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Research Article

Hölder-Type Inequalities for Norms of Wick Products

Alberto Lanconelli[1] and Aurel I. Stan[2]

[1] *Dipartimento di Matematica, Università di Bari, Campus Universitario, Via E. Orabona 4, 70125 Bari, Italy*
[2] *Department of Mathematics, Ohio State University at Marion, 1465 Mount Vernon Avenue, Marion, OH 43302, USA*

Correspondence should be addressed to Aurel I. Stan, stan.7@osu.edu

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Various upper bounds for the L^2-norm of the Wick product of two measurable functions of a random variable X, having finite moments of any order, together with a universal minimal condition, are proven. The inequalities involve the second quantization operator of a constant times the identity operator. Some conditions ensuring that the constants involved in the second quantization operators are optimal, and interesting examples satisfying these conditions are also included.

1. Introduction

It was proven in [1] that for any positive numbers p and q, such that $(1/p)+(1/q)=1$, any normally distributed random variable X, and any f and g complex-valued Borel measurable functions, such that both random variables $\Gamma(\sqrt{p}I)f(X)$ and $\Gamma(\sqrt{q}I)(X)$ are square integrable, the Wick product $f(X)\diamond g(X)$ is square integrable and the following inequality holds:

$$E\left[\left|f(X)\diamond g(X)\right|^2\right]\leq E\left[\left|\Gamma(\sqrt{p}I)f(X)\right|^2\right]E\left[\left|\Gamma(\sqrt{q}I)g(X)\right|^2\right]. \tag{1.1}$$

Here Γ denotes the second quantization operator and I the identity operator of the one-dimensional Hilbert space $\mathbb{C}X$. The authors' motivation was to find a Hausdorff-Young-type inequality for the theory of Bosonian Fock spaces and they believed that (1.1) was indeed an inequality of this type, based on their feeling that the Wick product is an analogue concept of the convolution product from the theory of Fourier transform. After discussing with other mathematicians and thinking more about it, they have become convinced that the Wick product is in fact a simpler product, playing for the theory of Bosonian Fock spaces a role similar to the classic product of two series. Together this reconsideration and the condition $(1/p) + (1/q) = 1$ strongly suggest that (1.1) is in fact a Hölder-type inequality for the theory of Gaussian Hilbert spaces (Bosonian Fock spaces).

We will generalize inequality (1.1) to other types of random variables X, and in some cases find the optimal constants p and q. Moreover, we will prove that no matter how we choose a nonconstant random variable X, having finite moments of any order, the condition $(1/p)+(1/q) = 1$ cannot be improved. In Section 2, we present a minimal background about the Szegö-Jacobi parameters of a random variable having finite moments of any order. We define a set of basic properties and prove some connections between these properties in Section 3. Section 4 is dedicated completely to proving a fundamental necessary condition that we call the universal minimal (unimprovable) condition. The main inequalities of the paper are proven in Section 5. Finally, in Section 6, we provide many examples in support of the results proven in the previous section. Some of these examples demonstrate that the estimates from Section 5 are optimal.

2. Background

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ a random variable having finite moments of all orders. That means, for all $p > 0$, $E[|X|^p] < \infty$, where E denotes the expectation with respect to P. Since X has finite moments of all orders, all the terms of the sequence, $1, X, X^2, \ldots$, are square integrable, and thus we can apply the Gram-Schmidt orthogonalization procedure to obtain a sequence of orthogonal polynomial random variables $f_0(X) = 1$, $f_1(X)$, $f_2(X), \ldots$. The inner product that we are using is $\langle f(X), g(X) \rangle := E[f(X)\overline{g(X)}]$ for all $f, g : \mathbb{R} \to \mathbb{C}$ measurable, such that $E[|f(X)|^2] < \infty$ and $E[|g(X)|^2] < \infty$. Also, $f_0, f_1, f_2, \ldots$ are polynomial functions chosen, such that for all $n \geq 0$, if f_n is not the null polynomial, then f_n has the degree equal to n and a leading coefficient of 1. In fact, if X is a discrete random variable taking on only k different values $a_1, a_2, \ldots, a_k$ with positive probabilities, then f_n is the null polynomial for all $n \geq k$. If X is not a discrete random variable, or X is a discrete random variable taking on a countable set of values with positive probabilities, then for all $n \geq 0$, f_n is a polynomial of degree n with a leading coefficient equal to 1.

It is well known that there exist two sequences of real numbers $\{\alpha_n\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 1}$, such that for all $n \geq 0$,

$$Xf_n(X) = f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}(X). \tag{2.1}$$

When $n = 0$, $f_{n-1} = f_{-1} := 0$ (the null polynomial) and we can choose $\omega_0 := 0$. Also, if X is a discrete random variable taking on only k different values with positive probabilities, then for $n = k - 1$, the equality (2.1) must be understood in the almost-sure sense, and we can choose $\alpha_n = 0$ and $\omega_n = 0$ for all $n \geq k$. The sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\omega_n\}_{n\geq 1}$ are called the *Szegö-Jacobi parameters* of X. Moreover, $\omega_1, \omega_2, \ldots$ are called the *principal* Szegö-Jacobi parameters of X. It is well known that for all $n \geq 1$, $E[f_n^2(X)] = \omega_1\omega_2 \cdots \omega_n$ (see, e.g., [2, 3]).

Let $N = k$ if X is discrete and takes on only k values with positive probabilities, and $N = \infty$ otherwise. We define the Hilbert space $\mathcal{H} := \{\sum_{n<N} c_n f_n(X) \mid \{c_n\}_{n<N} \subset \mathbb{C},\ \sum_{n<N} \omega_n!|c_n|^2 < \infty\}$, where $\omega_n! := \omega_1\omega_2 \ldots \omega_n$ for all $1 \leq n < N$ and $\omega_0! := 1$. $\mathcal{H}$ is in fact the closure of the space $F := \{f(X) \mid f \text{ is polynomial}\}$ in $L^2(\Omega, \mathcal{F}, P)$. For many classic probability measures $\mathcal{H} = L^2(\Omega, \sigma(X), P)$, where $\sigma(X)$ denotes the smallest sub-sigma-algebra of $\mathcal{F}$ with respect to which X is measurable. We denote by F_n the space of all random variables of the form $f(X)$, where f is a polynomial of degree at most n, and define $G_n := F_n \ominus F_{n-1}$, that is, G_n is the orthogonal complement of F_{n-1} into F_n for all $n \geq 0$. For convenience, we define F_{-1} and G_{-1} to be the null space. For all $n \geq 0$, $G_n = \mathbb{R} f_n$ and G_n is called the *homogenous chaos space of order n* generated by X. We will also call $\mathcal{H}$ the *chaos space* generated by X.

For any $m, n \geq 0$, we define the *Wick product* $f_m(X) \diamond f_n(X)$ of $f_m(X)$ and $f_n(X)$, as $f_m(X) \diamond f_n(X) := f_{m+n}(X)$. Observe that if $N = k$ is finite, then $f_m(X) \diamond f_n(X) = 0$ for all m and n, such that $m + n \geq k$. It is not hard to see that $f_m(X) \diamond f_n(X)$ is in fact the projection of the point-wise product $f_m(X) f_n(X)$ on the space G_{m+n}. We extend now the Wick product by bilinearity, defining formally for all $\varphi = \sum_{n<N} c_n f_n \in \mathcal{H}$ and $\psi = \sum_{n<N} d_n f_n \in \mathcal{H}$,

$$\varphi \diamond \psi := \sum_{n<N} \left(\sum_{p+q=n} c_p d_q \right) f_n. \tag{2.2}$$

Since it is not guaranteed that $\sum_{n<N} \omega_n! |\sum_{p+q=n} c_p d_q|^2 < \infty$, $\varphi \diamond \psi$ may not belong to $\mathcal{H}$.

Definition 2.1. For any complex number c, define the *second quantization operator of* cI, where I denotes the identity operator of the one-dimensional Hilbert space $\mathbb{C}X$, spanned by X, as a densely defined operator on $\mathcal{H}$, defined by

$$\Gamma(cI) \left(\sum_{0 \leq n<N} d_n f_n(X) \right) := \sum_{0 \leq n<N} c^n d_n f_n(X), \tag{2.3}$$

where $d_n \in \mathbb{C}$ for all $0 \leq n < N$.

A random variable $\varphi := \sum_{0 \leq n<N} d_n f_n(X)$ belongs to the domain of $\Gamma(cI)$ if and only if $\sum_{0 \leq n<N} (1 + |c|^{2n}) |d_n|^2 \omega_n! < \infty$.

3. Wick-Hölder property

Definition 3.1. Let M and t be two fixed positive numbers. Let X be a random variable, having finite moments of all orders, and let $\mathcal{H}$ denote the chaos space generated by X. X is said to be *(M, t)-Wick-Hölderian*, if, for all positive numbers p and q, such that $(1/p) + (1/q) = 1/t$, and for all $\varphi(X) \in \mathcal{H}$ and $\psi(X) \in \mathcal{H}$, such that $\Gamma(\sqrt{p}I)\varphi(X) \in \mathcal{H}$ and $\Gamma(\sqrt{q}I)\psi(X) \in \mathcal{H}$, there exists $\varphi(X) \diamond \psi(X) \in \mathcal{H}$, and the following inequality holds:

$$E\left[\left|\varphi(X) \diamond \psi(X)\right|^2\right] \leq M E\left[\left|\Gamma(\sqrt{p}I)\varphi(X)\right|^2\right] E\left[\left|\Gamma(\sqrt{q}I)\psi(X)\right|^2\right]. \tag{3.1}$$

Since for any fix $\varphi(X) \in \mathcal{H}$ the function $u : [0, \infty) \to [0, \infty]$, $u(t) = E[|\Gamma(\sqrt{t}I)\varphi(X)|^2]$ is non-decreasing, if X is an (M, t)-Wick-Hölderian random variable, then X is also (M, t')-Wick-Hölderian for all $t' > t$. By taking $\varphi = \psi = 1$, we conclude from (3.1) that $M \geq 1$.

Definition 3.2. Let t be a fixed positive number. Let X be a random variable, having finite moments of all orders. X is said to be t-Wick-Hölderian if X is $(1,t)$-Wick-Hölderian.

Again, if X is a t-Wick-Hölderian random variable, then X is also t'-Wick-Hölderian for all $t' > t$.

Definition 3.3. Let X be a random variable, having finite moments of all orders. X is said to be *Wick-Hölderian* if there exists a positive number t, such that X is t-Wick-Hölderian.

Proposition 3.4. *If X is a random variable, having finite moments of all orders, then the following two conditions are equivalent:*

(1) X *is Wick-Hölderian;*

(2) *there exist two positive numbers M and t, such that X is (M,t)-Wick-Hölderian.*

Proof. (1) ⇒ (2) This implication is obvious.

(2) ⇒ (1) Let us assume that X is (M,t)-Wick-Hölderian for some $M \geq 1$ and $t > 0$. Let $s := \max(3, 3Mt)$.

Claim 1. X is s-Wick-Hölderian.

Indeed, let $p > 0$ and $q > 0$, such that $(1/p) + (1/q) = 1/s$. Let $\varphi(X) = \sum_{n<N} c_n f_n(X) \in \mathscr{A}$, and $\psi(X) = \sum_{n<N} d_n f_n(X) \in \mathscr{A}$, such that $\Gamma(\sqrt{p}I)\varphi(X) \in \mathscr{A}$ and $\Gamma(\sqrt{q}I)\psi(X) \in \mathscr{A}$, where $\{c_n\}_{n\geq 0} \subset \mathbb{C}$, $\{d_n\}_{n\geq 0} \subset \mathbb{C}$, and $\{f_n\}_{n\geq 0}$ represents the sequence of orthogonal polynomials, having a leading coefficient equal to 1, generated by X. Let $\|\cdot\|$ denote the L^2-norm. Let $g(X) := \varphi(X) - c_0 1$ and $h(X) := \psi(X) - d_0 1$. We have $\varphi(X) = c_0 1 + g(X)$ and $\psi(X) = d_0 1 + h(X)$. Since $g(X) \perp 1$, $h(X) \perp 1$, and $g(X) \diamond h(X) \perp 1$, where $\perp$ denotes the orthogonality relation, applying the Pythagorean theorem, we obtain

$$\begin{aligned}
\|\varphi(X) \diamond \psi(X)\|^2 &= \left\| \left[c_0 1 + g(X)\right] \diamond \left[d_0 1 + h(X)\right] \right\|^2 \\
&= \left\| c_0 d_0 1 + c_0 h(X) + d_0 g(X) + g(X) \diamond h(X) \right\|^2 \\
&= \left|c_0\right|^2 \left|d_0\right|^2 + \left\| c_0 h(X) + d_0 g(X) + g(X) \diamond h(X) \right\|^2 \\
&\leq \left|c_0\right|^2 \left|d_0\right|^2 + \left[\left|c_0\right| \left\|h(X)\right\| + \left|d_0\right| \left\|g(X)\right\| + \left\|g(X) \diamond h(X)\right\| \right]^2 \\
&\leq \left|c_0\right|^2 \left|d_0\right|^2 + \left[\left|c_0\right| \left\|h(X)\right\| + \left|d_0\right| \left\|g(X)\right\| + \left\|g(X) \diamond h(X)\right\| \right]^2 + 3\left\|g(X) \diamond h(X)\right\|^2.
\end{aligned} \tag{3.2}$$

Because $(1/p) + (1/q) = 1/s \leq 1/3$, we have $1/p < 1/3$ and $1/q < 1/3$. Thus, $p > 3$ and $q > 3$. However, $p > 3$ implies $3\|g(X)\|^2 \leq p\|g(X)\|^2 \leq \|\Gamma(\sqrt{p}I)g(X)\|^2$. Similarly, we have $3\|h(X)\|^2 \leq \|\Gamma(\sqrt{q}I)h(X)\|^2$. Since $1/[p/(3M)] + 1/[q/(3M)] = 3M/s \leq 1/t$ and X is an (M,t)-Wick-Hölderian random variable, we have

$$\|g(X) \diamond h(X)\| \leq M \left\| \Gamma\left(\sqrt{p/(3M)}I\right) g(X) \right\| \left\| \Gamma\left(\sqrt{q/(3M)}I\right) h(X) \right\|. \tag{3.3}$$

Thus, since $3 < 9$ and $M \geq 1$, we have

$$\begin{aligned}
\|\varphi(X) \diamond \psi(X)\|^2 &\leq |c_0|^2|d_0|^2 + 3|c_0|^2\|h(X)\|^2 + 3|d_0|^2\|g(X)\|^2 + 9\|g(X) \diamond h(X)\|^2 \\
&\leq |c_0|^2|d_0|^2 + |c_0|^2\|\Gamma(\sqrt{q}I)h(X)\|^2 + |d_0|^2\|\Gamma(\sqrt{p}I)g(X)\|^2 \\
&\quad + 9M^2\left\|\Gamma\left(\sqrt{p/(3M)I}\right)g(X)\right\|^2\left\|\Gamma\left(\sqrt{q/(3M)I}\right)h(X)\right\|^2 \\
&\leq |c_0|^2|d_0|^2 + |c_0|^2\|\Gamma(\sqrt{q}I)h(X)\|^2 + |d_0|^2\|\Gamma(\sqrt{p}I)g(X)\|^2 \\
&\quad + \left\|\Gamma\left(\sqrt{3M\cdot p/(3M)I}\right)g(X)\right\|^2\left\|\Gamma\left(\sqrt{3M\cdot q/(3M)I}\right)h(X)\right\|^2 \\
&= \left[|c_0|^2 1 + \|\Gamma(\sqrt{p}I)g\|^2\right]\left[|d_0|^2 1 + \|\Gamma(\sqrt{q}I)h\|^2\right] \\
&= \|\Gamma(\sqrt{p}I)\varphi\|^2\|\Gamma(\sqrt{q}I)\psi\|^2.
\end{aligned} \tag{3.4}$$

Hence, X is s-Wick-Hölderian. □

Definition 3.5. Let $\{X_i\}_{i\in I}$ be a family of random variables, having finite moments of all orders. The family $\{X_i\}_{i\in I}$ is said to be *uniformly Wick-Hölderian* if there exists a positive number t_0, such that for all $i \in I$, X_i is t_0-Wick-Hölderian.

It follows from the proof of the previous proposition that a family $\{X_i\}_{i\in I}$ is uniformly Wick-Hölderian if and only if there exists a pair (M_0, t_0) of positive numbers, such that for all $i \in I$, X_i is (M_0, t_0)-Wick-Hölderian.

From now on, to make the notation easier, we say that a random variable X is of class (M, t)-W-H, class t-W-H, or class W-H if X is (M, t)-Wick-Hölderian, t-Wick-Hölderian, or Wick-Hölderian, respectively. We also say that a uniformly Wick-Hölderian family $\{X_i\}_{i\in I}$, of random variables, is of class unif.-W-H.

4. A universal minimal condition

In this section we prove a very important condition about any two corresponding multipliers involved in a Wick product inequality.

Lemma 4.1. *Let X be a random variable, having finite moments of all orders, such that the support of X contains at least two distinct points (that means X is not almost surely constant). Let $f_0 = 1$ and f_1 be the first two orthogonal polynomials, with a leading coefficient equal to 1, generated by X. If p and q are two positive numbers, such that for all $\varphi(X), \psi(X) \in \mathbb{C}f_0(X) + \mathbb{C}f_1(X)$ (i.e., φ and ψ are polynomial functions of degree at most 1), the following inequality holds:*

$$E\left[|\varphi(X) \diamond \psi(X)|^2\right] \leq E\left[|\Gamma(\sqrt{p}I)\varphi(X)|^2\right]E\left[|\Gamma(\sqrt{q}I)\psi(X)|^2\right], \tag{4.1}$$

then one must have:

$$\frac{1}{p} + \frac{1}{q} \leq 1. \tag{4.2}$$

Proof. The fact that the support of X contains at least two points guarantees that $f_1(X) \neq 0$. Let f_2 be next orthogonal polynomial and let ω_1 and ω_2 be the first two principal Szegö-Jacobi parameters of X. We have $\omega_1 = \|f_1(X)\|^2 > 0$ and $\|f_2(X)\|^2 = \omega_2! = \omega_1\omega_2 \geq 0$ (it is possible that $f_2(X) = 0$, in which case $\omega_2 = 0$). As before, $\|\cdot\|$ denotes the L^2-norm. Let us apply inequality (4.1) to the random variables $\varphi(X) = 1 + cf_1(X)$ and $\psi = 1 + c\lambda f_1(X)$, where c and λ are arbitrary real numbers, such that $c \neq 0$. Since

$$\begin{aligned}\varphi(X) \diamond \psi(X) &= \left[1 + cf_1(X)\right] \diamond \left[1 + c\lambda f_1(X)\right] \\ &= 1 + c(1+\lambda) f_1(X) + c^2 \lambda f_2(X),\end{aligned} \tag{4.3}$$

the inequality,

$$E\left[\left|\varphi(X) \diamond \psi(X)\right|^2\right] \leq E\left[\left|\Gamma(\sqrt{p}I)\varphi(X)\right|^2\right] E\left[\left|\Gamma(\sqrt{q}I)\psi(X)\right|^2\right], \tag{4.4}$$

means that

$$1 + c^2(1+\lambda)^2\omega_1 + c^4\lambda^2\omega_1\omega_2 \leq \left[1 + pc^2\omega_1\right]\left[1 + qc^2\lambda^2\omega_1\right] \tag{4.5}$$

for all $c \neq 0$ and all $\lambda \in \mathbb{R}$. Subtracting first 1 from both sides of this inequality, and then dividing both sides of the resulting inequality, by the strictly positive number $c^2\omega_1$, we conclude that the inequality,

$$(1+\lambda)^2 + c^2\lambda^2\omega_2 \leq p + q\lambda^2 + pqc^2\lambda^2\omega_1, \tag{4.6}$$

holds for all $c \neq 0$ and $\lambda \in \mathbb{R}$. Passing to the limit, as $c \to 0$, in the last inequality, we obtain

$$(1+\lambda)^2 \leq p + q\lambda^2 \tag{4.7}$$

for all real numbers λ. Moving all terms to the right, we conclude that the quadratic trinomial

$$(q-1)\lambda^2 - 2\lambda + (p-1), \tag{4.8}$$

must be nonnegative for all real values of λ. Therefore, q must be greater than one, and the discriminant $\Delta = 4 - 4(p-1)(q-1)$ must be less than or equal to zero. This is equivalent to $1 \leq (p-1)(q-1)$, which in turn means $p + q \leq pq$. Dividing both sides of this inequality by the positive number pq, we conclude that

$$\frac{1}{p} + \frac{1}{q} \leq 1. \tag{4.9}$$

□

Corollary 4.2. *If X is a nonconstant random variable of class t-W-H, then t is at least* 1.

We will call the condition $(1/p) + (1/q) \leq 1$ the *universal minimal* (*unimprovable*) *condition*. We will also say that a nonconstant random variable X of class 1-W-S satisfies *the best Wick-Hölder inequality*. We know from [1] that every Gaussian random variable satisfies the best Wick-Hölder inequality. However, there are many other random variables of class 1-W-H, and in the next section we will give some sufficient conditions that guarantee this property.

Let X be a random variable, having finite moments of any order, and let $\{f_n\}_{n\geq 0}$ and $\mathscr{H}$ be the sequence of orthogonal polynomials and chaos space generated by X, respectively. If $\alpha := \{\alpha_n\}_{n\geq 0}$ is a sequence of complex numbers, then we denote by $\mathcal{M}_\alpha$ the densely defined linear operator on $\mathscr{H}$ that maps $f_n(X)\to\alpha_n f_n(X)$ for all $n \geq 0$. We call the sequence α the *multiplier* of the operator $\mathcal{M}_\alpha$. It is clear that for all $n \geq 0$, $\mathcal{M}_\alpha G_n \subset G_n$. The converse is also true, namely, if T is a linear operator defined on the space of all polynomial functions of X that leaves all homogenous chaos spaces generated by X invariant, then there exists a sequence of complex numbers α, such that $T = \mathcal{M}_\alpha$. If $\alpha_0 = 1$, then $\mathcal{M}_\alpha 1 = 1$, and we say that the operator $\mathcal{M}_\alpha$ (or the multiplier α) *respects* the vacuum space G_0. Doing the same proof as in Lemma 4.1, we can prove the following result.

Lemma 4.3. *Let X be a random variable, having finite moments of all orders, such that the support of X contains at least $n+1$ distinct points, where $n \geq 1$ is fixed. Let $f_0 = 1$, and f_n be the orthogonal polynomial of degree n, generated by X. If $c = \{c_n\}_{n\geq 0}$ and $d = \{d_n\}_{n\geq 0}$ are two sequences of complex numbers, such that $c_0 = d_0 = 1$, and for all $\varphi(X), \psi(X) \in \mathbb{C}f_0(X) + \mathbb{C}f_n(X)$, the following inequality holds:*

$$E\left[\left|\varphi(X)\diamond\psi(X)\right|^2\right] \leq E\left[\left|\mathcal{M}_c\varphi(X)\right|^2\right]E\left[\left|\mathcal{M}_d\psi(X)\right|^2\right], \tag{4.10}$$

then one must have

$$\frac{1}{\left|c_n\right|^2} + \frac{1}{\left|d_n\right|^2} \leq 1. \tag{4.11}$$

We call inequality (4.11) the *generalized universal minimal* (*unimprovable*) *condition*. Even though we will not be using this generalized condition in this paper, we would like to reformulate it in words, so that some other mathematicians might use it in the future.

If the norm of the Wick product of $\varphi(X)$ and $\psi(X)$ is always bounded above by the product of the norms of $\mathcal{M}_c\varphi(X)$ and $\mathcal{M}_d\psi(X)$, where c and d are two multipliers respecting the vacuum space, then the sum of the reciprocals of the square of the modulus of any two corresponding terms of the sequences, c and d, must be at most 1.

We extend now the universal minimal condition from the L^2 case to the L^r case for $r \geq 2$. If we pay attention to the proof of the universal minimal condition, when dividing by c^2 and then passing to the limit as $c\to 0$, then we can observe that, in fact, we were differentiating an inequality twice with respect to c. Therefore, we will attack the L^r case in the same manner, based on two very simple observations.

Observation 1. If f and g are two functions from $\mathbb{R}$ to $\mathbb{R}$ that are twice differentiable, such that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, and there exists an $x_0 \in \mathbb{R}$, such that $f(x_0) = g(x_0)$, then $f'(x_0) = g'(x_0)$, and $f''(x_0) \leq g''(x_0)$.

Proof. This can be seen intuitively by drawing a picture for the graphs of f and g, or of $f - g$, and mathematically by using the formula

$$f''(x_0) = \lim_{h\to 0}\frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2}, \tag{4.12}$$

and a similar relation for $g''(x_0)$. □

We can formulate this observation in the following way: *we can differentiate twice an inequality between two functions at the points where the functions are equal (touching each other), and the inequality is preserved.*

Observation 2. If r is a real number, then the function $h(x) = |x|^r$ is differentiable if and only if $r > 1$, in which case $h'(x) = rx|x|^{r-2}$.

We are generalizing now Lemma 4.1 to powers $r > 2$.

Lemma 4.4. *Let X be a random variable, having finite moments of all orders, such that the support of X contains at least two distinct points. Let $f_0 = 1$ and f_1 be the first two orthogonal polynomials, with a leading coefficient equal to 1, generated by X. Let $r \geq 2$ be fixed. If p and q are two positive numbers, such that for all $\varphi(X)$, $\psi(X) \in \mathbb{C}f_0(X) + \mathbb{C}f_1(X)$, the following inequality holds:*

$$E\left[\left|\varphi(X) \diamond \psi(X)\right|^r\right] \leq E\left[\left|\Gamma(\sqrt{p}I)\varphi(X)\right|^r\right] E\left[\left|\Gamma(\sqrt{q}I)\psi(X)\right|^r\right], \tag{4.13}$$

then one must have

$$\frac{1}{p} + \frac{1}{q} \leq 1. \tag{4.14}$$

Proof. Let $\varphi_c(X) := 1 + cf_1(X)$ and $\psi_{c,\lambda}(X) := 1 + c\lambda f_1(X)$, where c and λ are arbitrary real numbers. Let us consider the functions $g_\lambda : \mathbb{R}\to\mathbb{R}$, $g_\lambda(c) := E[|\varphi_c(X) \diamond \psi_{c,\lambda}(X)|^r]$ and $h_\lambda : \mathbb{R}\to\mathbb{R}$, $h_\lambda(c) := E[|\Gamma(\sqrt{p}I)\varphi_c(X)|^r]E[|\Gamma(\sqrt{q}I)\psi_{c,\lambda}(X)|^r]$. Since $g_\lambda(c) \leq h_\lambda(c)$ for all $c \in \mathbb{R}$, and $g_\lambda(0) = h_\lambda(0)$, according to Observation 1, we have

$$g_\lambda''(0) \leq h_\lambda''(0). \tag{4.15}$$

Because X has finite moments of all orders, and $r \geq 2$, a simple application of dominated convergence theorem shows that we can put the derivatives inside the expectations. Thus, applying Observation 2 twice and the product rule of differentiation, we get that for all $c \in \mathbb{R}$,

$$\begin{aligned}
g_\lambda''(c) &= r(r-1)E\left[\left|1 + c(1+\lambda)f_1 + c^2\lambda f_2\right|^{r-2}\left((1+\lambda)f_1 + 2c\lambda f_2\right)^2\right] \\
&\quad + 2r\lambda E\left[\left|1 + c(1+\lambda)f_1 + c^2\lambda f_2\right|^{r-2}\left(1 + c(1+\lambda)f_1 + c^2\lambda f_2\right)f_2\right], \\
h_\lambda''(c) &= r(r-1)pE\left[\left|1 + c\sqrt{p}f_1\right|^{r-2}f_1^2\right]E\left[\left|1 + c\lambda\sqrt{q}f_1\right|^r\right] \\
&\quad + 2r^2\lambda\sqrt{pq}E\left[\left|1 + c\sqrt{p}f_1\right|^{r-2}\left(1 + c\sqrt{p}f_1\right)f_1\right] \\
&\quad \times E\left[\left|1 + c\lambda\sqrt{q}f_1\right|^{r-2}\left(1 + c\lambda\sqrt{q}f_1\right)f_1\right] \\
&\quad + r(r-1)q\lambda^2E\left[\left|1 + c\lambda\sqrt{q}f_1\right|^{r-2}f_1^2\right]E\left[\left|1 + c\sqrt{p}f_1\right|^r\right].
\end{aligned} \tag{4.16}$$

Setting $c = 0$, we get now

$$\begin{aligned}
g_\lambda''(0) &= r(r-1)(1+\lambda)^2E\left[f_1^2\right] + 2r\lambda E\left[f_2\right], \\
h_\lambda''(0) &= r(r-1)pE\left[f_1^2\right] + 2r^2\lambda\sqrt{pq}E\left[f_1\right]E\left[f_1\right] + r(r-1)q\lambda^2E\left[f_1^2\right].
\end{aligned} \tag{4.17}$$

Since $f_2(X) \perp f_0(X)$, $f_1(X) \perp f_0(X)$, and $f_0(X) = 1$, we have $E[f_2(X)] = 0$ and $E[f_1(X)] = 0$. Thus, the $g''_\lambda(0) \leq h''_\lambda(0)$ becomes, after dividing both sides by the positive number $r(r-1)E[f_1^2]$

$$(1+\lambda)^2 \leq p + q\lambda^2 \tag{4.18}$$

for all $\lambda \in \mathbb{R}$. Now moving all terms to the right and writing the condition that the discriminant of the quadratic function in λ be nonpositive, we obtain as before that

$$\frac{1}{p} + \frac{1}{q} \leq 1. \tag{4.19}$$

□

Our technique for proving the universal minimal condition for $r \geq 2$ is similar to the proof of Theorem 3 from [4].

Finally, we can also see that for multipliers, the inequality

$$E\left[\left|\varphi(X) \diamond \psi(X)\right|^r\right] \leq E\left[\left|\mathcal{M}_c\varphi(X)\right|^r\right] E\left[\left|\mathcal{M}_d\psi(X)\right|^r\right], \tag{4.20}$$

for a fix $r \geq 2$, also implies the condition that

$$\frac{1}{|c_n|^2} + \frac{1}{|d_n|^2} \leq 1 \tag{4.21}$$

for all $n \geq 1$.

5. Random variables with (M, α)-subadditive omega parameters

We present first the following lemma.

Lemma 5.1. *Let $\{\omega_n\}_{n\geq 1}$ be a sequence of positive numbers, such that there exist a number t, greater than or equal to 1, and a sequence $\{\alpha_n\}_{n\geq 2}$, of nonnegative numbers, such that the series $\sum_{n=2}^{\infty}\alpha_n$ is convergent, and for all n and k natural numbers, with $n > k$, it holds that*

$$\omega_n \leq t(1+\alpha_n)(\omega_k + \omega_{n-k}). \tag{5.1}$$

Then, for all nonnegative numbers $k \geq r \geq 0$, it holds that

$$\binom{\omega_k}{\omega_r} \leq Mt^k \binom{k}{r}, \tag{5.2}$$

where $M := \max\{1, (1/t)\prod_{n=2}^{\infty}(1+\alpha_n)\}$, $\binom{k}{r} := (1\cdot 2\cdots k)/[(1\cdot 2\cdots r)(1\cdot 2\cdots (k-r))]$, and $\binom{\omega_k}{\omega_r} := (\omega_1\omega_2\cdots\omega_k)/[(\omega_1\omega_2\cdots\omega_r)(\omega_1\omega_2\cdots\omega_{k-r})]$ for $0 < r < k$. If $r = 0$ or $r = k$, then $\binom{k}{r} := 1$ and $\binom{\omega_k}{\omega_r} := 1$.

Proof. Since $\alpha_n \geq 0$ for all $n \geq 2$ and $\sum_{n=2}^{\infty}\alpha_n < \infty$, we conclude that the product $\prod_{n=2}^{\infty}(1+\alpha_n)$ is convergent. For $k = 0$, the inequality (5.2) is obvious since $\binom{\omega_0}{\omega_0} = \binom{0}{0} = 1$. We prove now by induction on k that for all $k \geq 1$, we have

$$\binom{\omega_k}{\omega_r} \leq \left[\prod_{i=2}^{k}(1+\alpha_i)\right] t^{k-1} \binom{k}{r}, \tag{5.3}$$

for all $r \in \{0, 1, \ldots, k\}$. For $k = 1$, we have nothing to prove, since $\binom{\omega_k}{\omega_r} = \binom{k}{r} = 1$, and $\prod_{i=2}^{k}(1+\alpha_i)$ is defined to be 1 in this case.

Let us suppose now that (5.3) holds for $k = n$ and all $r \in \{0, 1, \ldots, n\}$ and prove that it continues to hold for $k = n+1$ and all $r \in \{0, 1, \ldots, n+1\}$. Indeed, we may assume that $1 \leq r \leq n$ since for $r = 0$ and $r = n+1$, (5.3) is trivial because $t \geq 1$. It follows from (5.2) that

$$\begin{aligned}
\binom{\omega_{n+1}}{\omega_r} &= \frac{\omega_{n+1}\omega_n!}{\omega_r\omega_{r-1}!\omega_{n+1-r}\omega_{n-r}!} \\
&\leq \frac{t(1+\alpha_{n+1})\left[\omega_r + \omega_{n+1-r}\right]\omega_n!}{\omega_r\omega_{r-1}!\omega_{n+1-r}\omega_{n-r}!} \\
&= t(1+\alpha_{n+1})\left[\frac{\omega_n!}{\omega_{r-1}!\omega_{n+1-r}!} + \frac{\omega_n!}{\omega_r!\omega_{n-r}!}\right] \\
&= t(1+\alpha_{n+1})\left[\binom{\omega_n}{\omega_{r-1}} + \binom{\omega_n}{\omega_r}\right].
\end{aligned} \tag{5.4}$$

Using now this inequality, the induction hypothesis, and the classic Pascal identity $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, we get

$$\begin{aligned}
\binom{\omega_{n+1}}{\omega_r} &\leq \left[\prod_{i=2}^{n+1}(1+\alpha_i)\right] t^n \left[\binom{n}{r-1} + \binom{n}{r}\right] \\
&= \left[\prod_{i=2}^{n+1}(1+\alpha_i)\right] t^n \binom{n+1}{r}.
\end{aligned} \tag{5.5}$$

Hence, for all $k \geq 1$, we have

$$\begin{aligned}
\binom{\omega_k}{\omega_r} &\leq \left[\prod_{i=2}^{k}(1+\alpha_i)\right] t^{k-1} \binom{k}{r} \\
&\leq \left[\prod_{i=2}^{\infty}(1+\alpha_i)\right] t^{k-1} \binom{k}{r} \\
&= \frac{1}{t}\left[\prod_{i=2}^{\infty}(1+\alpha_i)\right] t^{k} \binom{k}{r}.
\end{aligned} \tag{5.6}$$

In order to make this inequality also true, for $k = 0$, we have to replace $(1/t)[\prod_{i=2}^{\infty}(1+\alpha_i)]$ by $\max\{1, (1/t)[\prod_{i=2}^{\infty}(1+\alpha_i)]\}$. □

We introduce now three definitions.

Definition 5.2. Given a number t, greater than or equal to 1, and a sequence of nonnegative numbers $\alpha = \{\alpha_n\}_{n\geq 2}$, such that the series $\sum_{n=2}^{\infty}\alpha_n$ is convergent, then a sequence of nonnegative numbers $\{\omega_n\}_{n\geq 1}$ is said to be *(t, α)-subadditive* if the following inequality,

$$\omega_n \leq t(1+\alpha_n)(\omega_k + \omega_{n-k}), \tag{5.7}$$

holds for all natural numbers n and k, such that $n > k$.

Definition 5.3. Given a number t, greater than or equal to 1, then a sequence of nonnegative numbers $\{\omega_n\}_{n\geq 1}$ is said to be *t-subadditive*, if it is $(t, 0)$-subadditive, where $0 = \{0, 0, \ldots\}$ denotes the identical zero sequence, that is, if the following inequality,

$$\omega_{m+n} \leq t(\omega_m + \omega_n), \tag{5.8}$$

holds for all natural numbers m and n. In particular, if $t = 1$, then a 1-subadditive sequence is called simply subadditive.

Of course (t, α)-subadditivity implies t'-subadditivity for $t' := t(1 + \sup\{\alpha_n \mid n \geq 2\})$.

Definition 5.4. If t is a number, greater than or equal to 1, and $\{\omega_n\}_{n\geq 1}$ is a sequence of nonnegative numbers, then the sequence $\{\omega_n\}_{n\geq 1}$ is said to be *exp-t-subadditive* if

$$\omega_{m+n}^{1/t} \leq \omega_m^{1/t} + \omega_n^{1/t} \tag{5.9}$$

for all m and n positive integers.

Observation 3. An exp-t-subadditive sequence $\{\omega_n\}_{n\geq 1}$ of nonnegative numbers is also exp-s-subadditive for all $s > t$.

Proof. Applying the inequality $(a+b)^r \leq a^r + b^r$ for all $a \geq 0$, $b \geq 0$, and $0 < r \leq 1$, we get

$$\omega_{m+n}^{1/s} = \left[\omega_{m+n}^{1/t}\right]^{t/s} \leq \left[\omega_m^{1/t} + \omega_n^{1/t}\right]^{t/s} \leq \left[\omega_m^{1/t}\right]^{t/s} + \left[\omega_n^{1/t}\right]^{t/s} = \omega_m^{1/s} + \omega_n^{1/s} \tag{5.10}$$

for all m and n positive integers. □

Observation 4. For all $t \geq 1$, any exp-t-subadditive sequence $\{\omega_n\}_{n\geq 1}$ of nonnegative numbers is also 2^{t-1}-subadditive.

Proof. Since $t \geq 1$, the function $f : [0, \infty) \to \mathbb{R}$, $f(x) = x^t$ is convex. Thus, for all $m, n \geq 1$, we have

$$\begin{aligned} \omega_{m+n} &= \left[\omega_{m+n}^{1/t}\right]^t \leq \left[\omega_m^{1/t} + \omega_n^{1/t}\right]^t = 2^t \left[\frac{1}{2}\omega_m^{1/t} + \frac{1}{2}\omega_n^{1/t}\right]^t \\ &\leq 2^t \left[\frac{1}{2}\left(\omega_m^{1/t}\right)^t + \frac{1}{2}\left(\omega_n^{1/t}\right)^t\right] = 2^{t-1}(\omega_m + \omega_n). \end{aligned} \tag{5.11}$$

□

We present now the main result of this paper.

Theorem 5.5. *Let X be a random variable, having finite moments of any order, and let $\{\omega_n\}_{n\geq 1}$ be the principal Szegö-Jacobi parameters of X. Let M and t be two numbers that are greater than or equal to 1. If the sequence $\{\omega_n\}_{n\geq 1}$ satisfies the condition*

$$\binom{\omega_k}{\omega_r} \leq M t^k \binom{k}{r} \tag{5.12}$$

for all $0 \leq r \leq k < N$, where N denotes the dimension of the chaos space generated by X, then X is of class (M, t)-W-H.

Proof. Let $p > 0$ and $q > 0$, such that $(1/p) + (1/q) = (1/t)$. Let $\mathcal{H}$ be the chaos space generated by X. Let $\varphi \in \mathcal{H}$ and $\psi \in \mathcal{H}$, such that $\Gamma(\sqrt{p}I)\varphi \in \mathcal{H}$ and $\Gamma(\sqrt{q}I)\psi \in \mathcal{H}$. Let $\{f_n\}_{n<N}$ be the orthogonal sequence of polynomials, with a leading coefficient of 1, generated by X. We distinguish between two cases.

Case 1. If $N = \infty$, then there exist two unique sequences of complex numbers $\{c_n\}_{n\geq 0}$ and $\{d_n\}_{n\geq 0}$, such that

$$\begin{aligned} \varphi &= \sum_{n=0}^{\infty} c_n f_n(X), \\ \psi &= \sum_{n=0}^{\infty} d_n f_n(X). \end{aligned} \tag{5.13}$$

We have

$$\varphi \diamond \psi = \sum_{k=0}^{\infty}\left(\sum_{u+v=k} c_u d_v\right) f_k(X). \tag{5.14}$$

Using the triangle inequality, we obtain

$$\begin{aligned} E\left[|\varphi \diamond \psi|^2\right] &= \sum_{k=0}^{\infty}\left|\sum_{u+v=k} c_u d_v\right|^2 \omega_k! \\ &\leq \sum_{k=0}^{\infty}\left[\sum_{u+v=k} |c_u d_v|\right]^2 \omega_k! \\ &= \sum_{k=0}^{\infty} \omega_k! \left[\sum_{u+v=k} \frac{1}{\sqrt{\omega_u!\omega_v! p^u q^v}} \sqrt{p^u \omega_u!}|c_u| \sqrt{q^v \omega_v!}|d_v|\right]^2 \end{aligned} \tag{5.15}$$

From the Cauchy-Bunyakovsky-Schwarz inequality, inequality (5.2), and Newton's binomial formula, we obtain

$$\begin{aligned}
E\left[|\varphi \diamond \psi|^2\right] &\leq \sum_{k=0}^{\infty}\omega_k!\left[\sum_{u+v=k}\frac{1}{\omega_u!\omega_v!p^u q^v}\right] \times \left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&= \sum_{k=0}^{\infty}\left[\sum_{u+v=k}\frac{\omega_k!}{\omega_u!\omega_v!}\frac{1}{p^u}\frac{1}{q^v}\right] \times \left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&= \sum_{k=0}^{\infty}\left[\sum_{u+v=k}\binom{\omega_k}{\omega_u}\frac{1}{p^u}\frac{1}{q^v}\right] \times \left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&\leq \sum_{k=0}^{\infty}\left[\sum_{u+v=k}Mt^k\binom{k}{u}\frac{1}{p^u}\frac{1}{q^v}\right] \times \left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&= M\sum_{k=0}^{\infty}t^k\left[\sum_{u+v=k}\binom{k}{u}\frac{1}{p^u}\frac{1}{q^v}\right] \times \left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&= M\sum_{k=0}^{\infty}t^k\left[\frac{1}{p}+\frac{1}{q}\right]^k \times \left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&= M\sum_{k=0}^{\infty}t^k\left[\frac{1}{t}\right]^k\left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&= M\sum_{k=0}^{\infty}1\left[\sum_{u+v=k}p^u\omega_u!|c_u|^2 q^v\omega_v!|d_v|^2\right] \\
&= M\sum_{u=0}^{\infty}p^u\omega_u!|c_u|^2\sum_{v=0}^{\infty}q^v\omega_v!|d_v|^2 \\
&= ME\left[|\Gamma(\sqrt{p}I)f|^2\right]E\left[|\Gamma(\sqrt{q}I)g|^2\right].
\end{aligned} \tag{5.16}$$

Case 2. If $N < \infty$, then all the inequalities used in Case 1, remain the same or become strict inequalities, due to the fact that after a while, all ω's become zero. Thus, for example, some complete sums like $\sum_{u+v=k}\binom{k}{u}(1/p^u)(1/q^v)$ from Case 1 will become incomplete (that means some of the pais (u, v), with $u + v = k$, will be missing) in Case 2,and therefore instead of $\sum_{u+v=k}\binom{k}{u}(1/p^u)(1/q^v) = 1/t^k$, we will have $\sum_{u+v=k}\binom{k}{u}(1/p^u)(1/q^v) < 1/t^k$. Therefore, all the inequalities from Case 1 will also remain true in Case 2. Hence, X is of class (M, t)-W-H in this case too. □

Corollary 5.6. *Let t be a number greater than or equal to 1, and $\alpha = \{\alpha_n\}_{n\geq 2}$ a sequence of nonnegative numbers producing a convergent series $\sum_{n=2}^{\infty}\alpha_n$. Let X be a random variable, having finite moments of any order, and let $\{\omega_n\}_{n\geq 1}$ be the principal Szegö-Jacobi parameters of X. If the sequence $\{\omega_n\}_{n\geq 1}$ is (t, α)-subadditive, then X is of class (M, t)-W-H, where $M := \max\{1, (1/t)\prod_{n=2}^{\infty}(1 + \alpha_n)\}$.*

Corollary 5.7. *If X is a random variable, having finite moments of all orders, such that its principal Szegö-Jacobi sequence* $\{\omega_n\}_{n\geq 1}$ *is t-subadditive, then X is of class t-W-H.*

Corollary 5.8. *If X is a random variable, having finite moments of all orders, such that its principal Szegö-Jacobi sequence* $\{\omega_n\}_{n\geq 1}$ *is exp-t-subadditive, then X is of class* 2^{t-1}*-W-H. In particular, by taking* $p = q = \lambda = 2^t$*, it is concluded that if* $\mathcal{H}$ *denotes the chaos space generated by X, and if* $\varphi(X), \psi(X) \in \mathcal{H}$*, such that* $\Gamma(2^{t/2}I)\varphi(X) \in \mathcal{H}$ *and* $\Gamma(2^{t/2}I)\psi(X) \in \mathcal{H}$*, then* $\varphi(X) \diamond \psi(X) \in \mathcal{H}$ *and the following inequality holds:*

$$E\left[\left|\varphi(X) \diamond \psi(X)\right|^2\right] \leq E\left[\left|\Gamma(2^{t/2}I)\varphi(X)\right|^2\right] E\left[\left|\Gamma(2^{t/2}I)\psi(X)\right|^2\right]. \tag{5.17}$$

Observation 5. If $p = q = 2t$, then to prove that the inequality,

$$E\left[\left|\varphi(X) \diamond \psi(X)\right|^2\right] \leq E\left[\left|\Gamma(\sqrt{2t}I)\varphi(X)\right|^2\right] E\left[\left|\Gamma(\sqrt{2t}I)\psi(X)\right|^2\right], \tag{5.18}$$

holds whenever both expectations from the right-hand side are finite, we do not need the condition that each ω-binomial coefficient $\binom{\omega_k}{\omega_r}$ is less than or equal to $Mt^k\binom{k}{r}$, but it is enough to assume that for each $k \geq 0$, the sum of all binomial coefficients, having ω_k on the top, is less than or equal to Mt^k times the sum of all classic binomial coefficients having k on the top, that means

$$\sum_{r=0}^{k} \binom{\omega_k}{\omega_r} \leq Mt^k \cdot 2^k \tag{5.19}$$

for all $k \geq 0$.

Proof. The proof of this observation follows line by line the proof of Theorem 5.5, and it uses the fact that for all pairs (u, v) such that $u + v$ is equal to a fixed number k since $p = q = 2t$, the product $p^u q^v$ is always the same (independent of the pair) and equal to $(2t)^k$. □

The following proposition will be useful in showing later that $\lambda = 2^t$ is optimal for some particular random variables X.

Proposition 5.9. *Let X be a random variable having finite moments of all orders. We assume that the probability distribution of X has an infinite support. Let* $\mathcal{H}$ *be the chaos space generated by X and* $\{\omega_n\}_{n\geq 1}$ *the principal Szegö-Jacobi sequence of X. We define* $S_n := \sum_{k=0}^{n}\binom{\omega_n}{\omega_k}$ *for all* $n \geq 0$*, and* $\lambda_0 := \limsup_{n\to\infty} S_n^{1/n}$*. Then,*

(1) *if there exists a positive number* λ *such that the inequality,*

$$E\left[\left|\varphi(X) \diamond \psi(X)\right|^2\right] \leq E\left[\left|\Gamma(\sqrt{\lambda}I)\varphi(X)\right|^2\right] E\left[\left|\Gamma(\sqrt{\lambda}I)\psi(X)\right|^2\right], \tag{5.20}$$

holds for all $\varphi \in \mathcal{H}$ *and* $\psi \in \mathcal{H}$*, such that* $\Gamma(\sqrt{\lambda}I)\varphi \in \mathcal{H}$ *and* $\Gamma(\sqrt{\lambda}I)\psi \in \mathcal{H}$*, then* $\lambda \geq \lambda_0$*;*

(2) *if* λ_0 *satisfies* (5.20)*, then* λ_0 *is optimal (i.e., the smallest among all positive* λ*'s satisfying this inequality);*

(3) *if* $S_n \leq \lambda_0^n$ *for all* $n \geq 0$*, then* λ_0 *is optimal.*

Proof. (1) Let $\lambda > 0$ such that (5.20) holds whenever both expectations from its right-hand side are finite. For every $0 \leq r \leq k$, let us choose $\varphi := f_r(X)$ and $\psi := f_{k-r}(X)$. The inequality $E[|\varphi \diamond \psi|^2] \leq E[|\Gamma(\sqrt{\lambda}I)\varphi|^2]E[|\Gamma(\sqrt{\lambda}I)\psi|^2]$ is equivalent to $\omega_k! \leq \lambda^r\omega_r!\lambda^{k-r}\omega_{k-r}!$. This implies that $\binom{\omega_k}{\omega_r} \leq \lambda^k$. Summing up from $r = 0$ to $r = k$, we obtain $S_k \leq (k+1)\lambda^k$ for all $k \geq 0$. Taking the kth root in both sides of this inequality and then passing to the superior limit as $k\to\infty$, we get $\limsup_{k\to\infty}\sqrt[k]{S} \leq \lambda$. Thus, $\lambda_0 \leq \lambda$.

(2) It follows from 1.

(3) It follows from letting $M := 1$ and $t := \lambda_0/2$ in Observation 5. □

6. Examples

Example 6.1. Let $s \geq 1$ be a fixed real number and let X be a random variable, having finite moments of all orders, and the principal Szegö-Jacobi parameters $\omega_n = n^s > 0$ for all $n \geq 1$. It is clear that $\{\omega_n\}_{n\geq 1}$ is an exp-s-subadditive (in fact exp-s-additive) sequence. Therefore, $\{\omega_n\}_{n\geq 1}$ is also 2^{s-1}-subadditive. Thus, X is of class 2^{s-1}-W-H. Moreover, the following lemma holds.

Lemma 6.2. *For all $s \geq 1$, it holds that*

$$\left[\sum_{k=0}^{n}\binom{n}{k}^s\right]^{1/n} \leq 2^s, \tag{6.1}$$

$$\lim_{n\to\infty}\left[\sum_{k=0}^{n}\binom{n}{k}^s\right]^{1/n} = 2^s. \tag{6.2}$$

Formula (6.2) *holds even for* $0 < s < 1$.

Proof. If $s \geq 1$, then we have

$$\left[\sum_{k=0}^{n}\binom{n}{k}^s\right]^{1/n} \leq \left\{\left[\sum_{k=0}^{n}\binom{n}{k}\right]^s\right\}^{1/n} = [2^n]^{s/n} = 2^s. \tag{6.3}$$

Since $s \geq 1$, the function $h : (0,\infty)\to\mathbb{R}$, $h(x) = x^s$ is convex, and thus

$$\left[\sum_{k=0}^{n}\binom{n}{k}^s\right]^{1/n} = n^{1/n}\left[\sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}^s\right]^{1/n} \geq n^{1/n}\left\{\left[\sum_{k=0}^{n}\frac{1}{n}\binom{n}{k}\right]^s\right\}^{1/n} = n^{1/n}\left[\frac{2^n}{n}\right]^{s/n} = \frac{1}{\left(\sqrt[n]{n}\right)^{s-1}}2^s. \tag{6.4}$$

Hence,

$$\frac{1}{\left(\sqrt[n]{n}\right)^{s-1}}2^s \leq \left[\sum_{k=0}^{n}\binom{n}{k}^s\right]^{1/n} \leq 2^s. \tag{6.5}$$

Since $\sqrt[n]{n}\to 1$, as $n\to\infty$, we conclude that $\lim_{n\to\infty}\left[\sum_{k=0}^{n}\binom{n}{k}^s\right]^{1/n}$ exists and is equal to 2^s.

If $0 < s < 1$, then in a similar way, we can show that

$$\frac{1}{\left(\sqrt[n]{n}\right)^{s-1}} 2^s \geq \left[\sum_{k=0}^{n} \binom{n}{k}^s\right]^{1/n} \geq 2^s, \tag{6.6}$$

and it follows again that $\lim_{n\to\infty}\left[\sum_{k=0}^{n}\binom{n}{k}^s\right]^{1/n} = 2^s$. □

From formula (6.2), inequality (6.1), Corollary 5.8, and Proposition 5.9, we obtain the following proposition.

Proposition 6.3. *Let X be a random variable, having finite moments of any order whose principal Szegö-Jacobi parameters are $\omega_n = n^s$ for all $n \geq 1$, where s is a fixed real number, such that $s \geq 1$. Let $\mathcal{H}$ be the chaos space generated by X.*

(1) *If p and q are positive numbers, such that $(1/p) + (1/q) \leq (1/2^{s-1})$, then for all $\varphi \in \mathcal{H}$ and $\psi \in \mathcal{H}$, such that $\Gamma(\sqrt{p}I)\varphi \in \mathcal{H}$ and $\Gamma(\sqrt{q}I)\psi \in \mathcal{H}$, there exists $f(X) \diamond g(X) \in \mathcal{H}$ and the following inequality holds:*

$$E\left[|\varphi \diamond \psi|^2\right] \leq E\left[|\Gamma(\sqrt{p}I)\varphi|^2\right] E\left[|\Gamma(\sqrt{q}I)\psi|^2\right]. \tag{6.7}$$

(2) $\lambda_0 := 2^s$ *is the smallest among all positive numbers λ, for which the inequality,*

$$E\left[|\varphi \diamond \psi|^2\right] \leq E\left[|\Gamma(\sqrt{\lambda}I)\varphi|^2\right] E\left[|\Gamma(\sqrt{\lambda}I)\psi|^2\right], \tag{6.8}$$

holds for all $\varphi \in \mathcal{H}$ and $\psi \in \mathcal{H}$, such that $\Gamma(\sqrt{\lambda}I)\varphi \in \mathcal{H}$ and $\Gamma(\sqrt{\lambda}I)\psi \in \mathcal{H}$.

Example 6.4. This is a modification of the previous example. Let s be a fixed real number, such that $s > 2$. Let X_r be a random variable whose principal Szegö-Jacobi parameters are $\omega_n = n^s + rn$ for all $n \geq 0$, where $r > -1$. The condition $r > -1$ follows from the inequality $\omega_1 > 0$, and ensures that $\omega_n > 0$ for all $n \geq 2$. The reason why we have chosen $s > 2$ will become more transparent later.

Claim 1. If $r \geq 0$, then the sequence $\{\omega_n\}_{n\geq 1}$ is 2^{s-1}-subadditive.

Indeed, for all m and n natural numbers, we have

$$\begin{gathered}(m+n)^s \leq 2^{s-1}(m^s + n^s), \\ m + n < 2^{s-1}(m+n).\end{gathered} \tag{6.9}$$

Multiplying the second inequality by r and adding the resulting inequality to the first one, we get

$$\omega_{m+n} \leq 2^{s-1}(\omega_m + \omega_n). \tag{6.10}$$

Claim 2. If $-1 < r < 0$, then the sequence $\{\omega_n\}_{n\geq 1}$ is not 2^{s-1}-subadditive.

Indeed, if we define $r' := -r > 0$, then for all $n \geq 1$, we have

$$\begin{aligned} \omega_{2n} &= (2n)^s - 2nr' \\ &= 2^{s-1}\left(n^s - r'n + n^s - r'n\right) + 2n\left(2^{s-1} - 1\right)r' \\ &= 2^{s-1}\left(\omega_n + \omega_n\right) + 2n\left(2^{s-1} - 1\right)r' \\ &> 2^{s-1}\left(\omega_n + \omega_n\right). \end{aligned} \tag{6.11}$$

Claim 3. If $-1 < r < 0$, then there exists a sequence $\alpha = \{\alpha_n\}_{n\geq 2}$ of positive numbers, such that $\sum_{n=2}^{\infty}\alpha_n$ is convergent, and the sequence $\{\omega_n\}_{n\geq 1}$ is $(2^{s-1}, \alpha)$-subadditive.

As before, let $r' = -r \in (0, 1)$. Let $n \geq 2$ be a fixed natural number. Let us find a positive number α_n, such that

$$\omega_n \leq (1 + \alpha_n)2^{s-1}\left(\omega_k + \omega_{n-k}\right) \tag{6.12}$$

for all $1 \leq k < n$. This inequality is equivalent to

$$n^s - r'n \leq (1 + \alpha_n)2^{s-1}\left[k^s + (n-k)^s\right] - (1 + \alpha_n)2^{s-1}r'n \tag{6.13}$$

for all $1 \leq k < n$. Since $n^s \leq 2^{s-1}[k^s + (n-k)^s]$, if we choose α_n such that

$$n^s - r'n = (1 + \alpha_n)n^s - (1 + \alpha_n)2^{s-1}r'n, \tag{6.14}$$

then the last inequality holds, for all $1 \leq k < n$. In fact, for n even, if we choose $k = n - k$ (that means $k = n/2$), then we can see that α_n cannot be chosen smaller than the value of the solution of (6.14). Solving (6.14) for α_n, we get

$$\alpha_n = \frac{r'\left(2^{s-1} - 1\right)}{n^{s-1} - 2^{s-1}r'}. \tag{6.15}$$

Since $n \geq 2$ and $0 < r' < 1$, we can see that $\alpha_n > 0$ for all $n \geq 2$. Moreover, since we have chosen $s > 2$, the series $\sum_{n=2}^{\infty}\alpha_n$ has the same nature as the series $\sum_{n=2}^{\infty}1/n^{s-1}$, which is convergent.

Therefore, according to Theorem 5.8 for $r \geq 0$, X_r is of class 2^{t-1}-W-H, while for $-1 < r < 0$, X_r is of class $(M, 2^{t-1})$-W-H for some $M \geq 1$ (M depends on r). Moreover, the family $\{X_r\}_{r>-1}$ is not uniformly Wick-Hölderian (that means we cannot find the same M for all $r > -1$). This follows from the observation that $\lim_{r\to(-1)^+}\omega_1 = 0$, while $\lim_{r\to(-1)^+}\omega_2 = 2^s - 2 > 0$. If we assume the existence of two positive numbers p and q, such that the inequality: $E[|f_1(X_r) \diamond f_1(X_r)|^2] \leq E[|\Gamma(\sqrt{p}I)f_1(X_r)|^2]E[|\Gamma(\sqrt{q}I)f_1(X_r)|^2]$ holds for all $r > -1$ since $f_1(X_r) \diamond f_1(X_r) = f_2(X_r)$, where f_1 and f_2, are the orthogonal polynomials of degree 1 and 2, respectively, generated by X_r, then we would conclude that

$$\omega_2! \leq p\omega_1! \cdot q\omega_1!. \tag{6.16}$$

This inequality reduces to $\omega_2 \leq pq\omega_1$ for all $r > -1$, which is impossible since the left-hand side converges to a positive number, while the right-hand side tends to zero as r goes to -1.

Example 6.5. Let us take now $s = 2$, and consider the family of random variables X_r whose principal Szegö-Jacobi parameters are $\omega_n := n^2 + rn$ for all $n \geq 1$, where $r > -1$. If we take the other Szegö-Jacobi parameters to be $\alpha_n := \alpha n$ for all $n \geq 0$, where α is a fixed real number, then we can see that $\{X_r\}_{r>-1}$ are exactly the centered and rescaled Meixner random variables.

We can see exactly as in Example 6.5 that for $r \geq 0$, $\{\omega_n\}_{n\geq 1}$ is 2^{2-1}-subadditive, while for $-1 < r < 0$, it is not 2-suadditive. If $r' := -r$, unfortunately, for $\alpha_n = r'(2^{2-1} - 1)/(n^{2-1} - 2^{2-1}r') = r'/(n-2r')$, the series $\sum_{n=2}^{\infty}\alpha_n$ is not convergent. Moreover, as we saw in the previous example, since (6.14) cannot be avoided for n even, we can see that the sequence $\{\omega_n\}_{n\geq 1}$ is not $(2, \beta)$-subadditive, for any nonnegative sequence $\beta = \{\beta_n\}_{n\geq 2}$, such that $\sum_{n=2}^{\infty}\beta_n < \infty$.

Claim 1. If $-1/2 \leq r < 0$, then for all $n \geq k \geq 0$, we have

$$\binom{\omega_n}{\omega_k} \leq 2^n \binom{n}{k}. \tag{6.17}$$

Since $\binom{\omega_n}{\omega_k} = \binom{\omega_n}{\omega_{n-k}}$ and $\binom{n}{k} = \binom{n}{n-k}$, we may assume that $k \leq n/2$.

We fix $k \geq 1$, and prove by induction on n that for all $n \geq 2k$, $\binom{\omega_n}{\omega_k} \leq 2^n \binom{n}{k}$.

Let us prove first this inequality for $n = 2k$. Since $\omega_m = m(m - r')$ for all $m \geq 1$, we have

$$\begin{aligned}
\binom{\omega_{2k}}{\omega_k} &= \binom{2k}{k}\left[\frac{2k-r'}{k-r'}\cdot\frac{2k-1-r'}{k-r'}\right]\left[\frac{2k-2-r'}{k-1-r'}\cdot\frac{2k-3-r'}{k-1-r'}\right]\cdots \\
&\quad\times\left[\frac{2-r'}{1-r'}\cdot\frac{1-r'}{1-r'}\right] \\
&= \binom{2k}{k}\left[\left(2+\frac{r'}{k-r'}\right)\left(2-\frac{1-r'}{k-r'}\right)\right] \\
&\quad\times\left[\left(2+\frac{r'}{k-1-r'}\right)\left(2-\frac{1-r'}{k-1-r'}\right)\right]\cdots \\
&\quad\times\left[\left(2+\frac{r'}{1-r'}\right)\left(2-\frac{1-r'}{1-r'}\right)\right] \\
&= \binom{2k}{k}\left[4+2\frac{2r'-1}{k-r'}-\frac{r'(1-r')}{(k-r')^2}\right] \\
&\quad\times\left[4+2\frac{2r'-1}{k-1-r'}-\frac{r'(1-r')}{(k-1-r')^2}\right]\cdots \\
&\quad\times\left[4+2\frac{2r'-1}{1-r'}-\frac{r'(1-r')}{(1-r')^2}\right].
\end{aligned} \tag{6.18}$$

If $-1/2 \leq r < 0$, since $r' = -r$, we have $0 < r' \leq 1/2$, and thus $2r' - 1 \leq 0$. Hence,

$$4+2\frac{2r'-1}{j-r'}-\frac{r'(1-r')}{(j-r')^2} < 4, \tag{6.19}$$

for all $j \in \{1, 2, \ldots, k\}$. Hence, we conclude that

$$\binom{\omega_{2k}}{\omega_k} < \binom{2k}{k} 4^k = 2^{2k} \binom{2k}{k}. \tag{6.20}$$

Thus, we have succeeded in proving the induction hypothesis.

Let us suppose now that the inequality $\binom{\omega_n}{\omega_k} \le 2^n \binom{n}{k}$ holds for some $n \ge 2k$, and prove that the inequality $\binom{\omega_{n+1}}{\omega_k} \le 2^{n+1} \binom{n+1}{k}$ also holds. To prove this, it is enough to check that $\binom{\omega_{n+1}}{\omega_k} / \binom{\omega_n}{\omega_k} \le 2 \binom{n+1}{k} / \binom{n}{k}$. This is equivalent to $\omega_{n+1}/\omega_{n+1-k} \le 2(n+1)/(n+1-k)$. This new inequality means that $(n+1)(n+1-r')/[(n+1-k)(n+1-k-r')] \le 2(n+1)/(n+1-k)$, which reduces to checking that $(n+1-r') \le 2(n+1-k-r')$. Finally, this inequality is equivalent to $2k \le n+1-r'$, which is true since $n \ge 2k$ and $1 - r' > 0$.

It follows now from Theorem 5.5 that for all $r \ge 1/2$, X_r is of class 2-W-S.

Claim 2. If $r \in (-1, -1/2)$, then for all $\epsilon > 0$, there exists an M, which depends on both r and ϵ, such that for all $n \ge 2k$, we have

$$\binom{\omega_n}{\omega_k} \le M(2+\epsilon)^n \binom{n}{k}. \tag{6.21}$$

Indeed, we can see as before that for $n = 2k$, we have

$$\begin{aligned} \binom{\omega_{2k}}{\omega_k} = \binom{2k}{k} &\left[4 + 2\frac{2r'-1}{k-r'} - \frac{r'(1-r')}{(k-r')^2}\right] \\ &\times \left[4 + 2\frac{2r'-1}{k-1-r'} - \frac{r'(1-r')}{(k-1-r')^2}\right] \cdots \\ &\times \left[4 + 2\frac{2r'-1}{1-r'} - \frac{r'(1-r')}{(1-r')^2}\right]. \end{aligned} \tag{6.22}$$

Since $4 + 2(2r'-1)/(j-1-r') - r'(1-r')/(j-1-r')^2 \to 4$, as $j \to \infty$, there exists a natural number N, such that for all $j \ge N$, $4 + 2(2r'-1)/(j-1-r') - r'(1-r')/(j-1-r')^2 \le (2+\epsilon)^2$. Let $M := [1/(2+\epsilon)^{2|J|}] \prod_{j \in J} [4 + 2(2r'-1)/(j-1-r') - r'(1-r')/(j-1-r')^2]$, where $J := \{j \in \{1, 2, \ldots, N-1\} | 4 + 2(2r'-1)/(j-1-r') - r'(1-r')/(j-1-r')^2 > (2+\epsilon)^2\}$, and $|J|$ denotes the cardinality of J. Then, for all $k \ge 1$, we have

$$\binom{\omega_{2k}}{\omega_k} \le M(2+\epsilon)^{2k} \binom{2k}{k}. \tag{6.23}$$

Now, if we fix $k \ge 1$, we can prove, as before by induction on n that for all $n \ge 2k$, the inequality (6.21) holds.

Therefore, for any $r \in (-1, -1/2)$ and any $\epsilon > 0$, there exists $M > 0$, such that X_r is of class $(M, 2+\epsilon)$-W-H.

Claim 3. For all $r \in (-1, -1/2)$, X_r is of class $1/\sqrt{r(1-r)}$-W-H.

To see this, let us observe that since the roots of the trinomial $q(x) = (2+r'x)[2-(1-r')x]$ are $-2/r'$ and $2/(1-r')$, the maximum of this trinomial is attained at $x_v = 1/2[2/(1-r')-2/r'] = 1/(1-r')-1/r'$ and is equal to $q(x_v) = 1/[r'(1-r')]$. The closer the value of x to x_v, the greater the value of $q(x)$. As we saw before for all $k \geq 1$, we have

$$\begin{aligned}\binom{\omega_{2k}}{\omega_k} &= \binom{2k}{k}\left[\left(2+\frac{r'}{k-r'}\right)\left(2-\frac{1-r'}{k-r'}\right)\right] \\ &\quad\times\left[\left(2+\frac{r'}{k-1-r'}\right)\left(2-\frac{1-r'}{k-1-r'}\right)\right]\cdots \\ &\quad\times\left[\left(2+\frac{r'}{1-r'}\right)\left(2-\frac{1-r'}{1-r'}\right)\right] \\ &= \binom{2k}{k} q\left(\frac{1}{k-r'}\right) q\left(\frac{1}{k-1-r'}\right)\cdots q\left(\frac{1}{1-r'}\right) \\ &\leq \binom{2k}{k}[q(x_v)]^k \\ &= \binom{2k}{k}\left[\frac{1}{\sqrt{r'(1-r')}}\right]^{2k}.\end{aligned} \tag{6.24}$$

Since $1/\sqrt{r'(1-r')} \geq 2$, we can prove now by induction on n that for all $n \geq 2k$, we have

$$\binom{\omega_n}{\omega_k} \leq t^n \binom{n}{k}, \tag{6.25}$$

where $t := 1/\sqrt{r'(1-r')}$. It follows now from Corollary 5.6 that X_r is of class t-W-H.

One can improve a little bit this t, by observing that for all $-1 < r < -1/2$ (or, equivalently, $1/2 < r' < 1$), we have

$$\frac{1}{1-r'} > x_v > \frac{1}{2-r'} > \frac{1}{3-r'} > \cdots. \tag{6.26}$$

A simple computation shows that for $1/2 < r' < (7-\sqrt{17})/4$, $1/(2-r')$ is closer to x_v than $1/(1-r')$ is to x_v. Therefore, in this case, we can choose a smaller $t := \sqrt{q(1/(2-r'))}$. If $(7-\sqrt{17})/4 \leq r' < 1$, then we have that $1/(1-r') - x_v \leq x_v - 1/(2-r')$, and thus we can take $t := \sqrt{q(1/(1-r'))}$ to conclude that X_r is of class t-W-H.

Since $\lim_{r\to(-1)^+}\omega_1 = 0$ and $\lim_{r\to(-1)^+}\omega_2 = 2 > 0$, we can conclude, as in Example 6.5, that the family $\{X_r\}$ is not unif.-W-H. On the other hand, since for any $r'_0 > -1$, the set $\{1/\sqrt{r'(1-r')} \mid 1/2 < r' \leq r'_0\}$ is bounded above, we can see that for any $B \subset (-1, \infty)$, the family $\{X_r\}_{r\in B}$ is unif.-W-H if and only if $\inf B > -1$.

Claim 4. For any $r > -1$, if M and t are positive numbers, such that X_r is of class (M, t)-W-H, then $t \geq 2$.

To see this, let us observe first that by D'Alembert ratio theorem, we have

$$\lim_{n\to\infty}\sqrt[n]{\binom{\omega_{2n}}{\omega_n}} = \lim_{n\to\infty}\frac{\binom{\omega_{2(n+1)}}{\omega_{n+1}}}{\binom{\omega_{2n}}{\omega_n}} = \lim_{n\to\infty}\frac{(2n+2)(2n+1)(2n+2+r)(2n+1+r)}{(n+1)^2(n+1+t)^2} = 16. \tag{6.27}$$

On the other hand, since for $p = q = 2t$, $E[|f_n(X_r) \diamond f_n(X_r)|^2] \leq ME[|\Gamma(\sqrt{2t})f_n(X_r)|^2]^2$ and $f_n(X_r) \diamond f_n(X_r) = f_{2n}(X_r)$, we conclude that for all $n \geq 1$,

$$\binom{\omega_{2n}}{\omega_n} \leq M(2t)^{2n}. \tag{6.28}$$

Taking radical or order n from both sides of this inequality and passing to the limit as $n\to\infty$, we obtain that $16 \leq (2t)^2$. Thus, we get $t \geq 2$.

This last claim proves that at least for $r \geq -1/2$, we got the best possible t, which is 2, for the Wick-Hölder class of X_r.

Let us consider now the family of nondegenerate centered Meixner random variables: $X_{\alpha,\beta,t}$ whose Szegö-Jacobi parameters are $\alpha_n = \alpha n$ for all $n \geq 0$, and $\omega_n = \beta n^2 + (t-\beta)n$ for all $n \geq 1$, where the parameters α, β, and t satisfy the conditions: $\alpha \in \mathbb{R}$, $\beta \geq 0$, and $t > 0$. We distinguish between two cases.

Case 1. If $\beta = 0$, (i.e., $X_{\alpha,0,t}$ is gaussian or poissonian), then $\omega_n = tn$ for all $n \geq 1$, and therefore, $\{\omega_n\}_{n\geq 1}$ is additive. In this case, $X_{\alpha,0,t}$ is of class 1-W-H.

Case 2. If $\beta > 0$, then, $\omega_n = \beta\omega'_n$, where $\omega'_n = n^2 + [(t/\beta) - 1]n$, we can see that $\binom{\omega_n}{\omega_k} = \binom{\omega'_n}{\omega'_k}$ for all $n \geq k \geq 0$. Therefore, $X_{\alpha,\beta,t}$ satisfies the same Wick-Hölder inequality as X_r, where $r = t/\beta$.

Defining $t/\beta = \infty$ for $\beta = 0$, we can apply all the claims from this example to the family of Meixner distributions, and formulate the following theorem.

Theorem 6.6. *Let $\{X_{\alpha,\beta,t}\}_{\alpha\in\mathbb{R},\beta\geq 0,t>0}$ be the family of Meixner distributions. Then,*

(1) *for $X_{\alpha,0,t}$ is of class* 1-*W-H for all $\alpha \in \mathbb{R}$, and $t > 0$;*

(2) *if $\beta > 0$ and $t/\beta \geq 1/2$, then $X_{\alpha,\beta,t}$ is of class* 2-*W-H, and* 2 *is optimal;*

(3) *if $\beta > 0$ and $t/\beta < 1/2$, then $X_{\alpha,\beta,t}$ is of class $1/\sqrt{r'(1-r')}$-W-H, where $r' = 1 - (t/\beta)$;*

(4) *a family $\{X_{\alpha,\beta,t}\}_{(\alpha,\beta,t)\in I}$ of Meixner distributions is uniformly Wick Hölderian if and only if $\inf\{t/\beta \mid \exists\alpha \in \mathbb{R}, (\alpha,\beta,t) \in I\} > 0$.*

Example 6.7. Let X_q be the q-Gaussian random variable with parameter q, where $q \in [-1, 1]$. It means that for any $q \in (-1, 1)$, X_q is a symmetric random variable (so, $\alpha_n = 0$ for all $n \geq 0$) and $\omega_n = (1-q^n)/(1-q)$ for all $n \geq 1$. For $q = 1$, X_1 is the standard Gaussian since $\omega_n = \lim_{q\to 1^-}(1-q^n)/(1-q) = n$ for all $n \geq 1$. For $q = -1$, $\omega_1 = 1$ and $\omega_n = 0$ for all $n \geq 2$. Thus, X_{-1} is the centered Bernoulli random variable $(1/2)[\delta_{\{-1\}} + \delta_{\{1\}}]$. Despite the fact that $\lim_{q \to -1^+} \omega_{2n} = 0$ for all $n \geq 1$, the family $\{X_q\}_{-1 \leq q \leq 1}$ is uniformly Wick-Hölderian. In fact, every q-Gaussian random variable satisfies the best Wick-Hölder inequality since $\{\omega_n\}_{n \geq 1}$ is subadditive for all $-1 \leq q \leq 1$. This statement is trivial for $q = -1$ and $q = 1$. For $-1 < q < 1$ and all m and n natural numbers, we have

$$\omega_{m+n} - \omega_m - \omega_n = \frac{1 - q^{m+n}}{1 - q} - \frac{1 - q^m}{1 - q} - \frac{1 - q^n}{1 - q} = \frac{(1 - q^m)(q^n - 1)}{1 - q} < 0. \tag{6.29}$$

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