (\sum_{i=1}^{N} m_i e^{-at_i}) \| u - v \|_W .$$

Thus Φ is a contraction on W, which implies there is a unique $u \in W$ such that $u = \Phi(u)$. Since

$$u(0) = \Phi u(0) = \psi(x) - g(t_1, \ldots, t_n, S(t)u(0) + \int_0^t S(t-\tau)f(x,\tau)d\tau)$$

and

$$u(t) = S(t)u(0) + \int_0^t S(t-\tau)f(x,\tau)d\tau$$

it follows that $u(t)$ satisfies (1.6).

Since $S(t)$ has the smoothing property, $S(t)f \in D(A^{\frac{\alpha}{2}})$ for $t > 0$, $\alpha \geq 0$ and $f \in L^2(\Omega)$, we have the following regularity property:

Corollary 2.2: If the conditions of Theorem 2.1 are satisfied, $\psi(x) \in D(A^{\frac{\alpha}{2}})$, $\alpha \geq 0$; $f(x,t) \in L^\infty([0,T]; D(A^\mu)), \mu = max\{\frac{\alpha}{2} - 1 + \epsilon, 0\}$ for some $\epsilon > 0$; and $g(t_1, \ldots, t_N, \cdot)$ maps $C^0((0,T]; D(A^{\frac{\alpha}{2}}))$ into $D(A^{\frac{\alpha}{2}})$, then the solution $u(t)$ of (1.6) satisfies $u \in C^0([0,T]; D(A^{\frac{\alpha}{2}}))$.

Note: If $\sum_{i=1}^{N} m_i e^{-at} < \frac{1}{M^2}$ is not satisfied, there may not be a unique solution. For example, $u_t - u_{xx} + (a - \pi^2)u = 0$ on $(0,1), u(0,t) = 0 = u(1,t)$, and $u(x,0) - e^{-a}u(x,1) = 0$ has solutions $u(x,t) = 0$ and $u(x,t) = e^{-at}sin\pi x$.

3. THE SEMIDISCRETE APPROXIMATION

Let $\{V_h\}$ be a family of finite dimensional subspaces of $H^1(\Omega)$ such that for $f \in H^s(\Omega), 1 \leq s \leq r$,

$$\inf_{\chi \in V_h} \{ \| f - \chi \| + h \| f - \chi \|_1 \} \leq ch^s \| f \|_s, \tag{3.1}$$

where c is independent of h.

In this section we will assume (1.3) is satisfied with $\lambda_0 = 0$. If this is not the case, let $u = e^{\lambda_0 t}W$.

For fixed $\epsilon > 0$, assume $A_h: V_h \to V_h$ satisfies

$$(A_h f_h, f_h) \geq \sigma' \| f_h \|^2 \text{ if } f_h \in V_h, \tag{3.2}$$

where $0 < \sigma - \epsilon < \sigma' \leq \sigma$,

$$(A_h f_h, g_h) \leq c \| f_h \|_1 \| g_h \|_1 \text{ for all } f_h, g_h \in V_h \tag{3.3}$$

and

$$\| (P_h A^{-1} - A_h^{-1} P_h) f \| \leq c h^{\alpha+2} \| A^{\frac{\alpha}{2}} f \|, \quad 0 \leq \alpha \leq r-2, \tag{3.4}$$

where P_h is the L^2 projection of $L^2(\Omega)$ onto V_h.

Conditions (3.2), (3.3) and (3.4) are satisfied with $\sigma' = \sigma$ if the standard Galerkin method is used with $V_h \in H_0^1(\Omega)$ and A_h is defined by

$$(A_h f_h, g_h) = (A f_h, g_h), \quad f_n, v_n \in V_n.$$

The conditions are also satisfied if Nitsche's method is used, where $V_h \subseteq H^1(\Omega), V_h|_\Gamma \subseteq H^1(\Gamma)$, for $2 \leq s \leq r$,

$$\inf_{\chi \in V_h} \{ \| f - \chi \| + h \| f - \chi \|_1 + h^{\frac{1}{2}} \| f - \chi \|_{L^2(\Gamma)} + h^{\frac{3}{2}} \| f - \chi \|_{H^1(\Gamma)} \} \leq c h^s \| f \|_s$$

and $A_h : v_h \rightarrow v_n$ is defined by

$$(A_h f_h, g_h) = a(f_h, g_h) - (\frac{\partial f_h}{\partial n}, g_h)_{L^2(\Gamma)} - (f_h, \frac{\partial g_h}{\partial n})_{L^2(\Gamma)} + \beta h^{-1} (f_h, g_h)_{L^2(\Gamma)}$$

for β large enough such that (3.2) holds. See Lasiecka [9].

We will first show the following nonlocal system on V_h has a unique solution for $0 \leq t \leq T$:

$$\begin{aligned} u_h'(t) + A_h u_h &= P_h f(x,t), \\ u_h(0) + P_h g(t_1, \ldots, t_N, u_h) &= P_h \psi. \end{aligned} \tag{3.5}$$

Let $S_h(t) = e^{-A_h t}$, then (3.5) is equivalent to

$$u_h(t) = S_h(t) P_h \psi - S_h(t) P_n g(t_1, \ldots, t_N, u_h) + \int_0^t S_h(t-\tau) P_h f(x,\tau) d\tau. \tag{3.6}$$

Since $\| e^{-A_h t} f_h \| \leq M \sigma' e^{-\sigma' t} < \frac{1}{M_{\sigma'}^2}$, where $\lim_{\sigma' \rightarrow \sigma} M_{\sigma'} = M$, we can find $\epsilon > 0$ for (3.2) and $\delta > 0$ such that if $m_i' = m_i + \delta$ and $\sum_{i=1}^N m_i e^{-\sigma' t_i} < \frac{1}{M^2}$, then

$$\sum_{i=1}^N m_i' e^{-\sigma' t_i} < \frac{1}{M_{\sigma'}^2}. \tag{3.7}$$

Thus by a similar proof to that of Theorem 1.1, we can prove the following:

Theorem 3.1: *Assume the conditions in Theorem* 1.1 *are satisfied and* V_h *and* A_h *satisfy* (3.1) – (3.4), *where* σ' *from* (3.2) *is such that* (3.7) *holds and*

$$\| P_h(g(t_1,\dots,t_N,u_h) - g(t_1,\dots,t_N,v_h)) \| \le \sum_{i=1}^{N} m_i' \| u_h(t_i) - v_h(t_i) \| \tag{3.8}$$

for $u_h, v_h = w_n$ *of the form* $w_h(t) = S_h(t)w_h(0) + \int_0^t S_h(t-\tau)P_h f(x,\tau)d\tau$. *Then there is a unique solution* $u_h(t)$ *of* (3.6) *such that* $u_h \in C^0([0,T];V_h)$.

Since $\| P_h(h(x)f_h) \| \le (\sup_{x \in \Omega} |h(x)|) \| f_h \|$ *for* $f_h \in V_h$, *if* σ' *is close enough to* σ, *then* g *defined in* (1.9) *and* (1.10) *satisfy* (3.8).

Under the assumptions (3.1) – (3.4), *we have for* $a \le s \le r$ *and* $f \in D(A^{\frac{\alpha}{2}}), 0 \le \alpha \le s$ *the condition*

$$\| (S(t) - S_h(t)P_h)f \| \le \frac{Ch^s}{t^{\frac{s-\alpha}{2}}} \| A^{\frac{\alpha}{2}} f \| \tag{3.9}$$

and for $f(x,t) \in L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))$, $\quad 0 \le \alpha' \le r-2$

$$\| \int_0^t (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \| \le Ch^{\alpha'+2} ln(\tfrac{1}{h}) \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))} . \tag{3.10}$$

See for example Lasiecka [9] or Thomée [12].

We can now prove similar error estimates for the semidiscrete approximation to the nonlocal problems.

Theorem 3.2: *Let the assumptions of Theorems* 1.1 *and* 3.1 *be satisfied, and let the hypotheses of Corollary* 2.2 *be satisfied for* $\alpha \le r$, $f(x,t) \in L^\infty(0,T;D(A^{\frac{\theta}{2}})), \theta = max\{\mu, \alpha'\}$, $0 \le \alpha' \le r-2$, *and for* $u, v \in C^0([t_1,T], L^2(\Omega))$,

$$\| g(t_1,\dots,t_N,u) - g(t_1,\dots,t_N,v) \| \le k \| u - v \|_{L^\infty(t_1,T;L^2(\Omega))}. \tag{3.11}$$

Also assume that $u(t)$ *is the solution of* (1.6) *and* $u_h(t)$ *is the solution to* (3.6) *for* $\alpha \le s \le r$. *Then*

$$\| u(t) - u_h(t) \| \le Ch^s(\frac{1}{t^{\frac{s-\alpha}{2}}} + 1) + Ch^{\alpha'+2} ln(\tfrac{1}{h}) \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))} . \tag{3.12}$$

Proof: We have

$$\| u(t) - u_h(t) \| \leq \| (S(t) - S_h(t)P_h)\psi \| + \| (S(t) - S_h(t)P_h)g(t_1, \ldots t_N, u) \|$$

$$+ \| S_h(t)P_h(g(t_1, \ldots, t_N, u) - g(t_1, \ldots, t_N, u_h)) \|$$

$$+ \| \int_0^t (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \| \tag{3.13}$$

$$\leq \frac{Ch^s}{t^{\frac{s-\alpha}{2}}}(\| A^{\frac{\alpha}{2}}\psi \| + \| A^{\frac{\alpha}{2}}g(t_1, \ldots, t_N, u) \|) + Ch^{\alpha' + 2}ln\frac{1}{h} \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$

$$+ M_{\sigma'}e^{-\sigma' t} \| g(t_1, \ldots, t_N, u) - g(t_1, \ldots, t_N, u_h) \| .$$

Since A_h is bounded, $S_h(-t) = e^{A_h t}$ exists. Let $t \geq t_1$, then

$$\| g(t_1, \ldots, t_N, u) - g(t_1, \ldots, t_N, u_h) \|$$

$$\leq \| g(t_1, \ldots, t_N, u) - g(t_1, \ldots, t_N, S_h(t-t_1)P_h S(t_1)u(0) + \int_0^t S_h(t-\tau)P_h f(x,\tau)d\tau) \|$$

$$+ \| g(t_1, \ldots, t_N, S_h(t)(S_h(-t_1)P_h S(t_1)u(0)) + \int_0^t S_h(t-\tau)P_h f(x,\tau)d\tau)$$

$$- g(t_1, \ldots, t_N, u_h) \| \tag{3.14}$$

$$\leq k \sup_{t_1 \leq t \leq T}(\| (S(t-t_1) - S_h(t-t_1)P_h)S(t_1)u(0) \| + \| \int_0^t (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \|$$

$$+ \sum_{i=1}^N m_i' \| S_h(t_i)(S_h(-t_1)P_h S(t_1)u(0)) - S_h(t_i - t_1)S_h(t_1)u_h \|$$

$$\leq Ch^s \| A^{\frac{s}{2}}S(t_1)u(0) \| + Ch^{\alpha' + 2}ln\frac{1}{h} \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$

$$+ \sum_{i=1}^N m_i' M_{\sigma'}e^{-\sigma'(t_i - t_1)} \| S(t_1)u(0) - S_h(t_1)u_h(0) \|$$

$$\leq Ch^s \| A^{\frac{s}{2}}S(t_1)u(0) \| + Ch^{\alpha' + 2}ln\frac{1}{h} \| f \|_{L^\infty(0,T;D(A^{\frac{\alpha'}{2}}))}$$

$$+ \sum_{i=1}^N m_i M_{\sigma'}e^{-\sigma'(t_i - t_1)}(\| u(t_1) - u_h(t_1) \| + \| \int_0^{t_1} (S(t-\tau) - S_h(t-\tau)P_h)f(x,\tau)d\tau \|)$$

$$\leq Ch^{s} \| A^{\frac{s}{2}}S(t_1)u(0) \| + Ch^{\alpha' + 2}ln\frac{1}{h} \| f \|_{L^{\infty}(0,T;D(A^{\frac{\alpha'}{2}}))}$$
$$+ \sum_{i=1}^{N} m_i M_{\sigma'} e^{-\sigma'(t_i - t_1)} \| u(t_1) - u_h(t_1) \| .$$

Let $t = t_1$ in (3.13), then

$$\| u(t_1) - u_h(t_1) \| \leq C(\frac{h^s}{t_1^{\frac{s-\alpha}{2}}} + 1) + Ch^{\alpha' + 2}ln(\frac{1}{h}) \| f \|_{L^{\infty}(0,T;D(A^{\frac{\alpha'}{2}}))}$$
$$+ M_{\sigma'}^2 \sum_{i=1}^{N} m_i' e^{-\sigma' t_i} \| u(t_1) - u_h(t_1) \| . \qquad (3.15)$$

Since $M_{\sigma'}^2 \sum_{i=1}^{N} m_i' e^{-\sigma' t_i} < 1$, (3.12) holds for $t = t_1$. Therefore the theorem follows from (3.13) and (3.15).

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