Rothe's Method to Semilinear Hyperbolic Integrodifferential Equations[1]

D. BAHAGUNA
Department of Mathematics, Indian Institute of Technology, Kanpur
Kanpur - 208 016, India

A.K. PANI
Department of Mathematics, Indian Institute of Technology, Bombay
Powai, Bombay - 400076, India

V. RAGHAVENDRA
Department of Mathematics, Indian Institute of Technology, Kanpur
Kanpur - 208 016, India

ABSTRACT

In this paper we consider an application of Rothe's method to abstract semi-linear hyperbolic integrodifferential equations in Hilbert spaces. With the aid of Rothe's method we establish the existence of a unique strong solution.

Key words: Rothe's Method, Positive Definite Operator, V-Elliptic Operator, Lax-Milgram Lemma.

AMS Subject Classification: 34G20, 35H05.

1. INTRODUCTION

In this paper we are concerned with the application of Rothe's method to the following semi-linear hyperbolic integrodifferential equation

(1.1)
$$\frac{d^2u}{dt^2}(t) + Au(t) = \int_0^t a(t-s)k(s,u(s))ds + f(t), \quad a.e. \quad t \in I$$

$$u(0) = U_0 \in \mathcal{V}, \ \frac{du}{dt}(0) = U_1 \in \mathcal{V}$$

where u is an unknown function from $I := [0,T]$, $0 < T < \infty$, into a real Hilbert space $\mathcal{H}$, A is a bounded linear operator from another Hilbert space $\mathcal{V}$ into its dual space $\mathcal{V}^*$, k is a nonlinear mapping from $[0,T] \times \mathcal{V}$ into $\mathcal{H}$, a and f, respectively, are real-valued and $\mathcal{H}$-valued functions on $[0,T]$.

[1]Received: April 1989, Revised: September 1989

Earlier, some of the applications of Rothe's method to the homogeneous and nonhomogeneous linear hyperbolic problems have been considered by Rektorys [6], Putlar[5] and Streiblova [8] (other references are cited in these papers).

Kačur [4] has applied Rothe's method to a semilinear hyperbolic equation under a global Lipschitz-like condition on nonlonear forcing term. Recently, Bahuguna [1,2] has employed Rothe's method to a more general case of the problem considered by Kačur [4] and has proved the local existence under local Lipschitz condition on nonlinear forcing terms.

Similar kinds of nonlinear integral perturbations as in (1.1) have been investigated by Bahuguna and Raghavendra [3] (see also [2]) for nonlinear parabolic problems with the aid of Rothe's method.

2. ASSUMPTIONS AND MAIN RESULT

Let $\mathcal{V}$ and $\mathcal{H}$ be two real Hilbert spaces such that $\mathcal{V}$ is dense in $\mathcal{H}$ and the embedding of $\mathcal{V}$ in $\mathcal{H}$ is compact. We denote by $\|\cdot\|$ and $|\cdot|$ the respective norms of $\mathcal{V}$ and $\mathcal{H}$. Furthermore, the inner product in $\mathcal{H}$ and the usual duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ are denoted by (u,v), $u, v\in \mathcal{H}$; and $<f,v> \in \mathcal{V}^*$, $v\in \mathcal{V}$; respectively. Let I denote the interval $[0,T]$ where $0<T<\infty$ is arbitrary. We introduce the following hypotheses:

(H_1) The bounded linear operator A: $\mathcal{V} \rightarrow \mathcal{V}^*$ is symmetric and $\mathcal{V}$-elliptic, i.e.

$$<Au,v>=<Av,u> \text{ and } <Au,u>\geq\alpha\|u\|^2$$

for all $u, v\in \mathcal{V}$ and $\alpha>0$ is a constant.

(H_2) k: $\mathrm{I}\times\mathcal{V} \rightarrow \mathcal{H}$ is continuous in both variables and satisfies

$$|k(t,u)|\leq C_1\|u\|+C_2$$

for all $t\in$ I and all $u\in\mathcal{V}$, where C_1 and C_2 are positive constants.

(H_3) The mapping k satisfies

$$|k(t,u)-k(t,v)|\leq L(t)\|u-v\|$$

for $t\in$ I a.e. and all $u, v\in\mathcal{V}$, where $L\in L^1(\mathrm{I})$ is nonnegative.

(H_4) Functions f: I $\rightarrow \mathcal{H}$ and a: I $\rightarrow \mathbb{R}$ are Lipschitz continuous.

To apply Rothe's method to equation (1.1), we proceed as follows. For every positive integer n denote by $\{t_j^n\}$ the partition of the interval I defined by $t_j^n=j\cdot h$, $h=\frac{T}{n}$, $j=1,\ldots,n$. Setting

$$u_0^n=U_0\,,\; u_{-1}^n=U_0-hU_1 \tag{2.1}$$

$$u_{-2}^n=h^2(f(0)-AU_0)-2hU_1+U_0\,, \tag{2.2}$$

we successively look for a solution $u_j^n\in\mathcal{V}$ of the variational identity

(2.3) $$\left(\frac{u-2u_{j-1}^n+u_{j-2}^n}{h^2},\ v\right)+<Au,v>=(h\sum_{i=0}^{j-1}a(t_j^n-\ t_i^n)k(t_i^n,u_i^n)+f(t_j^n),v)$$

for all $v\in\mathcal{V}$ and $j=1,2,\ldots,n$. The existence of a unique solution satisfying (2.3) is a consequence of Lax-Milgram Theorem, see Rektorys [7, p. 383]. Denote

(2.4) $$z_j^n=\frac{u_j^n-u_{j-1}^n}{h},\ s_j^n=\frac{z_j^n-z_{j-1}^n}{h},\ j=0,1,\ldots,n$$

and define Rothe's sequences $\{U^n\}$ and $\{Z^n\}$ of Lipschitz continuous functions respectively from I into $\mathcal{V}$ and from I into $\mathcal{H}$ by

(2.5) $$\begin{cases} U^n(t)=u_{j-1}^n+\frac{1}{h}(t-t_{j-1}^n)(u_j^n-u_{j-1}^n) \\ Z^n(t)=z_{j-1}^n+\frac{1}{h}(t-t_{j-1}^n)(z_j^n-z_{j-1}^n) \end{cases}$$

and sequences $\{u^n\}$, $\{z^n\}$, $\{s^n\}$ of step functions from $(-h,T]$ into $\mathcal{V}$, by

(2.6) $$\begin{matrix} u^n(t)=u_0^n \\ z^n(t)=z_0^n \\ s^n(t)=s_0^n \end{matrix} \quad t\in(-h,0] \qquad \begin{matrix} u^n(t)=u_j^n \\ z^n(t)=z_j^n \\ s^n(t)=s_j^n \end{matrix} \quad t\in(t_{j-1}^n,t_j^n]$$

After proving some *a priori* bounds for the sequences of functions $\{U^n\}$, $\{Z^n\}$, $\{u^n\}$, $\{z^n\}$ and $\{s^n\}$ we prove the following main existence result for equation (1.1).

Theorem 2.1. *Assume that Hypotheses* (H_1), (H_2), *and* (H_4) *hold and let* $AU_0\in\mathcal{H}$. *Then there exists a function* u *in* $Lip(\mathrm{I},\mathcal{V})$ *with the properties*

$$\frac{du}{dt}\in L_\infty(\mathrm{I},\mathcal{V})\cap C(\mathrm{I},\mathcal{H}),\ \frac{d^2u}{dt^2}\in L_\infty(\mathrm{I},\mathcal{H})$$

$$Au\in L_\infty(\mathrm{I},\mathcal{H}),\ u(0)=U_0,\ \frac{du}{dt}(0)=U_1$$

and u *satisfies the identity*

(2.7) $$\left(\frac{d^2u}{dt^2}(t),v\right)+<Au(t),v>=(K(u)(t)+f(t),v)$$

for $t\in\mathrm{I}$ *a.e. and for all* $v\in\mathcal{V}$, *where*

(2.8) $$K(u)(t)=\int_0^t a(t-s)k(s,u(s))ds$$

In addition, if (H_3) *is also satisfied, then* u *is unique.*

For the notational convenience, we drop the superscript n and denote for $0\leq i,j\leq n$ by

$$\text{(2.9)} \qquad \begin{cases} a_{ji} = a(t_j - t_i) \\ k_j = k(t_j, u_j) \\ f_j = f(t_j) \end{cases}$$

Henceforth, C will represent a generic constant independent of j, h and n. Below we state and prove all lemmas required in the proof of Theorem 2.1 which is proved at the end.

Lemma 2.1. *Assume that hypotheses* (H_1), (H_2) *and* (H_4) *hold. Then there exists a positive integer* N *such that*

$$|z_j|^2 + \|u_j\|^2 \leq C\,,\ j=1,2,\ldots,n,\ n>N.$$

Proof. Using the notations of (2.4) and (2.9) in (2.3), for all $v \in \mathcal{V}$ and $j=1,2,\ldots,n$, we have

$$\text{(2.10)} \qquad (z_j - z_{j-1},\ v) + h < Au_j\,, u> = h^2\Big(\sum_{i=0}^{j-1} a_{ji}k_i\,, v\Big) + h(f_j\,, v).$$

Putting $v = z_j$ in (2.10), using (H_2) and the identities

$$2(z_j - z_{j-1}\,, z_j) = |z_j|^2 + |z_j - z_{j-1}|^2 - |z_{j-1}|^2,$$

$$2 < Au_j\,, u_j - u_{j-1} > = \|u_j\|_A^2 + \|u_j - u_{j-1}\|_A^2 - \|u_{j-1}\|_A^2,$$

we obtain

$$\text{(2.11)} \qquad |z_j|^2 - |z_{j-1}|^2 + \|u_j\|_A^2 - \|u_{j-1}\|_A^2 \leq Ch|z_j|^2 + Ch^2 \sum_{i=0}^{j-2} \|u_i\|_A^2 + Ch.$$

Choose a positive integer N such that $CT/N < 1$. Then for $n > N$ inequality (2.11) implies that

$$\text{(2.12)} \qquad (1-Ch)[|z_j|^2 + \|u_j\|_A^2 \leq (1+Ch^2)[|z_{j-1}|^2 + \|u_{j-1}\|_A^2] + Ch^2 \sum_{i=0}^{j-1} \|u_i\|_A^2 + Ch.$$

Applying inequality (2.12) recursively, we obtain

$$\text{(2.13)} \qquad (1-Ch)^j[|z_j|^2 + \|u_j\|_A^2] \leq (1+jCh^2)^j[|z_0|^2 + \|u_0\|_A^2] + jCh.$$

Inequality (2.13) implies

$$|z_j|^2 + \|u_j\|_A^2 \leq C$$

which together with the $\mathcal{V}$-ellipticity of A proves the assertion of the lemma.

Lemma 2.2. *Assume the hypotheses of Lemma* 2.1 *and let* $AU_0 \in \mathcal{H}$. *Then there exists a positive integer* N *such that*

$$\|z_j\|_A^2 + |s_j|^2 \leq C,\ j=1,2,\ldots,n,\ n>N.$$

Proof. We rewrite (2.10) as

$$\text{(2.14)} \qquad (s_j,\ v) + < Au_j,\ v > = h \sum_{i=0}^{j-1} (a_{ji}k_i,\ v) + (f_j,\ v).$$

Thus we have

$$(s_j,\ v) + < Au_j - Au_{j-1},\ v > = (s_{j-1},\ v) + h\,(a_{jj-1}k_{j-1},\ v)$$

$$+\, h \sum_{i=0}^{j-1} ([a_{ji} - a_{j-1i}]\ k_i,\ v)$$

$$+ (f_j - f_{j-1}, v). \tag{2.15}$$

Putting $v = s_j$ in (2.15) using (H_2) and (H_2) and (H_4) we obtain

$$|s_j|^2 - |s_{j-1}|^2 + \|z_j\|_A^2 - \|z_{j-1}\|_A^2$$

$$\leq C\,h\,|s_j|^2 + C\,h^2 \sum_{i=o}^{j-1} \|z_i\|_A^2 + C\,h. \tag{2.16}$$

We assume that N is large enough such that $C\frac{T}{N} < 1$. For $n > N$, inequality (2.16) then implies that

$$(1 - C\,h)\,[|s_j|^2 + \|z_j\|_A^2]$$

$$\leq (1 + C\,h^2)\,[|s_{j-1}|^2 + \|z_{j-1}\|_A^2]$$

$$+ C\,h^2 \sum_{i=0}^{j-2} \|z_i\|_A^2 + C\,h. \tag{2.17}$$

Proceeding similarly as in Lemma 1.1 we obtain the required result of the lemma.

Remark 2.1. Lemmas 2.1 and 2.2 imply the estimates

$$\|u^n(t)\| + \|U^n(t)\| + \|z^n(t)\| + \|Z^n(t)\| + |s^n(t)| \leq C,$$

$$\|U^n(t) - u^n(t)\| + |Z^n(t) - z^n(t)| \leq \frac{C}{n},$$

$$\|U^n(t) - U^n(s)\| + \|Z^n(t) - Z^n(s)\| \leq C\,|\,t - s|$$

for all $t, s \in \mathrm{I}$ and $n > N$.

Lemma 2.3. *Assume the hypotheses of Lemma 2.2. Then there exists* $u \in Lip(\mathrm{I}, \mathcal{V})$ with the properties

$$\frac{du}{dt} \in L_\infty(\mathrm{I}, \mathcal{V}) \cap \mathcal{C}(\mathrm{I}, \mathcal{H}),\ \frac{d^2u}{dt} \in L_\infty(\mathrm{I}, \mathcal{H})$$

such that

$$U^n \to u \text{ in } \mathcal{C}(\mathrm{I}, \mathcal{V}) \text{ and } Z^n \to \frac{du}{dt} \text{ in } \mathcal{C}(\mathrm{I}, \mathcal{H}).$$

Proof. Since $\{u^n\}$ and $\{z^n\}$ are uniformly bounded in $\mathcal{V}$, and $\mathcal{V}$ is compactly embedded in $\mathcal{H}$, there exists a subsequence $\{n_k\}$ of of the indices $\{n\}$ such that

$$u^{n_k}(t) \to u(t) \text{ and } z^{n_k}(t) \to z(t) \text{ in } \mathcal{H} \text{ as } k \to \infty$$

for some functions u and z from I into $\mathcal{H}$. Remark 2.1 implies that

$$U^{n_k}(t) \to u(t) \text{ and } Z^{n_k}(t) \to z(t) \text{ as } k \to \infty.$$

We notice that the families $\{U^{n_k}\}$ and $\{Z^{n_k}\}$ are equicontinuous in $\mathcal{C}(\mathrm{I}, \mathcal{H})$. Also, $\{U^{n_k}(t)\}$ and

$\{Z^{n_k}(t)\}$ are relatively compact in $\mathcal{H}$ for every $t \in \mathrm{I}$. Therefore

$$U^{n_k} \to u \text{ and } Z^{n_k} \to z \text{ in } \mathcal{C}(\mathrm{I}, \mathcal{H}) \text{ as } k \to \infty.$$

Now we show that $U^{n_k} \to u$ in $\mathcal{C}(\mathrm{I}, \mathcal{V})$ as $k \to \infty$. We denote by

$$K^n(0) := h\, a_{10} k_0, \quad K^n(t) := \sum_{i=0}^{j-1} a_{ji} k_i,$$

for $t \in (t_{j-1}, t_j]$.

$$f^n(0) := f(0), \qquad f^n(t) := f(t_j)$$

Clearly, $\{K^n(t)\}$ and $\{f^n(t)\}$ are uniformly bounded and $f^n(t) \to f(t)$ uniformly on I as $n \to \infty$. From (2.14) for positive integers p, $q > N$, $t \in (0, T]$ and all $v \in \mathcal{V}$, we get

$$(s^p(t) - s^q(t), v) + {<}\, Au^p(t) - Au^q(t), v{>}$$

$$= (K^p(t) - K^q(t) + f^p(t) - f^q(t), v). \tag{2.18}$$

Putting $v = u^p(t) - u^q(t)$ in (2.18) and rearranging the terms, we obtain

$$\|u^p(t) - u^q(t)\|_A^2 \le [|s^p(t) - s^q(t)| + |K^p(t) - K^q(t)| \tag{2.19}$$

$$+ | f^p(t) - f^q(t) |] \, | u^p(t) - u^q(t) | \le C \, | u^p(t) - u^q(t) |.$$

Since $\{u^{n_k}\}$ converges in $\mathcal{C}(\mathrm{I}, \mathcal{H})$, inequality (2.19) implies that $\{u^{n_k}\}$ is a Cauchy sequence in $\mathcal{C}(\mathrm{I}, \mathcal{V})$. From Remark 2.1 it follows that $u : \mathrm{I} \to \mathcal{V}$ and $z : \mathrm{I} \to \mathcal{H}$ are Lipschitz continuous hence

$\frac{du}{dt} \in L_\infty(\mathrm{I}, \mathcal{V})$ and $\frac{dz}{dt} \in L_\infty(\mathrm{I}, \mathcal{H})$. Now for all $v \in \mathcal{V}$,

$$(U^n(t), v) = \int_0^t \left(\frac{dU^{n_k}}{dt}(s), v\right) ds + (U_0, v)$$

$$= \int_0^t (Z^{n_k}(s), v)\, ds + (U_0, v). \tag{2.20}$$

We pass through the limit as $k \to \infty$ in (2.20) to obtain

$$(u(t), v) = \int_0^t (z(s), v)\, ds + (U_0, v).$$

Therefore $\frac{du}{dt}(t) = z(t)$ a.e. on I and hence $\frac{d^2u}{dt^2}(t) \in L_\infty(\mathrm{I}, \mathcal{H})$. The proof of the lemma is complete.

Lemma 2.4. *Assume the hypotheses of Lemma* 2.3 *and let* $u(t)$ *be defined as in Lemma* 2.3. *Then*

$$K^n(t) \to K(u)(t) \text{ as } k \to \infty \text{ in } \mathcal{H} \text{ uniformly on } \mathrm{I}.$$

The proof of Lemma 2.4 is same as the proof of Lemma 2.4 in [3] (also, see [2, Chapter IV]).

Proof of Theorem 1.1. For n_k, we write (2.3) as

(2.21) $$(\frac{d}{dt} Z^{n_k}(t), v) + \langle Au^{n_k}(t), v\rangle = (K^{n_k}(t) + f^{n_k}(t), v)$$

for all $v \in \mathcal{V}$ and all $t \in (0, T]$. Integrating (2.21) over $(0, t)$, we get

(2.22) $$(Z^{n_k}(t), v) - (U_1, v) + \int_0^t \langle Au^{n_k}(s), v\rangle\, ds = \int_0^t (K^{n_k}(s) + f^{n_k}(s), v)\, ds.$$

Passing throught the limit as $k \to \infty$, using Lemma 2.4 and bounded convergence theorem, we have

(2.23) $$(z(t), v) - (U_1, v) + \int_0^t \langle Au(s), v\rangle\, ds = \int_0^t (K(u)(s) + f(s), v)\, ds.$$

Differentiating (2.23) with respect to t, we get

(2.24) $$(\frac{dz}{dt}(t), v) + \langle Au(t), v\rangle = (K(u)(t) + f(t), v)$$

for all $v \in \mathcal{V}$ and a.e. $t \in I$ which implies identity (2.7). Now we prove the uniqueness under hypothesis (H_3). Let u_1 and u_2 be two functions satisfying the assertions of Theorem 2.1. Let $u := u_1 - u_2$ and let

(2.25) $$W := \frac{a_0}{(\alpha)^{1/2}} \int_0^T w(s)ds, \text{ where } a_0 = \max_I |a(t)|.$$

We divide the interval I into a finite number of subintervals of equal lengths p such that

(2.26) $$Wp^2 < \frac{1}{2}.$$

Let $t_1, t_2 \in [0, p]$ be such that

(2.27) $$\left|\frac{du}{dt}(t_1)\right| = \max_{[0, p]} \left|\frac{du}{dt}(t)\right|,$$

(2.28) $$\| u(t_2) \|_A = \max_{[0, p]} \| u(t) \|_A.$$

Then we have

(2.29) $$\int_0^{t_1} \frac{d}{dt} \left|\frac{du}{dt}(t)\right|^2 dt + \int_0^{t_2} \frac{d}{dt} \| u(t) \|_A^2\, dt \le \int_0^p \left[\frac{d}{dt} \left|\frac{du}{dt}(t)\right|^2 + \frac{d}{dt} \| u(t)\|_A^2\right] dt.$$

Now from identy (2.7) for $v = \frac{du}{dt}(t)$, we have

(2.30) $$\frac{d}{dt} \left|\frac{du}{dt}(t)\right|^2 + \frac{d}{dt} \| u(t) \|_A^2 = 2\left(K(u_1)(t) - K(u_2)(t), \frac{du}{dt}(t)\right).$$

Therefore

$$\int_0^p \left[\frac{d}{dt} \left| \frac{du}{dt}(t) \right|^2 + \frac{d}{dt} \| u(t) \|_A^2 \right] dt \leq 2 \frac{a_0}{(\alpha)^{1/2}} \int_0^p \left(\int_0^t w(s) \| u(s) \|_A \, ds \right) \left| \frac{du}{dt}(t) \right| dt$$

$$\leq 2 \, W p^2 \, \| u(t_2) \|_A \left| \frac{du}{dt}(t_1) \right| \leq W p^2 \left[\left| \frac{du}{dt}(t_1) \right|^2 + \| u(t_2) \|_A^2 \right]. \tag{2.31}$$

From inequalities (2.29), (2.26) and (2.31) we have

$$\frac{du}{dt}(t) \equiv 0, \; u(t) \equiv 0 \; on \; [0, p].$$

Repeating the above arguments for $[ip, (i+1)p]$, $i = 1, 2, \ldots$, we have that $u(t) \equiv 0$ on I. Therefore $u_1 \equiv u_2$. The proof of Theorem 2.1 is thus complete.

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