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MARKOV CHAINS WITH TRANSITION DELTA-MATRIX: ERGODICITY CONDITIONS, INVARIANT PROBABILITY MEASURES AND APPLICATIONS[1]

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ABSTRACT

A large class of Markov chains with so-called $\Delta_{m,n}$ - and $\Delta'_{m,n}$ - transition matrices ("delta-matrices") which frequently occur in applications (queues, inventories, dams) is analyzed.

The authors find some structural properties of both types of Markov chains and develop a simple test for their irreducibility and aperiodicity. Necessary and sufficient conditions for the ergodicity of both chains are found in the article in two equivalent versions. According to one of them, these conditions are expressed in terms of certain restrictions imposed on the generating functions $A_i(z)$ of the elements of the ith row of the transition matrix, $i = 0, 1, 2, \ldots$; in the other version they are connected with the characterization of the roots of a certain associated function in the unit disc of the complex plane. The invariant probability measures of Markov chains of both kinds are found in terms of generating functions. It is shown that the general method in some important special cases can be simplified and yields convenient and, sometimes, explicit results.

As examples, several queueing and inventory (dam) models, each of independent interest, are analyzed with the help of the general methods developed in the article.

Key words: Markov chain, ergodicity condition, invariant probability measure, queueing, inventory, dam.

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1. INTRODUCTION

In this article a general analytical method is proposed for the analysis of discrete Markov processes with, so-called, $\Delta_{m,n}$- or $\Delta'_{m,n}$- transition matrices[2] ("delta-matrices") which are frequently encountered in applications. (These Markov processes were first introduced in the earlier work of the authors [3]). A description of a delta-matrix is given in the following definitions.

Definition 1.1: A finite or an infinite stochastic matrix $A = \{a_{ij}\}$ is called a $\Delta_{m,n}$-*matrix* (resp., $\Delta'_{m,n}$-*matrix*), $n \geq m \geq 1$, if $a_{ij} = 0$ for $i > n$ and $i - j > m$ (resp., $a_{ij} = 0$ for $j > n$ and $j - i > m$).

Definition 1.2: If a matrix $A = (a_{ij})$ is either a $\Delta_{m,n}$- or a $\Delta'_{m,n}$-matrix then it is called a *delta-matrix*.

Thus, the transition probability matrix of Markov chains considered in the article has either the form of matrix

$$A = \begin{pmatrix} a_{00} & \cdots & a_{0n-m} & a_{0n-m+1} & \cdots \\ a_{10} & \cdots & a_{1n-m} & a_{1n-m+1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n0} & \cdots & a_{nn-m} & a_{nn-m+1} & \cdots \\ 0 & \cdots & 0 & a_{n+1n-m+1} & \cdots \\ 0 & \cdots & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \tag{1.1}$$

or the form of its transpose A'. Many discrete stochastic processes encountered in applications have transition matrices which are special cases of A (or A'): imbedded Markov chains describing the evolution of the queue in queueing systems $M^X/G^Y/1$, $G^X/M^Y/1$, $G^X/M^Y/n$ with bulk arrivals, and batch service, with state dependent parameters, with a threshold, with warm-up, with switching, with hysteresis service, with queue buildup, with preliminary service, and with vacations of the server. As examples, there are also Markov chains describing the state of a storage in the theory of inventory control, or the state of a dam (see, for instance, [2,4,9]).

In this paper the authors consider some properties of stochastic $\Delta_{m,n}$- and $\Delta'_{m,n}$-matrices and their applications to the analysis of the corresponding discrete Markov processes. A simple sufficient condition for the ergodicity of a finite Markov chain with

[2] $\Delta'_{m,n}$ denotes the transpose of $\Delta_{m,n}$.

transition $\Delta_{m,n}$ ($\Delta'_{m,n}$)-matrix is found. For processes with an infinite number of states the corresponding necessary and sufficient condition is determined in two equivalent versions (Sections 3,4). According to one of them, this condition is expressed in terms of certain restrictions imposed on the generating functions $A_i(z) = \sum_{j=0}^{\infty} a_{ij}z^j$, $i = 0,1,2,\ldots$; in the other version it is connected with the existence and characterization of the roots of the function $z^m - K(z)$, $K(z) = \sum_{i=0}^{\infty} k_i z^i$ in the closed unit disk of the complex plane.[3] In Section 5, the authors consider the problem of finding the invariant probability measure of infinite Markov chains with transition delta-matrix. It is shown that in the case of a $\Delta_{m,n}$-matrix this problem can be reduced to the problem of finding a unique solution of a linear $(n+1)\times(n+1)$ system of equations whose coefficients may contain the roots of the function $z^m - K(z)$. In some special cases this method can be considerably simplified and yields convenient and, sometimes, explicit results (see Examples 1-4). In Section 6, the authors analyze the problem of finding the invariant probability measure of a Markov chain with transition $\Delta'_{m,n}$-matrix. It is shown that the generating function of the stationary distribution of this chain can be implicitly found in terms of the first $(n+1)$ invariant probabilities, which, in turn, form a unique solution of the provided system of $(n+1)$ linear equations. Using this approach the authors succeed in finding a relatively simple, compact and explicit expression for the generating function of the stationary probabilities in a special case corresponding to a $G^X/M^Y/1$ bulk queueing system with continuously operating server. The method used in this section is based on the employment of Liouville's theorem for analytic functions of a complex variable. In the form of a Riemann boundary value problem, this method was previously introduced and developed by one of the authors in [6,7].

2. DELTA-MATRICES AND THEIR PROPERTIES

A general definition of a delta-matrix is given in Section 1. In this section, keeping in mind specific features of processes encountered in applications, we will introduce some additional notions and then mention some properties of delta-matrices.

First we will define a positive N-homogeneous $\Delta_{m,n}(\Delta'_{m,n})$-matrix. Discrete Markov processes with transition matrices of these two types are typical for queueing systems, inventory, and dam models.

Definition 2.1: Let $A = (a_{ij})$ be a stochastic $\Delta_{m,n}$-matrix (resp., $\Delta'_{m,n}$-matrix). If there exists a number N $(N \geq n)$ such that $a_{ij} = k_{j-i+m}$ for $i > N$ and $j \geq i-m$ (resp.,

[3]The results obtained in the previous work [3] of the authors were revised and included in Sections 3 and 4.

$a_{ij} = k_{i-j+m}$ for $j > N$ and $i \geq j - m$) then the matrix A is called *a N-homogeneous $\Delta_{m,n}$-matrix* (resp., *an N-homogeneous $\Delta'_{m,n}$-matrix*). If $N = n$, the matrix A is called *homogeneous.*

Another definition of an N-homogeneous $\Delta_{m,n}$-matrix (that is more convenient in the case of infinite matrices) can be given in the following way. Let $A_i(z) = \sum_{j=0}^{\infty} a_{ij}z^j$ be the generating function of elements a_{ij} of the ith row of a stochastic matrix $A = (a_{ij}, i, j = 0, 1, 2, \ldots)$. The set of functions $A_i(z)$, $i = 0, 1, 2, \ldots$, completely determines the matrix A.

Definition 2.2: An infinite stochastic matrix $A = (a_{ij}, i, j = 0, 1, 2, \ldots)$ is called an *N-homogeneous $\Delta_{m,n}$-matrix* if

$$A_i(z) = z^{i-N} \sum_{r=0}^{\infty} k_r z^r, \; i > N.$$

Definition 2.2 is equivalent to Definition 2.1.

Definition 2.3: A stochastic $\Delta_{m,n}$-matrix (resp., $\Delta'_{m,n}$-matrix) is called *positive* if $a_{ij} > 0$ for $i \leq n$ and any j, and for $i > n$, $j \geq i - m$ (resp., $j \leq n$ and any i, and for $j > n$, $i \geq j - m$).

We will also need a more general version of a positive delta-matrix which is given by the following definition.

Definition 2.4: A stochastic $\Delta_{m,n}$-matrix (resp., $\Delta'_{m,n}$-matrix) is called *essential* if $a_{nj} \neq 0$, $j = 0, 1, 2, \ldots, n - m$ and $a_{ij} \neq 0$ for all (i, j) such that $i - j = m$ (resp., $a_{in} \neq 0$, $i = 0, 1, \ldots, n - m$ and $a_{ij} \neq 0$ for all (i, j) such that $j - i = m$).

Now let us point out some properties of $\Delta_{m,n}$-matrices (analogous properties hold for $\Delta'_{m,n}$-matrices).

Property 1: Let A and B be two infinite essential (resp., positive) $\Delta_{m,n}$- and $\Delta_{m^*n^*}$-matrices, respectively. Then AB is an infinite essential (resp., positive) $\Delta_{m+m^*, n+m^*}$-matrix.

This fact can be shown by direct verification.

The following statement is an immediate consequence of Property 1.

Property 2: If A is an essential (resp., positive) infinite $\Delta_{m,n}$-matrix then for any integer k, $k > 0$ the matrix A^k is an essential (resp., positive) infinite $\Delta_{km, n+(k-1)m}$-matrix.

Applying this result to a finite positive $\Delta_{m,n}$-matrix we obtain:

Property 3: Every finite positive $\Delta_{m,n}$-matrix is primitive (i.e. a matrix M for which there is a $k > 1$ such that M^k contains strictly positive entries).

Based on Property 3, a simple test for the ergodicity of a Markov chain with a finite transition delta-matrix will be established.

Let $\{\xi_r\}$, $r = 0, 1, 2, \ldots$ be a homogeneous Markov chain with transition $\Delta_{m,n}$-matrix $A = (a_{ij})$; $i, j = 0, 1, 2, \ldots$. We denote:

$$p_i^{(r)} = P\{\xi_r = i\};\ \lim_{r \to \infty} p_i^{(r)} = p_i,\ i = 0, 1, 2, \ldots$$

$$P = (p_0, p_1, p_2, \ldots);\ P(z) = \sum_{i=0}^{\infty} p_i z^i,\ \mid z \mid \leq 1.$$

Similar notations will be used for a Markov chain $\{\zeta_r\}$, $r = 0, 1, 2, \ldots$ with transition $\Delta'_{m,n}$-matrix. We also denote: $\Gamma = \{z, \mid z \mid = 1\}$, $\Gamma^+ = \{z, \mid z \mid < 1\}$, $\Gamma^- = \{z, \mid z \mid > 1\}$, $\bar{\Gamma}^+ = \{z, \mid z \mid \leq 1\}$.

In the next section we will obtain necessary and sufficient conditions for the ergodicity of Markov chains $\{\xi_r\}$ and $\{\zeta_r\}$ with finite and infinite transition delta-matrices.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR THE ERGODICITY OF MARKOV CHAINS WITH TRANSITION DELTA-MATRIX

In this section we first mention a simple sufficient condition for the ergodicity of a finite Markov chain with transition probability delta-matrix.

Theorem 3.1: *Every Markov chain whose transition matrix can be represented as a finite positive delta-matrix is ergodic.*

Proof: The statement of the theorem follows directly from Property 3 of the previous section. □

Now we consider discrete Markov processes with an infinite number of states. First we prove that under some natural conditions, any Markov chain with transition delta-matrix is irreducible and aperiodic.

Definition 3.1: A Markov chain $\{\xi_k\}$ is called *queue-type* if

(a) $\{\xi_k\}$ takes on only nonnegative integers,

(b) $0 < P\{\xi_{k+1} > i \mid \xi_k = i\} < 1$, $i = 0, 1, 2, \ldots$,

(c) $0 < P\{\xi_{k+1} < i \mid \xi_k = i\} < 1$, $i = 1, 2, 3, \ldots$, $k = 0, 1, 2, \ldots$, (3.1)

Theorem 3.2: *A "queue-type" Markov chain $\{\xi_k\}$ is irreducible and aperiodic.*

Proof: Due to (3.1), $\{\xi_k\}$ can reach state $\{0\}$ starting from any state. Therefore, $\{0\}$ must belong to every class of essential states. Since these classes are classes of equivalence, then having a common element means that they coincide; therefore, $\{\xi_k\}$ has only one class of essential states.

From (b), taking $i = 0$ we find that $P\{\xi_{k+1} = 0 \mid \xi_k = 0\} > 0$, which immediately shows aperiodicity of $\{\xi_k\}$. □

Theorem 3.3: *Let $\{\xi_k\}$ be a Markov chain with transition $\Delta_{m,n}$ $(\Delta'_{m,n})$-matrix $A = (a_{ij})$, $i, j = 0, 1, 2, \ldots$. If*

(1) $0 < \sum_{j=0}^{i-1} a_{ij} < 1,\ 0 \leq i \leq n,$

(2) $0 < \sum_{j=i+1}^{\infty} a_{ij} < 1,\ 0 \leq i < n$ *and*

(3) $0 < \sum_{j=0}^{m-1} k_j < \sum_{j=0}^{m} k_j < 1$

then $\{\xi_k\}$ is queue-type and, therefore, irreducible and aperiodic.

Proof: Due to the definition of a $\Delta_{m,n}$ $(\Delta'_{m,n})$-matrix, conditions (1), (2) and (3) of Theorem 3.3 guarantee conditions (b) and (c) of Definition 3.1. Therefore, $\{\xi_k\}$ is irreducible and aperiodic. □

Remark 3.1: The fact that a Markov chain has only one class of essential states does not mean that all the states belong to this class (in which case the steady state probabilities are positive). Consider, for example, a standard $M^X/G^Y/1$ bulk queuing system in which customers arrive in batches of two and the server's capacity is also two. The imbedded Markov chain $\{\xi_n\}$, where ξ_n is the number of customers in the system at moments of successive service completions, is aperiodic and irreducible but only even states are essential.

Remark 3.2: Condition (b) of Definition 3.1 guarantees that infinitely many states can be reached starting from any state (say, $\{0\}$), so that the class of essential states is infinite.

Next we will establish the main result of this section. For the sake of simplicity, we will assume here and later that all considered $\Delta_{m,n}$-matrices are homogeneous (unless stated otherwise). However, all results obtained can be easily extended to the general case of N-homogeneous $\Delta_{m,n}$-matrices, $N > n$.

Theorem 3.4: *A queue-type Markov chain $\{\xi_r\}$ with transition $\Delta_{m,n}$-matrix $A = (a_{ij})$, $i, j = 0, 1, 2, \ldots$, $A_i'(1) < \infty$, $i = 0, 1, 2, \ldots$ is ergodic if and only if*

$$K'(1) < m. \tag{3.2}$$

Proof:

Sufficiency: Setting $x_j = j$, $j = 0, 1, 2, \ldots$ and using the definition of a homogeneous $\Delta_{m,n}$-matrix we obtain:

$$\sum_{j=0}^{\infty} a_{ij}x_j - x_i = \begin{cases} A_i'(1) - i < \infty, & i = 0,1,\ldots,n \\ K'(1) - m < 0, & i = n+1, n+2, \ldots \end{cases}.$$

By the corollary of Moustafa [8] to Foster's lemma we conclude that (3.2) is sufficient for the ergodicity of the chain $\{\xi_r\}$.

Necessity: Suppose, on the contrary, that $K'(1) \geq m$ and the chain $\{\xi_r\}$ is ergodic. Then there are stationary probabilities p_i, not all zero, such that

$$p_j = \sum_{i=0}^{\infty} p_i a_{ij},\ j = 0,1,2,\ldots \tag{3.3}$$

$$\text{or } P = PA, \text{ where } P = (p_0, p_1, p_2, \ldots). \tag{3.4}$$

Multiplying both sides of (3.4) from the right by vector $(\underbrace{0,0,\cdots,0}_{n+1},1,z,z^2,\ldots)^T$, we obtain:

$$\sum_{i=n+1}^{\infty} p_i z^{i-n-1} = \sum_{i=0}^{n} p_i \sum_{j=n+1}^{\infty} a_{ij} z^{j-n-1} + \sum_{j=n+1}^{\infty} a_{ij} z^{j-n-1},\ |z| \leq 1$$

which is equivalent to

$$\left(\sum_{i=n+1}^{\infty} p_i z^{i-n-1}\right) \frac{1 - z^{-m}K(z)}{z-1}$$

$$= \frac{\sum_{i=0}^{n} p_i \sum_{j=n+1}^{\infty} a_{ij} z^{j-n-1} - \sum_{i=n+1}^{\infty} \sum_{j=0}^{n} z^{j-n-1} k_{j-i+m}}{z-1}. \tag{3.5}$$

Taking the limit of both sides of (3.5) as $z \to 1$ we obtain:

$$\left(\sum_{i=n+1}^{\infty} p_i\right)[m - K'(1)] = \sum_{i=0}^{n} p_i \sum_{j=n+1}^{\infty} (j-n-1)a_{ij} + \sum_{i=n+1}^{\infty} p_i \sum_{j=0}^{n} (n-j+1)k_{j-i+m}. \tag{3.6}$$

While the left-hand side of (3.6) is supposed to be nonpositive, the right-hand side is a sum of nonnegative terms, and so (3.6) means that both sides must be zero. Taking into account that $\{\xi_r\}$ is queue-type, we conclude that there must be an essential state $i_0 < n+1$ and $j_0 \geq n+1$ such that $a_{i_0 j_0} > 0$ (see Property (b) of Definition 3.1). On the other hand, the right-hand side of (3.6) being zero means that j_0 can only be $n+1$, for otherwise the first sum in the right-hand side will be positive. It follows that $n+1$ is an essential state and $p_{n+1} > 0$, and so the right-hand side being zero requires that

$$\sum_{j=0}^{n} (n-j+1)k_{j-n+m-1} = \sum_{r=0}^{m-1} (m-r)k_r = 0.$$

Therefore, $k_r = 0$, $r = 0,1,\ldots,m-1$.

Since $\sum_{i=n+1}^{\infty} p_i(m-K'(1))=0$ and $p_{n+1}>0$, $K'(1)$ must be equal to m. This is possible only if $k_m=1$, which contradicts to inequality (c) of Definition 3.1. This contradiction proves part B of the theorem. □

A similar result holds for a Markov chain $\{\xi_r\}$, $r=0,1,2,\ldots$ with transition matrix of $\Delta'_{m,n}$-type:

Theorem 3.5: *A queue-type Markov chain $\{\zeta_r\}$ with transition n-homogeneous $\Delta'_{m,n}$-matrix $A=(a_{ij})$, $i,j=0,1,2,\ldots$ is ergodic if and only if*

$$K'(1)>m. \tag{3.7}$$

Proof:

Sufficiency: Setting $x_j=max\{j-n,0\}$, $j=0,1,2,\ldots$ and taking into account the structure of $\Delta'_{m,n}$-matrix A we obtain:

$$\sum_{j=0}^{\infty} a_{ij}x_j - x_i = \begin{cases} 0, & \text{if } i<n-m+1 \\ D_z^{(i-n+m+1)}\dfrac{K(z)-z^m}{(1-z)^2}, & \text{if } i\geq n-m+1 \end{cases}$$

where $D_x^{(r)}$ is an operator defined by

$$D_x^{(r)}F(y)=\frac{1}{r!}\frac{\partial^r F(x,y)}{\partial x^r}\bigg|_{x=0}. \tag{3.8}$$

Note that

$$D_z^{(i)}\frac{K(z)-z^m}{1-z} = \begin{cases} \sum_{j=0}^{i} k_j \geq 0, & \text{if } i<m \\ \sum_{j=0}^{i} k_j - 1 \leq 0, & \text{if } i\geq m \end{cases}$$

and therefore, if $i<m$,

$$D_z^{(i)}\frac{K(z)-z^m}{(1-z)^2} = \sum_{j=0}^{i} D_z^{(j)}\frac{K(z)-z^m}{1-z} \geq 0. \tag{3.9}$$

If $i\geq m$

$$D_z^{(i+1)}\frac{K(z)-z^m}{(1-z)^2} \leq D_z^{(i)}\frac{K(z)-z^m}{(1-z)^2}. \tag{3.10}$$

Hence, the sequence

$$\left\{D_z^{(i)}\frac{K(z)-z^m}{(1-z)^2}\right\},\ i=1,2,3,\ldots$$

either has a finite limit as $i\to\infty$ or tends to $-\infty$. If $K'(1)>1$ then by using a Tauberian theorem in case of a finite limit we obtain

$$\lim_{i\to\infty} D_z^{(i)}\frac{K(z)-z^m}{(1-z)^2} = \lim_{z\to 1-0}\frac{K(z)-z^m}{1-z} = -K'(1)+m<0.$$

In this case there exist $M < 0$ and τ such that

$$\sum_{j=0}^{\infty} a_{ij}x_j - x_i < M, \text{ if } i > \tau.$$

Applying again the criterion of Moustafa [8], we conclude that (3.7) is sufficient for the ergodicity of the chain $\{\zeta_r\}$.

Necessity: If the chain $\{\zeta_r\}$ is ergodic then there exists a strictly positive probability vector $P = (p_0, p_1, p_2, \ldots)$ such that $P = PA$. Multiplying this equation from the right by the vector $(x_0, x_1, x_2, \ldots)$ where $x_j = max\{j-n-1, 0\}$, $j = 0, 1, 2, \ldots$ after some transformations, we obtain:

$$\sum_{i=0}^{\infty} p_{n+i-m+2} D_z^{(i)} \frac{K(z) - z^m}{(1-z)^2} = 0. \tag{3.11}$$

Suppose that $K'(1) \leq m$. Then

$$\lim_{i \to \infty} D_z^{(i)} \frac{K(z) - z^m}{(1-z)^2} = \lim_{i \to \infty} \sum_{j=0}^{i} D_z^{(j)} \frac{K(z) - z^m}{1-z} = -K'(1) + m \geq 0.$$

Together with (3.9) and (3.10), this implies that $L_i = D_z^{(i)} \dfrac{K(z) - z^m}{(1-z)^2} \geq 0$, $i = 0, 1, 2, \ldots$, and it is readily seen that at least for $i = 0$, L_i is strictly positive which contradicts (3.11). □

4. ERGODICITY OF MARKOV CHAINS WITH TRANSITION DELTA-MATRIX AND CHARACTERIZATION OF THE ROOTS OF THE FUNCTION $z^m - K(z)$.

The conditions of ergodicity established in Section 3 are closely connected with the number and location of the roots of the function $z^m - K(z)$. More precisely, the following theorem is valid.

Theorem 4.1: *A.* *If $K'(1) < m$, then the function $z^m - K(z)$ has exactly m roots (counting multiplicities) in the closed unit disk $\bar{\Gamma}^+$. The roots lying on the boundary Γ are simple and, for some integer r, are all rth roots of 1.*

B. *If $K'(1) > m$, then the function $z^m - K(z)$ has exactly m roots (counting multiplicities) in the open unit disk Γ^+; on the boundary Γ there can be r additional simple roots which are all the rth roots of 1, where r is an integer, and $1 \leq r \leq m$.*

Proof: We will need the following auxiliary result.

Lemma 4.1: *The function $z^m - K(z)$ has roots on Γ if and only if there exists a* *common divisor r of m and all i such that $k_i \neq 0$. If this condition is satisfied, then all roots on Γ with this property coincide with the roots of the equation $z^r - 1 = 0$, where r is the maximal number having this property.*

Proof: Suppose that z_0 is a root of the equation $z^m - K(z) = 0$ such that $|z_0| = 1$. Then $|K(z_0)| = |z_0^m| = 1$. On the other hand,

$$K(z_0) = \left|\sum_{i=0}^{\infty} k_i z_0^i\right| \leq \sum_{i=0}^{\infty} k_i |z_0|^i = 1$$

and equality is attained if and only if $z_0^i = |z_0^i| = 1$ for any i, such that $k_i \neq 0$ and $z_0^m = 1$. The realization of both of these conditions simultaneously is possible only if the conditions of the lemma are satisfied. In this case, obviously, $z_0^r = 1$.

Suppose now that the conditions of the lemma are satisfied. Then for any root z_0 of the equation $z^r - 1 = 0$, we obtain

$$K(z_0) = 1 = z_0^m$$

which implies that the root z_0 is a root of the equation $K(z) - z^m = 0$. □

Now we return to the proof of the theorem.

A. First we suppose that the number r, appearing in the Lemma 4.1, is 1. Then the function $z^m - K(z)$ has only one root on the unit circle. This root is equal to 1 and it is simple, since $K'(1) = m$. We will prove that in this case $z^m - K(z)$ has exactly $m-1$ roots in Γ^+.

Consider an auxiliary function:

$$f(z) = \frac{1 - z^{-m}K(z)}{1 - z^{-1}}.$$

Clearly, $f(z) \neq 0$ for all z, $|z| = 1$ since the numerator of the expression for $f(z)$ may be zero only if $z = 1$, but $f(1) = m - K'(1) > 0$. Let $Ind_\Gamma f(z)$ denote the difference between the number of the roots and the number of the poles of the function $f(z)$ in Γ^+. By the argument principle

$$Ind_\Gamma f(z) = \frac{1}{2\pi}\Delta_\Gamma Arg f(z) = Ind_\Gamma[1 - z^{-m}K(z)] - Ind_\Gamma(1 - z^{-1}) \tag{4.1}$$

where $\Delta_\Gamma Arg f(z)$ is the increment of the argument of $f(z)$ when the argument φ of $z = e^{i\varphi}$ increases from 0 to 2π.

Consider the right-hand side of (4.1). Since $Arg f(1) = 0$, it is easy to notice that

$$\lim_{\varphi \to \pm 0} Arg[1 - e^{-im\varphi}K(e^{i\varphi})] = \lim_{\varphi \to \pm 0}[1 - e^{-i\varphi}] = \pm\frac{\pi}{2}.$$

At the same time, if $\varphi \in (0, 2\pi)$ then $|e^{-im\varphi}K(e^{i\varphi})| < 1$ and, hence $Arg[1 - e^{-m\varphi}K(e^{i\varphi})]$

$\in (-\frac{\pi}{2}, \frac{\pi}{2})$. It follows that

$$Ind_{\Gamma}[1 - z^{-m}K(z)] = -\tfrac{1}{2} = Ind_{\Gamma}[1 - z^{-1}]$$

and therefore

$$Ind_{\Gamma} f(z) = 0.$$

Taking into account that

$$f(z) = \frac{z^m - K(z)}{z^{m-1}(z-1)}$$

has exactly $m-1$ poles inside Γ, we can conclude that the number of the roots of $f(z)$ inside Γ is also $m-1$.

Thus, the total number of the roots (including $z = 1$) of the function $z^m - K(z)$ in the unit disk $\bar{\Gamma}^+$ is equal to m.

Now consider the general case $r > 1$. Introduce the function

$$F(z) = z^{\frac{m}{r}} - \sum_{i=0}^{\infty} k_i z^{\frac{i}{r}}. \tag{4.2}$$

From the definition of r it follows that all exponents of z in (4.2) are integers. Applying the previous reasoning to (4.2) we see that $F(z)$ has exactly $\frac{m}{r} - 1$ roots in Γ^+, and one root $z = 1$ on the boundary. It is clear that the set of all rth roots of $F(z)$ gives us all roots of $z^m - K(z)$. On the other hand, any root of $z^m - K(z)$ raised to the rth power is, obviously, a root of $F(z)$. Therefore, the set of all roots of $z^m - K(z)$ is described completely. The number of them is m; $m - r$ are in the region Γ^+, and r roots are on the boundary.

Suppose now that z_0 is a root of $z^m - K(z)$ such that $|z_0| = 1$. Then,

$$|K'(z_0)| = \left| \sum_{i=1}^{\infty} i k_i z^{i-1} \right| = \left| \frac{\sum_{i=1}^{\infty} i k_i z_0^i}{z_0} \right|$$

$$\leq \left| \frac{\sum_{i=1}^{\infty} i k_i}{z_0} \right| = \frac{K'(1)}{z_0} = |K'(1)| < m.$$

It follows that $K'(z_0) \neq m z_0^{m-1}$, which implies that all roots on the boundary are simple. Part A of the theorem is proved.

B. Since $K'(1) > m$, there exists $\delta > 0$ such that $K(\rho) < \rho^m$ for any $\rho \in [1-\delta, 1]$. Therefore, the inequality $|K(z)| < |z^m|$ is correct for any z, such that $|z| = \rho$. Using Rouche's theorem we obtain that $z^m - K(z)$ has exactly m roots in the region $|z| < 1$. The number of roots on the boundary, as before, depends on the value of r. If $r > 1$ then by Lemma 4.1 there are r roots on the boundary which represent a set of all rth roots of 1. The simplicity of these roots can be proved as before. □

5. FINDING THE ERGODIC DISTRIBUTION OF A MARKOV CHAIN WITH TRANSITION $\Delta_{m,n}$-MATRIX

Let $\{\xi_r\}$, $r = 0,1,2,\ldots$ be a queue-type Markov chain with transition homogeneous $\Delta_{m,n}$ - matrix $A = (a_{ij})$, $i,j = 0,1,2,\ldots$. Suppose that $A_i(z)$, $i = 0,1,2,\ldots,N$ and $K(z)$ satisfy necessary and sufficient conditions for the ergodicity of $\{\xi_r\}$ established in Section 3. Then the invariant probability measure $\boldsymbol{P} = \{p_0, p_1, p_2, \ldots\}$ of the matrix A exists and represents the only solution of the matrix equation $\boldsymbol{P} = \boldsymbol{P}A$. In the following theorems $\boldsymbol{P}$ is found in terms of the generating function $P(z) = \sum_{i=0}^{\infty} p_i z^i$.

Theorem 5.1: *Under conditions* (3.1) *and* (3.2), *the generating function* $P(z) = \sum_{i=0}^{\infty} p_i z^i$ *of the ergodic distribution of a queue-type Markov chain* $\{\xi_r\}$, $r = 0,1,2,\ldots$ *with transition homogeneous* $\Delta_{m,n}$-*matrix* $A_{ij} = \{a_{ij}\}$, $i,j = 0,1,2,\ldots$ *is determined by the following relations:*

$$P(z) = \frac{\sum_{i=0}^{n} p_i[A_i(z)z^m - K(z)z^i]}{z^m - K(z)}. \tag{5.1}$$

The unknown probabilities $p_0, p_1, \ldots, p_n$ *on the right-hand side of* (5.1) *form a unique solution of the system of* $n+1$ *linear equations:*

$$\frac{d^k}{dz^k} \sum_{i=0}^{n} p_i[A_i(z) - z^i] \bigg|_{z = z_r} = 0, \ k = 0,1,\ldots,\tau_r - 1, \ r = 1,2,\ldots,R \tag{5.2}$$

$$\sum_{i=0}^{N} p_i[m - i + A_i'(1) - K'(1)] = m - K'(1) \tag{5.3}$$

where z_r *are the roots of* $z^{n+1} - z^{n-m+1}K(z)$ *in the region* $\overline{\Gamma}^+ \backslash \{1\} = \{z, |z| \le 1, z \ne 1\}$ *with their multiplicities* τ_r *such that* $\sum_{r=1}^{R} \tau_r = n$.

Proof: Taking advantage of particular features of the homogeneous $\Delta_{m,n}$ - matrix A and the matrix equation $\boldsymbol{P} = \boldsymbol{P}A$ after elementary transformations we obtain:

$$P(z) = \sum_{i=0}^{\infty} p_i A_i(z) = \frac{\sum_{i=0}^{n} p_i[A_i(z)z^m - K(z)z^i]}{z^m - K(z)}. \tag{5.4}$$

In order to find $n+1$ relations necessary for determining the unknown probabilities on the right-hand side of (5.4), we represent this relation in the following form:

$$\sum_{i=n+1}^{\infty} p_i z^{i-n-1} = \frac{\sum_{i=0}^{n} p_i[A_i(z) - z^i]}{z^{n+1} - z^{n-m+1}K(z)}. \tag{5.5}$$

Since the function on the left-hand side of (5.5) is, analytic in the region Γ^+ and continuous on the boundary Γ of this region, so must be the function on the right-hand side of (5.5). On the other hand, due to Theorem 4.1 the function $z^{n+1} - z^{n-m+1}K(z)$ has exactly $n+1$ roots in $\bar{\Gamma}^+$ (including the simple root 1). Using the analyticity of the function on the right-hand side of (5.5) and condition $P(1) = 1$ we obtain (5.2) and (5.3).

It can be proved that the system of equations (5.2) – (5.3) has a unique solution. Suppose that there is another solution $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n$. Let us substitute $\hat{p}_i$, $i = 0, 1, \ldots, n$ into the right-hand side of (5.1). Due to (5.2) – (5.3) the function $\hat{P}(z)$ defined by (5.1) is analytic in Γ^+ and continuous on Γ. Consider the Maclaurin expansion of $\hat{P}(z)$: $\hat{P}(z) = \sum_{i=0}^{\infty} \hat{p}_i z^i$. It can be shown that since $A_i'(1)$, $i = 0, 1, 2, \ldots, n$ and $K'(1)$ are finite the sequence $\{\hat{p}_i\}$ is absolutely summable (see Lemma 5.1). Rearranging (5.1) for $\hat{p}(z)$ in the form

$$\hat{P}(z) = \sum_{i=0}^{n} \hat{p}_i A(z) + \sum_{i=n+1}^{\infty} \hat{p}_i K(z) z^{i-m} \tag{5.6}$$

and equating the corresponding coefficients we obtain from (5.6), that $\hat{\boldsymbol{P}} = \hat{\boldsymbol{P}} A$.

It also follows from (5.3) that $\sum_{i=0}^{\infty} p_i = 1$. Therefore, we have two different absolutely summable solutions of the matrix equation $\boldsymbol{X} = \boldsymbol{X} A$ with $(\boldsymbol{X}, 1) = 1$. This contradicts to the Chung Kai-Lai theorem [5]. □

Remark 1: The results of Theorem 5.1 can be easily extended to the case of N-homogeneous $\Delta_{m,n}$-matrices, $N > n$.

Remark 2: It is obvious that $n - m + 1$ roots of the function $z^{n+1} - z^{n-m+1}K(z)$ are zeros. Utilizing these roots and the analyticity of the right-hand side of (5.5) we obtain, as one would expect, the first $n - m + 1$ component-wise equations of the matrix equation $\boldsymbol{P} = \boldsymbol{P}A$. This part of the system (5.3) – (5.4) in some cases can be simplified. One of these special cases occurs when the transition N-homogeneous $\Delta_{m,n}$-matrix of the chain $\{\xi_r\}$ is essential. Due to the structural properties of this matrix, every p_j, $j = n+1, n+2, \ldots, N$ can be uniquely expressed as a linear combination of p_i, $i = 0, 1, 2, \ldots, n$ which enables us to reduce the number of equations in (5.2) – (5.3). This fact, in combination with the so-called "method of continuation" [1], makes it possible in some cases to obtain simple and even explicit results for the generating function $P(z)$ and other characteristics of the ergodic distribution of $\{\xi_r\}$ (see Example 4).

Another special case is a Markov chain with, a so-called transition Δ_m-matrix, which was first introduced and studied by one of the authors in [1]. A Δ_m-matrix is a special case of $\Delta_{m,n}$-matrix when $n = m$ and is also frequently encountered in applications (see Examples 1, 2, 3). Using Theorem 5.1 for $n = m$, we can find an expression for the generating function of the ergodic distribution of a Markov chain with transition Δ_m-matrix:

$$P(z) = \frac{\sum_{i=0}^{m} p_i (z^m A_i(z) - z^i K(z))}{z^m - K(z)} \tag{5.7}$$

where the unknown probabilities on the right-hand side of (5.7) can be found from (5.2) – (5.3) where $n = m$. However, in this case, since the number of the unknown probabilities is only $m+1$, it is enough to express p_m in terms of $p_0, p_1, \ldots, p_{m-1}$ to equate the number of the roots of $z^m - K(z)$ and the number of the unknown probabilities on the right-hand side of (5.7). It follows from the matrix equation $\boldsymbol{P} = \boldsymbol{PA}$ that

$$p_m = \sum_{i=0}^{m-1} \alpha_i p_i, \text{ where } \alpha_0 = \frac{1 - a_{00}}{a_{m0}},\ \alpha_i = -\frac{a_{i0}}{a_{m0}}, i = 1, 2, 3, \ldots, m-1. \tag{5.8}$$

Therefore, eliminating p_m from (5.5) we conclude that all probabilities $p_0, p_1, p_2, \ldots, p_{m-1}$ can be found from the condition of the absence of poles of the function

$$z \mapsto \frac{\sum_{i=0}^{m-1} p_i[A_i(z) - z^i + \alpha_i(A_m(z) - z^m)]}{z^m - K(z)}$$

in the closed unit disk $\bar{\Gamma}^+$ of the complex plane. The uniqueness of the solution of the corresponding system of equations is proved in Theorem 5.1.

EXAMPLES AND SPECIAL CASES

First we introduce an auxiliary Markov chain $\{\hat{\xi}_r\}$, $r = 1, 2, \ldots$ and a polynomial $R(z)$ which will play an important role in the analysis of some practical problems.

Consider a Markov chain with transition probability matrix $A = \{a_{ij}\}$, $i, j = 0, 1, 2, \ldots$ such that

$$A_i(z) = \begin{cases} K(z), & \text{if } i \le m, \\ z^{i-m}K(z), & \text{if } i > m \end{cases} .$$

We denote this chain by $\{\hat{\xi}_r\}$. The transition matrix of $\{\hat{\xi}_r\}$ is a homogeneous Δ_m-matrix.

Assume that $0 < \sum_{i=0}^{m-1} k_i < \sum_{i=0}^{m} k_i < 1$ and $K'(1) < m$. Due to Theorems 3.3 and 3.4, the chain $\{\hat{\xi}_r\}$ is irreducible, aperiodic and ergodic. According to (5.7), the generating function of the stationary distribution $\{\pi_0, \pi_1, \pi_2, \ldots\}$ of the chain $\{\hat{\xi}_r\}$ is

$$\pi(z) = \sum_{i=0}^{\infty} \pi_i z^i = \frac{K(z)R(z)}{z^m - K(z)}, \quad |z| \le 1 \tag{5.9}$$

where $R(z) = \sum_{i=0}^{m-1} \pi_i(z^m - z^i)$.

It follows that

$$\sum_{i=0}^{m-1} \pi_i + \sum_{i=m}^{\infty} \pi_i z^{i-m} = R(z)[z^m - K(z)]^{-1}. \tag{5.10}$$

Since the left-hand side of (5.10) is a Taylor series with absolutely summable coefficients, all roots of $R(z)$ are the roots of $z^m - K(z)$ in $\bar{\Gamma}^+$ with the same multiplicities. Therefore, $R(z)$ is completely determined up to a constant factor fixed by $R'(1) = m - K'(1)$ that follows from (5.10). As was mentioned above, the polynomial $R(z)$ can be used in the analysis of some practical problems. One of them is the following lemma.

Lemma 5.1: *Let $\hat{p}_i$, $i = 0, 1, 2, \ldots, n$ be any solution of the system of equations* (5.2) – (5.3). *Then the function*

$$\hat{P}(z) = \sum_{i=0}^{n} \hat{p}_i [z^m A_i(z) - z^i K(z)] \, [z^m - K(z)]^{-1}$$

can be expanded in a Maclaurin series in z with absolutely summable coefficients.

Proof: We can represent $\hat{P}(z)$ in the form

$$\hat{P}(z) = \sum_{i=0}^{n} \hat{p}_i z^i + z^m f(z) \cdot \pi(z)$$

where

$$f(z) = [R(z)]^{-1} \sum_{i=0}^{n} \hat{p}_i [A_i(z) - z^i].$$

Due to (5.2), $f(z)$ is analytic on Γ^+ and continuous on Γ. Let us rewrite $f(z)$ as follows:

$$f(z) = (z-1)[R(z)]^{-1} \sum_{i=0}^{n} \hat{p}_i [A_i(z) - z^i](z-1)^{-1}.$$

The function $(z-1)[R(z)]^{-1}$ is rational, with all its poles belonging to Γ^+, and so it can be expanded into a Maclaurin series in $\frac{1}{z}$ with absolutely summable coefficients. In other words, Maclaurin series is absolutely convergent on Γ.

On the other hand, all the functions $[A_i(z) - 1](z-1)^{-1}$, $i = 0, 1, \ldots, n$ under the original assumptions that $A_i'(1) < \infty$, are expandable into Maclaurin series in z with absolutely summable coefficients, or equivalently the series are absolutely convergent on Γ.

Consequently, the same is true for

$$\sum_{i=0}^{n} \hat{p}_i [A_i(z) - z^i](z-1)^{-1} = \sum_{i=0}^{n} \hat{p}_i [A_i(z) - 1](z-1)^{-1} + \sum_{i=0}^{n} \hat{p}_i [1 - z^i](z-1)^{-1}.$$

Therefore, $f(z)$ on Γ equals to the product of two series in z and in $\frac{1}{z}$ with absolutely summable coefficients which is a Laurent series with absolutely summable coefficients. However, since $f(z)$ is analytic in Γ^+ and continuous on Γ the principal part of this Laurent series must be zero. □

Example 1: Consider a $M/G^Y/1$ queueing system with bulk service, ordinary input, and additional exponential service phase provided by the server when the number of customers in the system at the beginning of a service act is less than the server capacity m (it is supposed that the server is never idle). The imbedded Markov chain $\{\xi_r\}$ describing the number of customers in the system at $t_n + 0$ where t_n, $n = 1, 2, \ldots$ are successive moments of service completions, has the transition matrix of Δ_m type with

$$A_i(z) = \begin{cases} K(z)\gamma(\gamma + \lambda - \lambda z)^{-1}, & i < m \\ z^{i-m}K(z), & i \geq m \end{cases}$$

where $K(z) = B^*(\lambda - \lambda z)$, B is the distribution function of service time, B^* is the Laplace-Stieltjes transform of B, λ is the intensity of Poisson arrival process, and γ is the rate of the additional phase. Condition $\lambda \neq 0$ guarantees that $k_i > 0$ for $i = 0, 1, 2, \ldots$ and, therefore, the chain $\{\xi_r\}$ is queue-type. Due to Theorem 3.3 the chain is ergodic if and only if $K'(1) < m$. Using relation (5.7), it can be shown that

$$P(z) = \frac{K(z)\sum_{i=0}^{m-1} p_i[z^m - z^i(\gamma + \lambda - \lambda z)\gamma^{-1}]}{z^m - K(z)} \cdot \frac{\gamma}{\gamma + \lambda - \lambda z}.$$

It follows that $\sum_{i=0}^{m-1} p_i[z^m - z^i(\gamma + \lambda - \lambda z)\gamma^{-1}]$, an mth degree polynomial in z, must have the same roots with the same multiplicities as $z^m - K(z)$ in $\bar{\Gamma}^+ \backslash \{1\}$ and must assume value $m - K'(1)$ at $z = 1$. Therefore, this polynomial must be equal to $R(z)$ and we get:

$$P(z) = \frac{\gamma}{\gamma + \lambda - \lambda z} \frac{K(z)R(z)}{z^m - K(z)} = \frac{\gamma}{\gamma + \lambda - \lambda z}\pi(z).$$

Example 2: Retaining all assumptions of Example 1, suppose in addition that customers can also arrive in pairs, so that the generating function of the arriving groups of customers is $a(z) = pz + qz^2$, $p + q = 1$. Then

$$A_i(z) = \begin{cases} K(z)\gamma(\gamma + \lambda - \lambda pz - \lambda qz^2)^{-1}, & i < m \\ z^{i-m}K(z), & i \geq m \end{cases}$$

where $K(z) = B^*(\lambda - \lambda a(z))$.

Assuming that $K'(1) < m$ and using (5.7), we obtain

$$P(z) = \frac{K(z)}{z^m - K(z)} \cdot \frac{\gamma}{\gamma + \lambda - \lambda a(z)} \sum_{i=0}^{m-1} p_i[z^m - z^i\gamma^{-1}(\gamma - \lambda pz - \lambda qz^2)]. \tag{5.11}$$

Repeating the arguments of Example 1, we conclude that the sum in (5.11) must be divisible by $R(z)$, and, therefore, being an $(m+1)$th degree polynomial, can be factored into the product of $R(z)$ and a first degree polynomial that equals 1 at $z = 1$:

$$\sum_{i=0}^{m-1} p_i[z^m - z^i\gamma^{-1}(\gamma + \lambda - \lambda pz - \lambda qz^2)] = (z - c)(1 - c)^{-1}R(z), \tag{5.12}$$

where c $(\neq 1)$ is a constant. To find c, substitute for z respective roots β_1 and β_2 of the quadratic polynomial $\gamma+\lambda-\lambda pz-\lambda qz^2$. Then

$$\beta_1^m\sum_{i=0}^{m-1}p_i=(\beta_1-c)(1-c)^{-1}R(\beta_1)$$

$$\beta_2^m\sum_{i=0}^{m-1}p_i=(\beta_2-c)(1-c)^{-1}R(\beta_2)$$

and we find:

$$c=\frac{\beta_2R(\beta_2)\beta_1^m-\beta_1R(\beta_1)\beta_2^m}{R(\beta_2)\cdot\beta_1^m-R(\beta_1)\beta_2^m}. \tag{5.13}$$

The final formula for $P(z)$ is

$$P(z)=\frac{\gamma}{\gamma+\lambda-\lambda a(z)}\,\frac{z-c}{1-c}\cdot\frac{K(z)R(z)}{z^m-K(z)}=\frac{\gamma}{\gamma+\lambda-\lambda a(z)}\,\frac{z-c}{1-c}\,\pi(z)$$

with c given by (5.13).

Example 3: Under the assumptions of Example 1 consider two additional exponential phases instead of one. Then

$$A_i(z)=\begin{cases}K(z)\gamma^2(\gamma+\lambda-\lambda z)^{-2}, & i<m\\ z^{i-m}K(z), & i\geq m\end{cases}$$

Assuming again that $K'(1)<m$ and using (5.7) we obtain:

$$P(z)=\frac{\gamma^2}{(\gamma+\lambda-\lambda z)^2}\,\frac{K(z)}{z^m-K(z)}\sum_{i=0}^{m-1}p_i[z^m-z^i\gamma^{-2}(\gamma+\lambda-\lambda z)^2].$$

Repeating the arguments of Examples 1 and 2, we conclude that

$$\sum_{i=0}^{m-1}p_i[z^m-z^i\gamma^{-2}(\gamma+\lambda-\lambda z)^2]=\frac{z-c}{1-c}R(z). \tag{5.14}$$

To find c, we substitute $\beta=1+\gamma\lambda^{-1}$ for z in (5.14) and in the derivative of both sides of (5.14) at $z=\beta$:

$$\beta^m\sum_{i=0}^{m-1}p_i=(\beta-c)(1-c)^{-1}R(\beta)$$

$$m\beta^m\sum_{i=0}^{m-1}p_i=R(\beta)(1-c)^{-1}+(\beta-c)(1-c)^{-1}R'(\beta).$$

It follows that

$$c=\frac{(m-1)\beta R(\beta)-\beta^2R'(\beta)}{mR(\beta)-\beta R'(\beta)}. \tag{5.15}$$

The final formula for $P(z)$ is:

$$P(z)=\frac{\gamma^2}{(\gamma+\lambda-\lambda z)^2}\,\frac{z-c}{1-c}\cdot\frac{K(z)R(z)}{z^m-K(z)}=\frac{\gamma^2}{(\gamma+\lambda-\lambda z)^2}\cdot\frac{z-c}{1-c}\pi(z)$$

where c is found in (5.15).

Example 4: Consider a Markov chain describing the evolution of a dam (or a storage) with a single-level control policy [2]. Suppose that X_k units of volume of water (or any kind of other material) enter a reservoir during the interval $(k,k+1)$, $k=0,1,2,\ldots$, and Y_k units flow out at the instant $k+1$. Suppose also that $X_0,X_1,X_2,\ldots$ are independent identically distributed random variables and that $Y_k = min\{Z_k + X_k, m\}$, where Z_k is the supply of water at instant $k+0$ and m is a constant. A single-level control policy requires that for all k

$$P\{X_k \le x | Z_k = j\} = \begin{cases} G(x), & \text{if } j = 0,1,2,\dots,n \\ K(x), & \text{if } j = n+1, n+2, \dots \end{cases}$$

and

$$m_j = \begin{cases} r \text{ if } Z_k = 0,1,2,\dots,n \\ m \text{ if } Z_k = n+1, n+2, \dots, m \ge r, \end{cases}$$

so that

$$Y_k = \begin{cases} min\{Z_k + X_k, r\}, & \text{if } Z_k = 0,1,2,\dots,n \\ min\{Z_k + X_k, m\}, & \text{if } Z_k = n+1, n+2, \dots \end{cases}.$$

Under these conditions $\{Z_k\}$, $k = 0,1,2,\dots$ constitutes a homogeneous Markov chain with the following transition $\Delta_{m,n}$-matrix:

$$A = \begin{array}{cc}
 & \begin{array}{cccccccccccc} 0 & 1 & 2 & \cdots & \cdots & n-m+1 & \cdots & \cdots & \cdots & n-r & & \end{array} \\
\begin{array}{c} 0 \\ 1 \\ \cdots \\ r \\ r+1 \\ \cdots \\ n \\ n+1 \\ n+2 \\ n+3 \\ \cdots \end{array} &
\left(\begin{array}{cccccccccccc}
f_r & g_{r+1} & g_{r+2} & \cdots & & \vdots & & & & \vdots & & \\
f_{r-1} & g_r & g_{r+1} & \cdots & & \vdots & & & & \vdots & & \\
\vdots & \vdots & \vdots & & & \vdots & & & & \vdots & & \\
f_0 & g_1 & g_2 & \cdots & & \vdots & & & & \vdots & & \\
0 & g_0 & g_1 & \cdots & & \vdots & & & & \vdots & & \\
\vdots & \vdots & \vdots & & & \vdots & & & & \vdots & & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots 0 & g_0 & g_1 & \cdots \\
0 & 0 & 0 & \cdots & 0 & k_0 & k_1 & k_2 & \cdots & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & k_0 & k_1 & \cdots & & & \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & k_0 & \cdots & & & \\
\vdots & \vdots & \vdots & & & & & & & & &
\end{array}\right)
\end{array}$$

where $g_i = P\{X_k = i \mid Z_k \le n\}$, $k_i = P\{X_k = i \mid Z_k > n\}$, $i = 0,1,2,\dots$, $f_j = \sum_{i=0}^{j} g_i$, $j = 0,1,2,\dots,r$. Denote $G(z) = \sum_{i=0}^{\infty} g_i z^i$, $K(z) = \sum_{i=0}^{\infty} k_i z^i$. Assuming $K'(1) < m$ and applying Theorem 5.1 we obtain the following expression for the generating function $P(z)$ of the ergodic distribution of the chain $\{Z_k\}$, $k = 0,1,2,\dots$:

$$P(z) = \frac{z^{m-r}\sum_{i=0}^{r-1} p_i[A_i(z)z^r - G(z)z^i] + [G(z)z^{m-r} - K(z)]\sum_{i=0}^{n} p_i z^i}{z^m - K(z)}. \tag{5.16}$$

The expression in the right-hand side of (5.16) can be simplified by noticing that if $G(z) = K(z)$ and $m = r$, the relation (5.16) turns into

$$P(z) = \frac{\sum_{k=0}^{r-1} p_i(A_i(z)z^r - G(z)z^i)}{z^r - G(z)}. \tag{5.17}$$

At the same time, the matrix A turns into a Δ_r-matrix and, therefore, by Theorem 2 in [1], the generating function $P(z)$ in (5.17) can be represented in the form

$$P(z) = \sum_{i=0}^{r-1} p_i \alpha_i(z) \tag{5.18}$$

where $\alpha_i(z) = (z-1)\varphi_{r-1}(z)[z^r - G(z)]^{-1}$ $\varphi_{r-i-1}(z) = \mathfrak{T}_{r-i-1}[z^i G(z)(1-z)^{-1}]$ (the operator $\mathfrak{T}_i F(z)$ gives the ith truncation of the Taylor's series of $F(z)$).

Comparing (5.17) and (5.18), we conclude that

$$A_i(z)z^r - G(z)z^i = (z-1)\varphi_{r-i-1}(z)$$

and, therefore,

$$P(z) = \frac{z^{m-r}(z-1)\sum_{i=0}^{r-1} p_i \varphi_{r-i-1}(z) + [G(z)z^{m-r} - K(z)]\sum_{i=0}^{n} p_i z^i}{z^m - K(z)}. \tag{5.19}$$

As was mentioned earlier (see Remark 2), in order to find the unknown probabilities $p_0, p_1, p_2, \ldots, p_n$ in the right-hand side of (5.19), we can use the m equations obtained from the existence of m roots of $z^m - K(z)$ and the analyticity of $P(z)$ in Γ^+, and $n-m+1$ additional equations obtained from the matrix equation $\boldsymbol{P} = \boldsymbol{P}A$. However, in this case due to a special structure of the transition matrix A, it is possible to simplify this procedure and obtain more convenient and even explicit solutions. It can be noticed that although matrix A, strictly speaking, is not essential in the sense of Definition 2.4, all elements $a_{i,i-r}$ of the rth subdiagonal, $i = r, r+1, \ldots, n$ and all elements $a_{i,i-m}$ of the mth subdiagonal, $i = n+1, n+2, \ldots$ are not zeros, and therefore matrix A possesses some properties of essential matrices. In particular, it can be noticed that $\sum_{k=0}^{n-m+r} p_i z^i$ can be expressed with the help of the matrix equations $\boldsymbol{P} = \boldsymbol{P}A$ solely in terms of $p_0, p_1, \ldots, p_{r-1}$. It implies that

$$\sum_{k=0}^{n-m+r} p_i z^i = \mathfrak{T}_{n-m+r} \sum_{i=0}^{r-1} p_i \alpha_i(z).$$

Using this fact, we can represent $P(z)$ in the following form:

$$\begin{aligned} P(z) = \Bigg\{ & \sum_{i=0}^{r-1} p_i \Big\{ z^{m-r}[(z-1)\varphi_{r-i-1}(t) + (G(z)z^{m-r} - K(z))]\mathfrak{T}_{n-m+r}\alpha_i(z) \Big\} + \\ & + (G(z)z^{m-r} - K(z)) \sum_{i=n-m+r+1}^{n} p_i z^i \Bigg\} (z^m - K(z))^{-1}. \end{aligned} \tag{5.20}$$

Now the number of unknown probabilities in the right-hand side of (5.20) is $r + n - (n - m + r + 1) + 1 = m$ and, hence, to determine them we do not need any additional equations. (This approach is especially advantageous when $n \gg m$).

Let us consider, for example, the case $m = r = 1$. Then

$$P(z) = p_0 g_0\left[z - 1 + (G(z) - K(z))\mathcal{T}_n\left\{\frac{z-1}{z - G(z)}\right\}\right] \cdot (z - K(z))^{-1}$$

$$p_0 = (1-\rho)\left[g_0\left(1 - \mathcal{T}_n\left\{\frac{\delta - \rho}{G(z) - z}\right\}\right)\right]^{-1}$$

where $\delta = G'(1)$, $\rho = K'(1) < 1$.

In particular, if $g_k = ab^k$, $k = 0,1,2,\ldots$; $0 < a < 1$, $b = 1 - a$ (a geometric distribution), then $G(z) = a(1 - bz)^{-1}$, $\delta = \frac{b}{a}$

$$\mathcal{T}_n\left\{\frac{z-1}{z - g(z)}\right\} = \frac{1 - bz - b(\delta z)^{n+1}}{a(1 - \delta z)}$$

and, therefore,

$$P(z) = p_0 \frac{a(z-1)(1-\delta z) + [G(z) - K(z)](1 - bz - b(\delta z)^{n+1})}{(z - K(z))(1 - \delta z)}$$

$$p_0 = (1-\rho)[a(1-\rho)\frac{1 - \delta^{n+2}}{1 - \delta} + a\delta^{n+2}]^{-1}.$$

6. FINDING THE STATIONARY DISTRIBUTION OF A MARKOV CHAIN WITH TRANSITION $\Delta'_{m,n}$-MATRIX

Let $\{\zeta_r\}$, $r = 0,1,2,\ldots$ be a Markov chain with transition homogeneous $\Delta'_{m,n}$-matrix. Throughout this section, we assume that conditions of Theorem 3.3 are satisfied, so $\{\zeta_r\}$ is queue-type and, therefore, irreducible and aperiodic. If, in addition, $K'(1) > m$ then, according to Theorem 3.5 and Theorem 4.1, $\{\zeta_r\}$ is ergodic and inside Γ, $z^m - K(z)$ has exactly m roots ω_r, $r = 1,2,\ldots,R$ with multiplicities τ_r, $\sum_{r=1}^{R} \tau_r = m$.

We introduce the following polynomial:

$$Q(z) = \prod_{r=1}^{R} (z - \omega_r)^{\tau_r} = q_0 + q_1 z + \ldots + q_m z^m.$$

Consider an auxiliary Markov chain $\{\hat{\zeta}_r\}$ with transition matrix $\hat{A} = (\hat{a}_{ij})$, $i,j = 0,1,2,\ldots$ where

$$\hat{a}_{ij} = \begin{cases} \sum_{l=i+m}^{\infty} k_l, & \text{if } j = 0, \\ k_{i+m-j}, & \text{if } 0 < j \leq i+m, \\ 0, & \text{if } j > i+m. \end{cases} \tag{6.1}$$

(This matrix is the transition matrix of the imbedded Markov chain describing the evolution of the number of customers in a bulk queueing system $G/M^Y/1$ with continuously operating server and customers arriving in groups of m).

Theorem 6.1: *The generating function of the ergodic distribution of* $\{\hat{\zeta}_r\}$ *is determined by*

$$P(z) = \frac{Q(1)}{z^m Q(\frac{1}{z})}. \tag{6.2}$$

Proof: Multiplying both sides of the matrix equation $P = PA$ from the right by vector $(1, z, z^2, \ldots)^T$, $|z| = 1$ and using (6.1) we get:

$$P(z) - p_0 = \sum_{i=0}^{\infty} p_i \sum_{j=1}^{i+m} k_{i+m-j} z^j, \ |z| = 1. \tag{6.3}$$

Now multiplying (6.3) by $z^m Q(\frac{1}{z})[1 - z^{-m} K(\frac{1}{z})]^{-1}$ after some transformations, we finally obtain:

$$P(z) z^m Q(\tfrac{1}{z}) = \frac{(p_0 - \Phi(z)) Q(\frac{1}{z})}{z^{-m} - K(\frac{1}{z})}, \ |z| = 1 \tag{6.4}$$

where $\Phi(z) = \sum_{i=0}^{\infty} p_i \sum_{l=i+m}^{\infty} k_l z^{m+i-l}$.

Since $z^m Q(\frac{1}{z}) = \sum_{i=0}^{m} q_i z^{m-i}$, the left-hand side of (6.4) is analytic in Γ^+ and continuous on Γ.

On the other hand, the right-hand side of (6.4) by the definition of $Q(z)$ and $\Phi(z)$ and due to the obvious relation $p_0 = \Phi(1)$ is analytic in Γ^- and continuous on Γ. By Liouville's theorem, it means that (6.4) holding true on Γ is only possible if both sides are identically equal to the same constant, say, C. Therefore, $P(z) z^m Q(\frac{1}{z}) = C$. Since $P(1) = 1$, $C = Q(1)$ which yields (6.2).

Now we formulate the main result of this section.

Theorem 6.2: *The generating function of the stationary distribution of a Markov chain* $\{\zeta_r\}$ *with transition homogeneous* $\Delta'_{m,n}$*-matrix* $A = (a_{ij})$ *is determined by*

$$P(z) = \sum_{i=0}^{n} p_i z^i - z^{n+1} \sum_{i=1}^{m} p_{n+1-i} \left(\sum_{j=0}^{m-i} q_j z^{m-i-j} [z^m Q(\tfrac{1}{z})]^{-1} \right). \tag{6.5}$$

The unknown probabilities $p_0, p_1, \ldots, p_n$ *on the right-hand side of* (6.5) *form a unique solution of the system of* $(n+1)$ *linear equations*

$$p_j = \sum_{i=0}^{n} p_i a_{ij} - \sum_{i=1}^{m} p_{n+1-i} b_{ij}, \ j = 1, 2, \ldots, n \tag{6.6}$$

$$Q(1) = \sum_{i=0}^{n} p_i \, Q(1) - \sum_{i=1}^{m} p_{n+1-i} \sum_{j=0}^{m-i} a_{ij} \tag{6.7}$$

where $b_{ij} = \sum_{l=0}^{\infty} a_{n+1+lj} D_z^{(l)} \sum_{k=0}^{m-1} q_k z^{m-i-k} [z^m Q(\frac{1}{z})]^{-1}$. $D_z^{(l)}$ *is the operator defined by*

$$D_z^{(l)} = \frac{1}{l!}\frac{\partial^l}{\partial z^l}(\cdot)\bigg|_{z=0}.$$

Proof: Multiplying both parts of the matrix equations $P = PA$ from the right by vector $(\underbrace{0,0,\ldots,0}_{n+1},1,z,z^2,\ldots)^T$, $|z| = 1$, and using the definition of a homogeneous $\Delta_{m,n}$-matrix, we obtain

$$\sum_{i=n+1}^{\infty} p_i z^{i-n-1} = \sum_{i=n+1-m}^{\infty} p_i \sum_{j=n+1}^{i+m} k_{i-j+m} z^j, \ |z| = 1.$$

Rearranging this relation and multiplying both sides by $z^m Q(\frac{1}{z})[1 - z^m K(\frac{1}{z})]^{-1}$, we get

$$\sum_{i=n+1}^{\infty} p_i z^{i-n-1} z^m Q(\tfrac{1}{z}) = -\sum_{i=1}^{m} p_{n+1-i} z^{-i} z^m Q(\tfrac{1}{z}) + \frac{\Phi(z)Q(\frac{1}{z})}{z^{-m} - K(\frac{1}{z})}, \tag{6.8}$$

where $\Phi(z) = \sum_{i=n+1-m}^{n} p_i z^{i-n} - \sum_{i=n+1-m}^{\infty} p_i \sum_{l=i+m-n}^{\infty} k_l z^{i+m-n-l-1}$.

Now we represent

$$\sum_{i=1}^{\infty} p_{n+1-i} z^{-i} \cdot z^m Q(\tfrac{1}{z}) = \sum_{i=1}^{m} p_{n+1-i} \sum_{j=0}^{m-i} q_j z^{m-i-j} + \sum_{i=1}^{m} p_{n+1-i} \sum_{j=m-i+1}^{m} q_j z^{m-i-j},$$

so (6.8) can be transformed into

$$\begin{aligned} &\sum_{i=n+1}^{\infty} p_i z^{i-n} z^m Q(\tfrac{1}{z}) + \sum_{i=1}^{m} p_{n+1-i} \sum_{j=0}^{m-i} q_j z^{m-i-j} \\ &= -\sum_{i=1}^{m} p_{n+1-i} \sum_{j=m-i+1}^{m} q_j z^{m-i-j} + \frac{\Phi(z)Q(\frac{1}{z})}{z^{-m} - K(\frac{1}{z})}, \ |z| < 1. \end{aligned} \tag{6.9}$$

The right-hand side of (6.9) is, obviously, analytic in Γ^+ and continuous on Γ. At the same time the right-hand side of (6.9) is analytic in Γ^- and continuous on Γ. To show this fact, it is enough to notice that the first part of the right-hand side of (6.9) is a polynomial in $\frac{1}{z}$ while the function $\Phi(z)$ in the second part is, by inspection, a Maclaurin series in $\frac{1}{z}$ with absolutely summable coefficients. Since the left-hand side of (6.9) exists at $z = 1$, so does the right-hand side of it. Also, by definition of $Q(z)$, $Q(\frac{1}{z})/(z^{-m} - K(\frac{1}{z}))$ is obviously analytic in Γ^-. Now, applying again the Liouville's theorem to (6.9), we conclude that both sides of this relation must be identically equal to the same constant C. The limit of the expression in the right-hand side of (6.9) as $z \to \infty$ is 0, because $\Phi(z)$ contains only negative powers of z. Therefore, $C = 0$ and the left-hand side of (6.9) gives

$$\sum_{i=n+1}^{\infty} p_i z^{i-n-1} = -\sum_{i=1}^{m} p_{n+1-i} \frac{\sum_{j=0}^{m-i} q_j z^{m-i-j}}{z^m Q(\frac{1}{z})} \tag{6.10}$$

which yields (6.5).

Applying operators $D_z^{(l)}$, $l = 0, 1, 2, \ldots$ to (6.10), we obtain

$$p_{n+1+l} = -\sum_{i=1}^{m} p_{n+1-i} D_z^{(l)} \sum_{j=0}^{m-i} q_j z^{m-i-j} [z^m Q(\tfrac{1}{z})]^{-1}, l = 0,1,2,\ldots \quad (6.11)$$

Now (6.6) follows from transition equations

$$p_j = \sum_{i=0}^{\infty} p_i a_{ij},\ j = 1,2,\ldots,n \quad (6.12)$$

and relations (6.11); relations (6.7) follow from (6.5) at $z = 1$, and the condition $P(1) = 1$.

It can be proved that (6.6) – (6.7) has a unique solution. Indeed, suppose $\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_n$ is another solution. Then the sequence $\{\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_n, \hat{p}_{n+1}, \ldots\}$ where $\hat{p}_l$, $l > n$, are recursively defined by (6.11) satisfies transition equations (6.12). Reversing the arguments that led to (6.5), one can easily verify that this sequence also satisfies (6.12) for $j > n$ and, due to matrix A being stochastic, for $j = 0$. Since the right-hand side of (6.10) is a rational function of z and all its poles belong to Γ^-, the sequence $\{\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_{n+1}, \ldots\}$ generated by (6.11) is absolutely summable. Therefore, we have two different absolutely summable solutions of $\boldsymbol{P} = \boldsymbol{P}A$, the components of which add up to 1 which contradicts the Chung Kai-Lai theorem [5]. This contradiction proves the uniqueness of the solution of the system (6.6) – (6.7). □

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