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THE $M^X/G/1$ QUEUE WITH QUEUE LENGTH DEPENDENT SERVICE TIMES

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We deal with the $M^X/G/1$ queue where service times depend on the queue length at the service initiation. By using Markov renewal theory, we derive the queue length distribution at departure epochs. We also obtain the transient queue length distribution at time t and its limiting distribution and the virtual waiting time distribution. The numerical results for transient mean queue length and queue length distributions are given.

Key words: $M^X/G/1$ Queue, Queue Length Dependent Service Time, Transient Queue Length Distribution, Waiting Time Distribution.

AMS subject classifications: 60K25, 90B22.

1. Introduction

Our analysis of the $M^X/G/1$ queue with queue length dependent service times is motivated by overload control on a multiplexer for a voice packet where less significant bits in the voice packet are dropped during congestion [4, 14]. One of the overload control schemes can be incorporated into the model by making the service times state-dependent; when the queue lengths exceed certain thresholds, the service time of the voice packets is decreased by dropping bits (see [3, 9, 11, 14]).

In this paper, we deal with the stable $M^X/G/1$ queue with a number of types of service times which depend on the queue length at the service initiation. Following the general approach based on structured Markov renewal processes of M/G/1 type discussed in [12], we derive the queue length distribution at departure epochs, the transient queue length distribution at time t, its limiting distribution and the virtual waiting time distribution.

Sriram and Lucantoni [14] examined the performance of a multiplexer for a voice packet in which the less significant bits in voice packet are dropped during states of congestion in the multiplexer so as to reduce the queueing delay at the expense of a slight reduction in voice quality. They modeled the multiplexer as a $M/\widetilde{D}/1/K$ queueing system in which $\widetilde{D}$ denotes the deterministic but state-dependent nature of service. Choi [4] considered $MMPP/G_1,G_2/1/K$ with queue length dependent service times and obtained the queue length distribution both at departure epochs and at arbitrary time. There are an abundance of studies of queues with length dependent arrival rates and/or service times. For comprehensive surveys of them, see Dshalalow [6]. We describe the model and results in Harris [8] and Ivnitskiy [10], since they are closely related with our model. Harris [8] investigated the $M^X/G/1$ queue with queue length dependent service times. To be more specific, when there are i customers in the system, the service time distribution of a customer entering into service is $S_i(x)$. Harris [8] derived the stationary condition and the probability generating function for the stationary queue length distribution at the departure epochs by using the embedded Markov chain technique. By employing the supplementary variable technique, Ivnitskiy [10] derived the transient and stationary queue length distributions for the $M^X/G/1$ queue in which the interarrival rates, group size probabilities and service rates all depend on the queue length in the system. The generating function obtained in [8] in general contains infinitely many unknown constants so that closed forms were presented only for some special cases, such as (*i*) $S_1(x) = 1 - e^{-\mu_1 x}$, $S_i(x) = 1 - e^{-\mu x}$, $i \geq 2$, (*ii*) $S_1(x) = 1 - (\mu_1 x + 1)e^{-\mu_1 x}$, $S_i(x) = 1 - e^{-\mu x}$, $i \geq 2$, (*iii*) $S_i(x) = 1 - e^{-i\mu x}$, $i \geq 1$. The queue length distribution obtained in [10] is so complex that the results are not appropriate for numerical computation. While the model we deal with here is a special case of the models of Harris [8] and Ivnitskiy [10], the formulae we obtain lend themselves rather more to computation.

In a practical system, a finite number of thresholds are used for overload control [11, 14] and hence it is useful to obtain computable form of queue length distribution for the model with a finite number of types of service times. The merit of our approach is to present explicit formulae for the transient queue length distribution and the first and second moments of the queue length in the steady state.

This paper is organized as follows. In Section 2, we present a Markov renewal process formed by the queue length at the departure epochs and the inter-departure times. Section 3 is devoted to obtaining the queue length distribution at departure epochs. In Section 4, we investigate the first passage times of Markov renewal process constructed in Section 2. Using the results of Section 4, the transient and stationary queue length distributions at an arbitrary time are derived in Section 5. Some numerical results for mean queue length and the queue length distribution in transient state and stationary state are given in Section 6. In Section 7, we deal with the virtual waiting time.

2. The Embedded Markov Renewal Process at Departures

We consider a queueing system in which customers arrive in bulk according to a time-homogeneous Poisson process with rate $\lambda > 0$. The bulk size X is a random variable with probability mass function $P(X = k) = x_k$ $(k = 1, 2, \ldots)$ and with probability generating function $X(x) = \sum_{k=1}^{\infty} x_k z^k$, $|z| \leq 1$. We assume that $\overline{x} = E(X) < \infty$ and $E(X^2) < \infty$. In fact, the arrivals form a compound Poisson process so that the probability $c_i(t)$ that i customers arrive in $(0, t]$ is

$$c_i(t) = \begin{cases} e^{-\lambda t}, & \text{if } i = 0, \\ e^{-\lambda t} \sum_{k=1}^{i} \frac{(\lambda t)^k}{k!} x_i^{(k)}, & \text{if } i = 1, 2, \ldots, \end{cases}$$

where $x_i^{(k)}$ is the k-fold convolution of $\{x_i\}$ with itself. We note that for $i \geq 1$

$$c_i(t) = \sum_{k=1}^{i} x_k \int_0^t \lambda e^{-\lambda u} c_{i-k}(t-u) du$$

and the probability generating function of $c_i(t)$ is

$$\sum_{i=0}^{\infty} c_i(t) z^i = \exp(-\lambda t(1 - X(z))).$$

The service time of a customer is determined by the queue length at his service initiation epoch. When the queue length is n at service initiation epochs, the service time is $S_n(x)$ and service times of customers beginning service with the same number n in the system are independent and identically distributed random variables with distribution function $S_n(x)$, $n = 1, 2 \ldots$. We assume that $S_1(x)$, $S_2(x)$, $\ldots$, $S_{N+1}(x)$ may be different but for $k \geq N+1$, $S_k(x)$ takes a common form $S(x)$. Let $\overline{s}_k$ and $\overline{s}$ denote the means of $S_k(x)$ and $S(x)$, respectively. Let τ_n $(n \geq 0)$ represent the succession of departure instants (with $\tau_0 = 0$) and I_n the number of customers in the system immediately after the nth departure. It is readily seen that the sequence $\{(I_n, \tau_n - \tau_{n-1}), n \geq 1\}$ is a Markov renewal sequence on the state space $\{0, 1, 2, \ldots\} \times [0, \infty)$. The transition probability matrix $Q(x)$ of $\{(I_n, \tau_n - \tau_{n-1}), n \geq 1\}$ has the special form

$$Q(x) = \begin{pmatrix} A_{0,0}(x) & A_{0,1}(x) & A_{0,2}(x) & A_{0,3}(x) & \cdots & \cdots & \cdots \\ A_{1,0}(x) & A_{1,1}(x) & A_{1,2}(x) & A_{1,3}(x) & \cdots & \cdots & \cdots \\ 0 & A_{2,0}(x) & A_{2,1}(x) & A_{2,2}(x) & \cdots & \cdots & \cdots \\ 0 & 0 & A_{3,0}(x) & A_{3,1}(x) & \cdots & \cdots & \\ \vdots & \vdots & \ddots & \ddots & \ddots & & \\ 0 & 0 & \cdots & A_{N,0}(x) & A_{N,1}(x) & A_{N,2}(x) & \cdots \\ 0 & 0 & \cdots & 0 & B_0(x) & B_1(x) & \cdots \\ 0 & 0 & \cdots & 0 & 0 & B_0(x) & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \end{pmatrix},$$

where

$$A_{i,j}(x) = \int_0^x c_j(t)dS_i(t), 1 \le i \le N, j \ge 0,$$

$$B_j(x) = \int_0^x c_j(t)dS(t), \quad j \ge 0,$$

$$A_{0,j}(x) = \begin{cases} \sum_{i=1}^{j+1} x_i \int_0^x A_{i,j-i+1}(x-u)\lambda e^{-\lambda u}du, & \text{if } j \le N-1, \\ \sum_{i=1}^{N} x_i \int_0^x A_{i,j-i+1}(x-u)\lambda e^{-\lambda u}du & \\ + \sum_{i=N+1}^{j+a} x_i \int_0^x B_{j-i+1}(x-u)\lambda e^{-\lambda u}du, & \text{if } j \ge N. \end{cases}$$

We introduce some useful transforms and notations:

$$\widetilde{A}_{i,j} = \int_0^\infty e^{-st}dA_{i,j}(t), \quad \widetilde{B}_j(s) = \int_0^\infty e^{-st}dB_j(t),$$

$$\widetilde{A}_i(z,s) = \sum_{j=0}^\infty z^j \widetilde{A}_{ij}(s), \quad \widetilde{B}(z,s) = \sum_{j=0}^\infty z^j \widetilde{B}_j(s),$$

$$\widetilde{A}_i(z) = \widetilde{A}_i(z,0), \quad \widetilde{B}(z) = \widetilde{B}(z,0),$$

$$A_{i,j} = A_{i,j}(\infty), \quad B_j = B_j(\infty),$$

$$\alpha_i = \sum_{k=1}^\infty kA_{i,k}, \quad \beta = \sum_{k=1}^\infty kB_k.$$

By a simple calculation we have

$$\widetilde{A}_i(z,s) = \widetilde{S}_i(s + \lambda(1 - X(z))), \quad i = 1,2,\ldots,N,$$

$$\widetilde{B}(z,s) = \widetilde{S}(s + \lambda(1 - X(z))), \tag{2.1}$$

$$\widetilde{A}_0(z,s) = \frac{\lambda}{\lambda+s}\frac{1}{z}\left(X(z)\widetilde{B}(z,s) + \sum_{i=1}^N x_i z^i(\widetilde{A}_i(z,s) - \widetilde{B}(z,s))\right),$$

where $\widetilde{S}_i(\theta) = \int_0^\theta e^{-\theta t}dS_i(t)$ and $\widetilde{S}(\theta) = \int_0^\infty e^{-\theta t}dS(t)$.

By differentiating equations (2.1) with respect to z and letting $z \to 1-$ and $s \to 0+$, we obtain

$$\alpha_i = \lambda \bar{x}\,\bar{s}_i, \quad i = 1,2,\ldots,N, \tag{2.2a}$$

$$\beta = \lambda \bar{x}\,\bar{s}, \tag{2.2b}$$

$$\alpha_0 = \bar{x} - 1 + \beta + \sum_{i=1}^{N} x_i(\alpha_i - \beta). \tag{2.2c}$$

3. The Queue Length Distribution at Departures

Let $\mathbf{p} = (p_0, p_1, \ldots)$ be the invariant probability vector of $Q(\infty)$, that is,

$$\mathbf{p}Q(\infty) = \mathbf{p} \text{ and } \sum_{i=0}^{\infty} p_i = 1. \tag{3.1}$$

Then (3.1) can be written as

$$p_k = \begin{cases} p_0 A_{0k} + \sum_{i=1}^{k+1} p_i A_{i,k-i+1}, & \text{if } k \le N-1 \\ p_0 A_{0k} + \sum_{i=1}^{N} p_i A_{i,k-i+1} + \sum_{i=N+1}^{k+1} p_i B_{k-i+1}, & \text{if } k \ge N. \end{cases} \tag{3.2}$$

Set $P(z) = \sum_{k=0}^{\infty} p_k z^k$. On multiplying both sides of (3.2) by z^k and summing over k, we have

$$\begin{aligned} P(z) &= \frac{1}{z - \widetilde{B}(z)} \left[p_0(z\widetilde{A}_0(z) - \widetilde{B}(z)) + \sum_{i=1}^{N} p_i(\widetilde{A}_i(z) - \widetilde{B}(z))z^i \right] \\ &= \frac{1}{z - \widetilde{B}(z)} \left[p_0(X(z) - 1)\widetilde{B}(z) + \sum_{i=1}^{N} (p_0 x_i + p_i)(\widetilde{A}_i(z) - \widetilde{B}(z))z^i \right]. \end{aligned} \tag{3.3}$$

Thus the probability generating function $P(z)$ is completely determined by finding the unknowns $p_0, p_1, \ldots, p_N$.

Consider the following stochastic matrix Q^*

$$Q^* = \begin{pmatrix} A_{00} & A_{01} & A_{02} & \cdots & A_{0,N-1} & \bar{A}_{0,N} \\ A_{10} & A_{11} & A_{12} & \cdots & A_{1,N-1} & \bar{A}_{1,N} \\ 0 & A_{20} & A_{21} & \cdots & A_{2,N-2} & \bar{A}_{2,N-1} \\ 0 & 0 & A_{30} & \cdots & A_{3,N-3} & \bar{A}_{3,N-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & A_{N,0} & \bar{A}_{N,1} \end{pmatrix}$$

where $\bar{A}_{ij} = \sum_{k=j}^{\infty} A_{ik}$, $0 \le i \le N$.

Let $\mathbf{q} = (q_0, q_1, \ldots, q_N)$ be the invariant probability vector of Q^*, that is, $\mathbf{q}Q^* = \mathbf{q}$ and $\sum_{i=0}^{N} q_i = 1$. Then since the vector $\mathbf{p}^* = (p_0, p_1, \ldots, p_N)$ is an eigenvector of Q^* corresponding to the eigenvalue 1, $\mathbf{p}^*$ is given by

$$\mathbf{p}^* = c\mathbf{q} \tag{3.4}$$

where c is a constant. Substituting (3.4) into (3.3) and using the normalizing condition $P(1) = 1$ and the relation (2.2c), the constant c is given by

$$c = \frac{1-\beta}{q_0\overline{x} + \sum_{i=1}^{N}(q_0x_i + q_i)(\alpha_i - \beta)}. \tag{3.5}$$

Now we derive the first two factorial moments $P'(1)$ and $P''(1)$ of the queue length at departures. Referring to formula (3.3), we set

$$Y(z) = p_0(X(z)-1)\widetilde{B}(z) + \sum_{i=1}^{N}(p_0x_i + p_i)z^i(\widetilde{A}_i(z) - \widetilde{B}(z))$$

and for later use we introduce the notations

$$\alpha_{i,j} = \frac{\partial^j}{\partial z^j}\widetilde{A}_i(z,s)\Big|_{z=1-,\, s=0+} \quad \text{and} \quad \beta_j = \frac{\partial^j}{\partial z^j}\widetilde{B}(z,s)\Big|_{z=1-,\, s=0+}$$

for moments.

Then (3.3) can be written as

$$P(z)(z - \widetilde{B}(z)) = Y(z). \tag{3.6}$$

Differentiating (3.6) and setting $z = 1-$, we obtain the first two factorial moments of the queue length at departures as follows.

$$P'(1) = \frac{1}{2(1-\beta)}(\beta_2 + Y''(1)), \tag{3.7}$$

$$P''(1) = \frac{1}{3(1-\beta)}(3P'(1)\beta_2 + \beta_3 + Y'''(1)), \tag{3.8}$$

where

$$Y'(1) = p_0\overline{x} + \sum_{i=1}^{N}(p_0x_i + p_i)(\alpha_i - \beta),$$

$$Y''(1) = p_0(2\overline{x}\beta + X''(1)) + \sum_{i=1}^{N}(p_0x_i + p_i)[2i(\alpha_i - \beta) + (\alpha_{i,2} - \beta_2)],$$

$$Y'''(1) = p_0(X'''(1) + 3X''(1)\beta + 3\overline{x}\beta_2)$$
$$+ \sum_{i=1}^{N}(p_0x_i + p_i)\big(3i(i-1)(\alpha_i - \beta) + 3i(\alpha_{i,2} - \beta_2) + (\alpha_{i,3} - \beta_3)\big).$$

4. Hitting Times

4.1 First Passage Times from State $i+1$ to State i

Define $G_i(x)$, for $x \geq 0$, $i = 0, 1, 2, \ldots$, to be the probability that the first passage from state $i+1$ to state i takes no more than time x. It follows that the spatial homogeneity of the transition matrix $Q(\cdot)$ except for the first $N+1$ rows and the *skip-free-to-the-left* property of $Q(\cdot)$ that for all $i \geq N$, the values $G_i(x)$ are the same. Thus we denote $G_i(x)$ by $G(x)$ for all $i \geq N$. By conditioning on the time and destination of the first transition and applying for the law of total probability, we have the following equation for $G(x)$:

$$G(x) = B_0(x) + \sum_{i=1}^{\infty} B_i * G^{(i)}(x), \tag{4.1}$$

where $*$ denotes convolution and $G^{(i)}(x)$ is the i-fold convolution of $G(x)$ with respect to x.

Routine calculation of the LST $\widetilde{G}(s) = \int_0^\infty e^{-sx} dG(x)$, $Re\, s > 0$ from (4.1) yields the equation

$$\widetilde{G}(s) = \widetilde{B}(\widetilde{G}(s), s). \tag{4.2}$$

Similarly, we obtain LST $\widetilde{G}_i(s) = \int_0^\infty e^{-sx} dG_i(x)$ for $i = 0, \ldots, N-1$ as follows:

$$\widetilde{G}_i(s) = \widetilde{A}_{i+1,0}(s) + \sum_{j=1}^{\infty} \widetilde{A}_{i+1,j}(s) \prod_{k=i}^{i+j-1} \widetilde{G}_k(s), \tag{4.3}$$

and consequently,

$$\widetilde{G}_i(s) = \widetilde{A}_{i+1,0}(s) \left[1 - \sum_{j=1}^{N-i-1} \widetilde{A}_{i+1,j}(s) \left(\prod_{k=1}^{j-1} \widetilde{G}_{i+k}(s) \right) \right. \tag{4.4}$$

$$\left. - \left(\prod_{k=1}^{N-i-1} \widetilde{G}_{i+k}(s) \right) \widetilde{G}(s)^{-N+i} \left\{ \widetilde{A}_{i+1}(\widetilde{G}(s), s) - \sum_{j=0}^{N-i-1} \widetilde{A}_{i+1,j}(s) \widetilde{G}(s)^j \right\} \right]^{-1}.$$

Remarks: (1) Following the procedure in Theorem 1.2.2 of [12], we have that a necessary and sufficient condition for $G \equiv G(\infty)$ to be unity is $\beta \leq 1$.

(2) Letting $s = 0+$ in (4.4), we have

$$G_{N-1} = A_{N,0} + A_{N,1} G_{N-1} + \sum_{j=2}^{\infty} A_{N,j} G^{j-1} G_{N-1}, \tag{4.5}$$

where $G_{N-1} \equiv \widetilde{G}_{N-1}(0+)$. Thus if $G = 1$, then (4.5) becomes $G_{N-1} = A_{N,0}(1 - G_{N-1}) + G_{N-1}$. Since $A_{N,0} = \widetilde{S}_N(\lambda) > 0$, we have that $G = 1$ implies $G_{N-1} = 1$. By induction, it is easily shown that if $G = 1$, then $G_{N-i} = 1$ $(i = 1, 2, \ldots, N)$.

By differentiating (4.2) and (4.3) with respect to s, we derive the following recursive formulae for mean times:

$$g \equiv -\widetilde{G}'(0+) = \frac{\bar{s}}{1-\beta},$$

$$g_j \equiv -\widetilde{G}'_j(0+)$$

$$= \frac{1}{A_{j+1,0}}\left[\bar{s}_{j+1} + g\left(\alpha_{j+1} - N + j + \sum_{k=0}^{N-j-1}(N-j-k)A_{j+1,k}\right)\right. \tag{4.6}$$

$$\left. + \sum_{k=j+1}^{N-1} g\left(1 - \sum_{i=0}^{k-j} A_{j+1,i}\right)\right], \quad j = N-1, N-2, \ldots, 0.$$

We note from (4.6) that if $\beta < 1$ and $\alpha_{j+k} < \infty$ $(k = 1, 2, \ldots, i)$, then $g_j < \infty$.

4.2 Recurrence Time for State i

Denote by T_{ij}, $(i < j)$ the first passage time from state i to state j in the Markov renewal process with transition probability matrix $Q(\cdot)$. Let $H_{ij}(x) = P(T_{ij} \le x)$ be the distribution function of T_{ij}. Following arguments similar to those leading to (4.1) and (4.2), we can show that the transforms $\widetilde{H}_{ij}(s) = E(e^{-sT_{ij}})$, $Re\, s \ge 0$, satisfy the following equations: for each $j \ge 1$,

$$\widetilde{H}_{0j}(s) = \sum_{i=0}^{j-1} \widetilde{A}_{0i}(s)\widetilde{H}_{ij}(s) + \widetilde{A}_{0j}(s) + \sum_{i=j+1}^{\infty} \widetilde{A}_{0j}(s)\left(\prod_{k=j}^{i-1} \widetilde{G}_k(s)\right), \tag{4.7a}$$

$$\widetilde{H}_{ij}(s) = \sum_{k=0}^{j-i} \widetilde{A}_{ik}(s)\widetilde{H}_{i+k-1,j}(s) + \widetilde{A}_{i,j-i+1}(s)$$

$$+ \sum_{k=j-i+2}^{\infty} \widetilde{A}_{ik}(s)\left(\prod_{n=j}^{i+k-1} \widetilde{G}_n(s)\right), \quad 1 \le i \le j-1. \tag{4.7b}$$

Rewriting the formulae (4.7), we have the following linear system of equations for $\widetilde{\boldsymbol{H}}_j(s) = (\widetilde{H}_{0j}(s), \ldots, \widetilde{H}_{j-1,j}(s))^t$,

$$\boldsymbol{C}_j(s)\widetilde{\boldsymbol{H}}_j(s) = -\boldsymbol{b}_j(s), \quad j \ge 1 \tag{4.8}$$

where $\boldsymbol{C}_j(s)$ is the $j \times j$-matrix given by

$$\boldsymbol{C}_j(s) = \begin{pmatrix} A_{0,0}(s) - 1 & A_{01}(s) & A_{02}(s) & \cdots & A_{0,j-2}(s) & A_{0,j-1}(s) \\ A_{10}(s) & A_{11}(s) - 1 & A_{12}(s) & \cdots & A_{1,j-2}(s) & A_{1,j-1}(s) \\ 0 & A_{20}(s) & A_{21}(s) - 1 & \cdots & A_{2,j-2}(s) & A_{2,j-1}(s) \\ 0 & 0 & A_{30}(s) & \cdots & A_{3,j-2}(s) & A_{3,j-1}(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & A_{j-1,0}(s) & A_{j-1,1}(s) - 1 \end{pmatrix}$$

and $\boldsymbol{b}_j(s) = (b_{0j}(s), \ldots, b_{j-1,j}(s))^t$ with

$$b_{0j}(s) = \widetilde{A}_{0j}(s) + \sum_{i=j+1}^{\infty} \widetilde{A}_{0j}(s)\left(\prod_{k=j}^{i-1} \widetilde{G}_k(s)\right), \quad j \geq 1$$

$$b_{ij}(s) = \widetilde{A}_{i,j-i+1}(s) + \sum_{k=j-i+2}^{\infty} \widetilde{A}_{ik}(s)\left(\prod_{n=j}^{i+k-1} \widetilde{G}_n(s)\right), \ 1 \leq i \leq j-1.$$

Now we consider the recurrence times. Let R_i be the recurrence time of the state i. Let $K_i(x) = P(R_i \leq x)$ be the distribution function of R_i. By applying the law of total probability with conditioning on the time and the destination of the first transition, we see that the LSTs $\widetilde{K}_i(s) = E(e^{-sR_i})$, $\operatorname{Re} s \geq 0$ satisfy the equations

$$\widetilde{K}_0(s) = \widetilde{A}_{0,0}(s) + \sum_{j=1}^{\infty} \widetilde{A}_{0,j}(s) \prod_{k=0}^{j-1} \widetilde{G}_k(s), \tag{4.9a}$$

$$\widetilde{K}_i(s) = \widetilde{A}_{i,0}(s)\widetilde{H}_{i-1,i}(s) + z\widetilde{A}_{i,1}(s) + \sum_{j=2}^{\infty} \widetilde{A}_{i,j}(s) \prod_{k=i}^{i+j-2} \widetilde{G}_k(s), 1 \leq i < N, \tag{4.9b}$$

$$\widetilde{K}_N(s) = \widetilde{A}_{N,0}(s)\widetilde{H}_{N-1,N}(s) + \sum_{j=1}^{\infty} \widetilde{A}_{N,j}(s)\widetilde{G}^{j-1}(s)$$

$$= \widetilde{A}_{N,0}(s)\widetilde{H}_{N-1,N}(s) + \frac{1}{\widetilde{G}(s)}\left(\widetilde{A}_N(\widetilde{G}(s), s) - \widetilde{A}_{N,0}(s)\right), \tag{4.9c}$$

$$\widetilde{K}_i(s) = \widetilde{B}_0(s)\widetilde{H}_{i-1,i}(s) + \sum_{j=1}^{\infty} \widetilde{B}_j(s)\widetilde{G}^{j-1}(s)$$

$$= 1 + \widetilde{B}_0(s)\left[\widetilde{H}_{i-1,i}(s) - \frac{1}{\widetilde{G}(s)}\right], \ 1 \geq N+1. \tag{4.9d}$$

The mean $\kappa_0 = E(R_0)$ is easily obtained as

$$\kappa_0 = \frac{1}{\lambda} + \bar{s}\left(1 - \sum_{i=1}^{N} x_i\right) + \sum_{i=1}^{N} x_i \bar{s}_i + \sum_{k=0}^{N-1} g_k\left(1 - \sum_{j=0}^{k} A_{0,j}\right)$$
$$+ g\left(\alpha_0 - N + \sum_{j=0}^{N} (N-j)A_{0j}\right). \tag{4.10}$$

Remark: Under the conditions $\bar{s} < \infty$, $\bar{s}_i < \infty$ $(i = 1, 2, \ldots, N)$, the following are equivalent.

(*i*) The Markov renewal process with transition probability matrix $Q(\cdot)$ is positive recurrent.

(*ii*) $\kappa_0 < \infty$.

(*iii*) $\alpha_0 < \infty$, $g < \infty$, $g_k < \infty$ $(k = 0, 1, 2, \ldots, N-1)$.

(*iv*) $\beta < 1$, $\alpha_i < \infty$ $(i = 0, 1, 2, \ldots, N)$.

5. The Queue Length Distribution at Time t

In this section, we present the transient queue length distribution at time t and a relationship between the stationary queue length distributions at an arbitrary time and at departure epochs. This is accomplished by a classical argument based on the key renewal theorem for Markov renewal processes. Let $M_{i,j}(t)$ denote the conditional expected number of visits to state j in the interval $[0,t]$ given $I_0 = i$. We assume that time $t = 0$ corresponds to a departure epoch and that there are i customers in the system at that time. By $\pi_{ij}(t)$ we denote the conditional probability that there are j customers in the system at time t given $I_0 = i$. By considering the state of the Markov renewal process at the epoch of the last departure before time t and using the law of total probability, we have that

$$\pi_{i0}(t) = \int_0^t dM_{i,0}(u)e^{-\lambda(t-u)},$$

$$\pi_{ij}(t) = \int_0^t dM_{i,0}(u)\sum_{k=1}^{j} x_k \int_0^{t-u} dv\lambda e^{-\lambda v} c_{j-k}(t-u-v)(1-S_k(t-u-v)) \tag{5.1}$$

$$+\sum_{k=1}^{j}\int_0^t dM_{i,k}(u)c_{j-k}(t-u)(1-S_k(t-u)), j \geq 1,$$

where $S_k(x) = S(x)$ if $k \geq N+1$. We introduce the necessary transforms

$$\tilde{\pi}_{ij}(s) = \int_0^\infty e^{-st}\pi_{ij}(t)dt, \quad \Pi_i(z,s) = \sum_{j=0}^{\infty} z^j \tilde{\pi}_{ij}(s), \ |z| \leq 1, Re\, s \geq 0,$$

$$m_{i,j}(s) = \int_0^\infty e^{-st}dM_{i,j}(t), m_i(z,s) = \sum_{j=0}^{\infty} z^j m_{i,j}(s), \ |z| \leq 1, Re\, s \geq 0.$$

We have from (5.1) that

$$\tilde{\pi}_{i0}(s) = \frac{1}{\lambda+s}m_{i,0}(s),$$

$$\tilde{\pi}_{ij}(s) = \sum_{k=1}^{j}\left(\frac{\lambda}{\lambda+s}x_k m_{i,0}(s) + m_{i,k}(s)\right)\int_0^\infty e^{-st}c_{j-k}(t)(1-S_k(t))dt, \ j \geq 1, \tag{5.2}$$

where $S_k(x) = S(x)$ for $k \geq N+1$. Multiplying both sides of (5.2) by z^j and summing over j, we have

$$\Pi_i(z,s) = \frac{1}{\lambda+s}m_{i,0}(s)$$

$$+\frac{\lambda}{\lambda+s}m_{i,0}(s)\frac{1}{\theta}\left(X(z)(1-\widetilde{S}(\theta))-\sum_{j=1}^{N}x_j(\widetilde{S}_j(\theta)-\widetilde{S}(\theta))z^j\right)$$

$$+\frac{1}{\theta}\left((m_i(z,s)-m_{i,0}(s))(1-\widetilde{S}(\theta))-\sum_{j=1}^{N}m_{i,j}(s)(\widetilde{S}_j(\theta)-\widetilde{S}(\theta))z^j)\right) \tag{5.3}$$

$$=\frac{1}{\theta}m_i(z,s)(1-\widetilde{B}(z,s))-\frac{1}{\theta}m_{i,0}(s)(z\widetilde{A}_0(z,s)-\widetilde{B}(z,s))$$

$$-\frac{1}{\theta}\sum_{j=1}^{N}m_{i,j}(s)(\widetilde{A}_j(z,s)-\widetilde{B}(z,s))z^j,$$

where $\theta = s+\lambda(1-X(z))$.

To determine $\Pi_i(z,s)$ completely, we have to find $m_i(z,s)$ and $m_{i,j}(s)$, $j=0,1,\ldots,N$. From the theory of Markov renewal processes (see, for example, Chapter 10 of [5]), we know that

$$M(t)=D(t)+Q(\cdot)*M(t),$$

where $M(t)=\{M_{i,j}(t), i=0,1,\ldots, j=0,1,2,\ldots\}$ is the Markov renewal matrix of $Q(\cdot)$, $D(t)=\{\delta_{i,j}\delta(t),\ i=0,1,\ldots,\ j=0,1,2,\ldots\}$, $\delta_{i,j}$ is Kronecker delta and $\delta(t)=0$ for $t<0$ and $\delta(t)=1$ for $t\geq 0$.

We have the transform equations

$$m_{i,j}(s)=\delta_{i,j}+m_{i,0}(s)\widetilde{A}_{0,j}(s)+\sum_{k=1}^{j+1}m_{i,k}(s)\widetilde{A}_{k,j+1-k}(s), 0\leq j\leq N-1,$$

$$m_{i,j}(s)=\delta_{i,j}+m_{i,0}(s)\widetilde{A}_{0,j}(s)+\sum_{k=1}^{N}m_{i,k}(s)\widetilde{A}_{k,j+1-k}(s) \tag{5.4}$$

$$+\sum_{k=N+1}^{j+1}m_{i,k}(s)\widetilde{B}_{j+1-k}(s), j\geq N.$$

By routine calculation we have from (5.4) that

$$(z-\widetilde{B}(z,s))m_i(z,s)=z^{i+1}+m_{i,0}(s)(z\widetilde{A}_0(z,s)-\widetilde{B}(z,s))$$

$$+\sum_{j=1}^{N}m_{i,j}(s)\left[\widetilde{A}_j(z,s)-\widetilde{B}(z,s)\right]z^j \tag{5.5}$$

$$=z^{i+1}-\frac{1}{\lambda+s}m_{i0}(s)(s+\lambda-\lambda X(z))\widetilde{B}(z,s)$$

$$+\sum_{j=1}^{N}\left(\frac{\lambda}{\lambda+s}m_{i0}(s)x_i+m_{ij}(s)\right)\left[\widetilde{A}_j(z,s)-\widetilde{B}(z,s)\right]z^j.$$

We derive readily from the theory of delayed renewal processes that

$$m_{i,j}(s) = \begin{cases} \prod_{k=j}^{i-1} \widetilde{G}_k(s) \frac{1}{1-\widetilde{K}_j(s)} & \text{if } i > j, \\ \frac{1}{1-\widetilde{K}_j(s)} & \text{if } i = j, \\ \widetilde{H}_{ij}(s) \frac{1}{1-\widetilde{K}_j(s)} & \text{if } i < j, \end{cases} \tag{5.6}$$

where $\widetilde{G}_k(s) = \widetilde{G}(s)$ for $k \geq N$. Thus we have determined $m_i(z,s)$ (5.5) and $m_{ij}(s)$ (5.6). Therefore, we have from (5.3) and (5.5) that

$$\Pi_i(z,s) = \frac{1}{s+\lambda-\lambda X(z)}(z^{i+1} + (1-z)m_i(z,s)). \tag{5.7}$$

By differentiating (5.7) at $z=1$ we have, for the mean queue length,

$$\widetilde{L}(s \mid i) = \frac{1}{s^2}(\lambda \bar{x} + (i+1-m_i(1,s))s), \tag{5.8}$$

where

$$m_i(1,s) = \frac{1 - \frac{s}{\lambda+s} m_{i0}(s)\widetilde{S}(s) + \sum_{j=1}^{N}\left(\frac{\lambda}{\lambda+s} m_{i0}(s)x_i + m_{ij}(s)\right)[\widetilde{S}_j(s) - \widetilde{S}(s)]}{1-\widetilde{S}(s)}.$$

- **Stationary probabilities $\pi_j = \lim_{t\to\infty}\pi_j(t)$**

For the derivation of the relationship between the stationary queue length distribution and the stationary queue length at departures, we use the fundamental mean E. The fundamental mean E of the Markov renewal process with transition matrix $Q(\cdot)$ is the inner product of the invariant probability vector $\mathbf{p} = (p_0, p_1, p_2, \ldots)$ of $Q(\infty)$ and the vector $\int_0^\infty x dQ(x)\mathbf{e}$ of $Q(\cdot)$, where $\mathbf{e} = (1,1,\ldots,1)^t$. That is, $E = \mathbf{p}\int_0^\infty x dQ(x)\mathbf{e}$. Note that the quantity $1/E$ may be interpreted as the rate at which transitions occur in the stationary version of the Markov renewal process $Q(\cdot)$. We have from the theory of Markov renewal processes (see, for example, Chapter 10 of [5]) that

$$\lim_{t\to\infty} M_{i,j}(t) = \lim_{s\to 0} s m_{i,j}(s) = p_j/E. \tag{5.9}$$

We get from (5.2) and (5.9) that

$$\pi_0 = \frac{1}{\lambda}\frac{p_0}{E},$$

$$\pi_j = \sum_{k=1}^{j}\left(\frac{p_0}{E}x_k + \frac{p_k}{E}\right)\int_0^\infty c_{j-k}(t)(1-S_k(t))dt, \; j \geq 1. \tag{5.10}$$

We have from (5.5), (5.9) and (3.3) that

$$\lim_{s\to 0} s m_i(z,s) = P(z)/E. \tag{5.11}$$

Employing (5.7) and (5.11), the probability generating function $\Pi(z) = \lim_{s\to 0} s\Pi(z,s)$ of $\{\pi_j\}$ is obtained as

$$\Pi(z) = \frac{1}{\lambda E}\frac{1-z}{1-X(z)}P(z). \tag{5.12}$$

We have from the fact $1 = \Pi(1-) = P(1-)$ and (5.12) that $E = 1/(\lambda \overline{x})$. Thus we have the well-known formula (e.g. see Dshalalow [6, pp. 68])

$$\Pi(z) = \overline{x}\frac{1-z}{1-X(z)}P(z), \tag{5.13}$$

linking the probability generating function $\Pi(z)$ for the limiting distribution of the number of customers present at an arbitrarily selected instant of time and the corresponding probability generating function $P(z)$ no the embedded chain. The mean queue length is given by

$$L = \Pi'(1) = \frac{2\overline{x}\,P'(1) - X''(1)}{2\overline{x}}. \tag{5.14}$$

Remark: If $N = 0$, that is, in the ordinary $\mathrm{M^X/G/1}$ queue, then we have $p_0 = (1-\beta)/\overline{x}$ and hence (5.13) becomes

$$\begin{aligned}\Pi(z) &= (1-\beta)\frac{(z-1)\widetilde{B}(z)}{z-\widetilde{B}(z)}\\ &= (1-\beta)\frac{(z-1)\widetilde{S}(\lambda - \lambda X(z))}{z - \widetilde{S}(\lambda - \lambda X(z))},\end{aligned}$$

which coincides with the classical result (e.g., see Takagi [16]).

6. Numerical Results

In this section, we present some numerical results graphically for the transient mean queue length and the queue length distributions to demonstrate the computability of our results. The parameters for arrival process used here are as follows:

- Arrival rate of batches is $\lambda = 0.4$;
- Batch size distribution is geometric with mean 2.5, that is,

$$x_n = 0.4(0.6)^{n-1}, \quad n = 1, 2, \ldots.$$

We use the threshold $N = 6$ and the LSTs of service times are as follows:

- $\widetilde{S}_1(s) = \frac{2}{2+s}$,
- $\widetilde{S}_2(s) = \widetilde{S}_3(s) = \sum_{i=1}^{5} a_i\left(\frac{\mu_i}{\mu_i + s}\right)$,
  where $\boldsymbol{a} = (0.05, 0.15, 0.2, 0.25, 0.35)$ and $\boldsymbol{\mu} = (0.25, 0.75, 1{,}0, 1.25, 1.75)$,
- $\widetilde{S}_4 = \widetilde{S}_5(s) = \widetilde{S}_6(s) = 0.4\left(\frac{4}{4+s}\right)^2 + 0.1\left(\frac{1}{1+s}\right)^5 + 0.1\left(\frac{10}{10+s}\right)$,
- $\widetilde{S}_7(s) = \left(\frac{10}{10+s}\right)\left(\frac{5}{5+s}\right)^2$.

The mean service times are given as $\overline{s}_1 = 1.5$, $\overline{s}_2 = \overline{s}_3 = 1.0$, $\overline{s}_4 = \overline{s}_5 = \overline{s}_6 = 0.75$ and $\overline{s} = 0.5$.

For finding the stationary distribution and mean queue length, the following procedure can be used:

1. Calculate $\boldsymbol{p}^* = (p_0, \ldots, p_N)$ by solving $qQ^* = q$ and using (3.4) and (3.5).
2. The probabilities p_k, $k \geq N+1$ can be obtained recursively from (3.2) as

$$p_{k+1} = \frac{1}{B_0}\left(p_k - (p_0 A_{0k} + \sum_{i=1}^{N} p_i A_{i,k-i+1} + \sum_{i=N+1}^{k} p_i B_{k-i+1} \right). \tag{6.1}$$

However, this formula is usually found to be numerically unstable as noted in Ramaswami [13] so instead of (6.1), the formula

$$p_k = \frac{p_0 \bar{A}_{0k} + \sum_{j=1}^{N} p_j \bar{A}_{j,k-j+1} + \sum_{j=N+1}^{k-1} p_j \bar{B}_{k-j+1}}{1 - \bar{B}_1}, \quad k \geq N+1, \tag{6.2}$$

where $\bar{A}_{ij} = \sum_{k=j}^{\infty} A_{ik}$ and $\bar{B}_j = \sum_{k=j}^{\infty} B_k$, seems to be more appropriate to have numerical results ([13]).

3. For the mean queue length L, use (3.7) and (5.14).

Since our transient results are complicated LSTs, we need to invert them numerically. There are many algorithms for numerical inversion of Laplace transforms (see [1]). Here we adopt Algorithm 368 in *Commun. ACM* [15], called the Gaver-Stehfest method, which seems to be easily available for our formulae. Briefly discussing the Gaver-Stehfest method (for further details, see [15]), an approximate numerical inversion $f(t)$ of $f^*(s)$ at time t is given by

$$f(t) = \frac{\ln 2}{t} \sum_{i=1}^{M} V_i f^*\!\left(\frac{\ln 2}{t} i\right), \tag{6.3}$$

where the coefficients

$$V_i = (-1)^{M/2+i} \sum_{k=\lfloor \frac{i+1}{2} \rfloor}^{\min(i, M/2)} \frac{k^{M/2}(2k)!}{(M/2-k)!k!(k-1)!(i-k)!(2k-i)!}$$

depends only on the constant M.

Inversion procedures for $\tilde{L}(s \mid i)$ and $\tilde{\pi}_{ij}(s)$ are as follows:

1. For $k = 1, 2, \ldots, M$,
   (1) $s_k = \frac{\ln 2}{t} k$.
   (2) Find $\tilde{G}(s_k)$ by solving (4.2) and then compute $\tilde{G}_i(s_k)$, $i = N-1, \ldots, 0$ recursively.
   (3) For each j, solve the linear system $\boldsymbol{C}_j(s_k)\tilde{\boldsymbol{H}}_j(s_k) = -\boldsymbol{b}_j(s_k)$ in (4.8).
   (4) Compute $K_j(s_k)$, $j = 0, 1, \ldots, N$.
   (5) Compute $\tilde{m}_{ij}(s_k)$ from (5.6).
   (6) Calculate $\tilde{L}(s_k \mid i)$ from (5.8) and $\tilde{\pi}_{ij}(s_k)$ from (5.2).
2. Then calculate $L(t \mid i)$ and $\pi_{ij}(t)$ using (6.3).

Figures 1 and 2 plot the transient queue length distributions for $\pi_{i0}(t)$ and $\pi_{i6}(t)$ and show how the transient results approach to the stationary ones. Figure 3 shows the transient behavior of mean queue length as time varies with various initial conditions. We observe that, as expected, as time t increases, the initial distribution gradually spreads out to approach the stationary distribution.

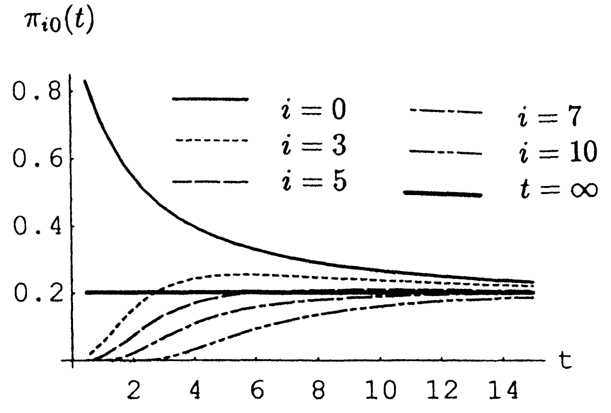

Figure 1: Queue length distribution $\pi_{i0}(t)$ with $i = 0, 3, 5, 7, 10$

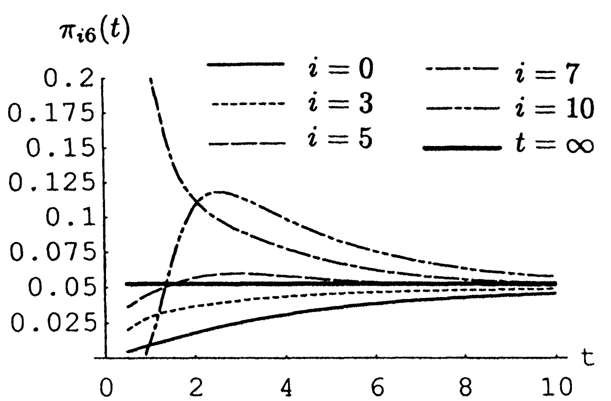

Figure 2: Queue length distribution $\pi_{i6}(t)$ with $i = 0, 3, 5, 7, 10$

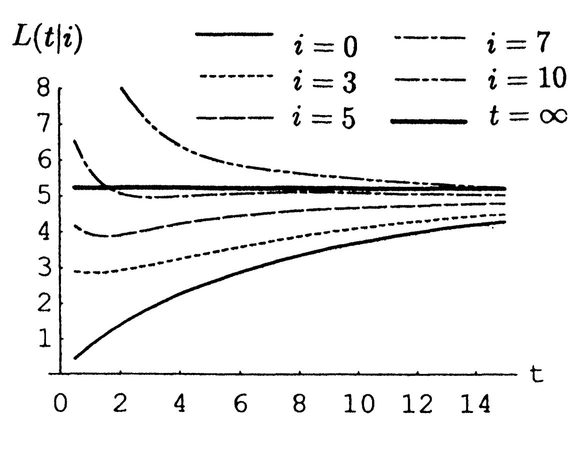

Figure 3: Mean Queue Length $L(t \mid i)$ with $i = 0, 3, 5, 7, 10$

7. The Virtual Waiting Time

We assume the first-come first-served discipline. The virtual waiting time $U(t)$ si the length of time the first customer in the batch arriving at time t would have to wait before entering service.

Let $W(t,x) = P(U(t) \leq x)$. For the event $\{U(t) \leq x\}$, there are three possibilities to consider (see Figure 4).

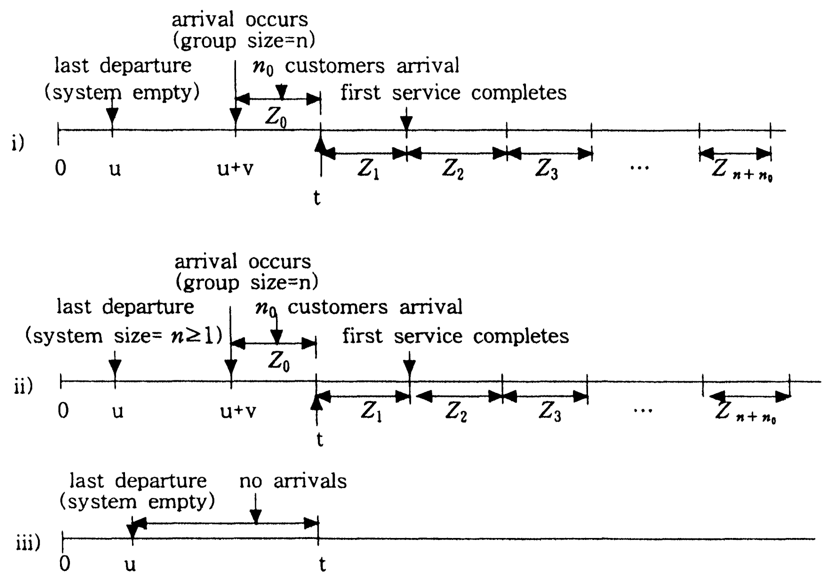

Figure 4: Scenarios for the virtual waiting time at time t

Case (*i*): At time t, the server is busy and the last state visited by the embedded Markov renewal process is 0. This means t falls during a first service of a busy period.

Case (*ii*): The server is busy at time t and the last state visited by the embedded Markov renewal process is some state $k \geq 1$. In this case, t falls during the second or later service of a busy period.

Case (*iii*): The server is idle at time t.

We first consider each case separately and then combine them.

Case (*i*): Suppose that embedded Markov renewal process visits the state 0 at time $u < t$ and a batch of size n arrives at $u+v < t$. Let n_0 be the number of customers who arrive during the time interval $[u+v, t]$. Since there are no departures in $[u+v, t]$, the remaining service time of the customer being serve at time t, show service time is Z, is $Z_1 = Z - (t-(u+v)) = Z + u + v - t$. Let Z_j, $(j = 2, 3, \ldots, n+n_0)$ be the service time of the jth customer in the system at time t. Then the virtual waiting time is the time period $\sum_{j=1}^{n+n_0} Z_j$. To specify the discussion above, we introduce some notations:

T: interarrival time of groups

n_j: number of customers arriving during Z_j with $Z_0 = t - (u+T)$

$l(n, n_0) = n + n_0$: (In cases of no confusions, we write l instead of $l(n, n_0)$ for the notational simplicity.)

$k_j = n + n_0 + n_1 + \ldots + n_{j-1} - (j-1),\ j = 1, 2, \ldots, l-1.$

By the law of total probability, the probability of $\{U(t) \le x\}$ in Case (i) is given by

$$\sum_{n=1}^{\infty} x_n \sum_{n_0=0}^{\infty} \int_{u=0}^{t} \int_{v=0}^{t-u} P\left(\sum_{j=1}^{n+n_0} Z_j \le x \,\middle|\, n, n_0, , u, T = v \right) \times \lambda e^{-\lambda v} c_{n_0}(t-u-v) dv\, dM_{i,0}(u), \tag{7.1}$$

where $P(E \mid n, n_0, u, T = v)$ means the conditional probability of E given that the Markov renewal process visits the state 0 at time u and a group of size n arrives at $u+v$ and n_0 customers arrive during $[u+v, t]$. Given that $n, n_0, n_1, \ldots, n_l,\ u, T = v$, the random variables $Z_1, Z_2, \ldots, Z_l$ are independent and have the conditional distributions

$$\begin{aligned} P(w_1 < Z_1 \le w_1 + dw_1 \mid n, n_0, u, T = v) &= dS_n(t-u-v+w_1), \\ P(w_j < Z_j \le w_j + dw_j \mid n, n_0, n_1, \ldots, n_{j-1}, u, T = v) &= dS_{k_j}(w_j), j = 2, 3, \ldots, l. \end{aligned} \tag{7.2}$$

Repeatedly applying the law of total probability to (7.1) yields

$$\sum_{n=1}^{\infty} x_n \sum_{n_0=0}^{\infty} \int_{u=0}^{t} \int_{v=0}^{t-u} \lambda e^{-\lambda v} c_{n_0}(t-u-v) V_{n,n_0}(t-u-v, x) dv\, dM_{i,0}(u), \tag{7.3}$$

where

$$V_{n,n_0}(t,x) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{l-1}=0}^{\infty} \int_{w_1=0}^{x} dS_n(t+w_1) c_{n_1}(w_1) \int_{w_2=0}^{x-w_1} dS_{k_2}(w_2) c_{n_2}(w_2) \cdots \int_{w_{l-1}=0}^{x-\sum_{j=0}^{l-2} w_j} dS_{k_{l-1}}(w_{l-1}) c_{n_{l-1}}(w_{l-1}) \int_{w_l=0}^{x-\sum_{j=0}^{l-2} w_j} dS_{k_l}(w_l). \tag{7.4}$$

Case (ii): By arguments similar to those of Case (i), Case (ii) contributes the formula

$$\sum_{n=1}^{\infty} \sum_{n_0=0}^{\infty} \int_{u=0}^{t} c_{n_0}(t-u) V_{n,n_0}(t-u, x) dv dM_{i,n}(u). \tag{7.5}$$

Case (iii): In this case, the waiting time is clearly 0.

Combining the results obtained for each case, we have that the distribution function $W(t,x)$ of $U(t)$ given $I_0 = i$ is

$$W(t,x) = \pi_0(t)$$

$$+\sum_{n=1}^{\infty} x_n \sum_{n_0=0}^{\infty} \int_{u=0}^{t} \int_{v=0}^{t-u} \lambda e^{-\lambda v} c_{n_0}(t-u-v) V_{n,n_0}(t-u-v,x)\, dv\, dM_{i,0}(u)$$

$$+\sum_{n=1}^{\infty} \sum_{n_0=0}^{\infty} \int_{u=0}^{t} c_{n_0}(t-u) V_{n,n_0}(t-u,x)\, dM_{i,n}(u). \tag{7.6}$$

In terms of the double Laplace transform $\widetilde{W}^*(\tau,s) = \int_0^\infty \int_0^\infty e^{-\tau t - sx} W(dt,dx)$, we have

$$\widetilde{W}^*(\tau,s) =$$

$$\sum_{n=1}^{\infty} \sum_{n_0=0}^{\infty} \left(x_n \frac{\lambda}{\lambda+\tau} m_{i,0}(\tau) + m_{i,n}(\tau) \right) \int_0^\infty e^{-\tau t} c_{n_0}(t) \widetilde{V}_{n,n_0}(t,s)\, dt, \tag{7.7}$$

where

$$\widetilde{V}_{n,n_0}(t,s) = \int_0^\infty e^{-sx} V_{n,n_0}(t,dx) + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{l-1}=0}^{\infty} \int_{x=0}^{\infty} e^{-sx} c_{n_1}(x)\, dS_n(t-x)$$

$$\cdot \prod_{j=2}^{l-1} \int_0^\infty e^{-sx} c_{n_j}(x)\, dSk_j(x) \widetilde{S}_{k_l}(s). \tag{7.8}$$

The Laplace transform $\widetilde{W}(s)$ of limiting distribution $W(x) = \lim_{t\to\infty} W(t,x)$ is given by

$$\begin{aligned} \widetilde{W}(s) &= \lim_{\tau\to 0} \tau \widetilde{W}^*(\tau,s) \\ &= \pi_0 + \lambda \bar{x} \sum_{n=1}^{\infty} \sum_{n_0=0}^{\infty} (p_0 x_n + p_n) \int_0^\infty c_{n_0}(t) \widetilde{V}_{n,n_0}(t,s)\, dt. \end{aligned} \tag{7.9}$$

Remark: Formula (7.9) seems to be quite complicated, so it is hard to have a closed form for general N. Here we write down the special cases $N = 0$ and $N = 1$.

Case $N = 0$: When $N = 0$, (7.8) becomes

$$\widetilde{V}_{n,n_0}(t,s) = \int_0^\infty e^{-sx} dS(t+x) [\widetilde{S}(s)]^{n+n_0-1}. \tag{7.10}$$

For notational simplicity, let

$$\zeta = \widetilde{S}(s), \quad \eta = \lambda(1 - X(\widetilde{S}(s))).$$

Routing calculation yields from (7.10) that

$$\begin{aligned} \sum_{n=1}^{\infty} x_n \sum_{n_0=0}^{\infty} c_{n_0}(t) \widetilde{V}_{n,n_0}(t,s) &= \frac{X(\zeta)}{\zeta} e^{-\eta t} \int_0^\infty e^{-sx} dS(t+x), \\ \sum_{n=1}^{\infty} p_n \sum_{n_0=0}^{\infty} c_{n_0}(t) \widetilde{V}_{n,n_0}(t,s) &= \frac{P(\zeta) - p_0}{\zeta} e^{-\eta t} \int_0^\infty e^{-sx} dS(t+x). \end{aligned} \tag{7.11}$$

From (7.9) and (7.11), we obtain

$$\begin{aligned}\widetilde{W}(s) &= \pi_0 + \lambda\overline{x}\,\frac{p_0(X(\zeta)-1)+P(\zeta)}{\zeta}\int_0^\infty e^{-\eta}\int_0^\infty e^{-sx}dS(t+x)dt \\ &= \pi_0 + \lambda\overline{x}\,\frac{p_0(X(\zeta)-1)+P(\zeta)}{\zeta}\,\frac{1}{s-\eta}(\widetilde{S}(\eta)-\zeta) \qquad (7.12)\\ &= \frac{(1-\beta)s}{s-\lambda+\lambda X(\widetilde{S}(s))},\end{aligned}$$

where the third equality is obtained from $p_0 = (1-\beta)/\overline{x}$, $\pi_0 = 1-\beta$ and

$$P(\zeta)+p_0(X(\zeta)-1) = \frac{p_0\zeta(1-X(\zeta))}{\widetilde{S}(\eta)-\zeta},$$

which in turn is obtained from (3.3). If $X(z) = z$, the case of the ordinary M/G/1 queue, then (7.12) becomes

$$\widetilde{W}(s) = \frac{(1-\beta)s}{s-\lambda+\lambda\widetilde{S}(s)},$$

which is the well-known Pollaczek-Khinchin formula.

Case $N = 1$: After a calculation similar to but more lengthy than that for $N = 0$, we get

$$\begin{aligned}\widetilde{V}_{n,n_0}(t,s) &= (\widetilde{S}_1(s)-\widetilde{S}(s)) \\ &\times\Big(\widetilde{S}(\lambda+s)\Big)^{n+n_0-2}\int_0^\infty e^{-(\lambda+s)x}dS_n(t+x)1(n+n_0\geq 2) \\ &+\Big((\widetilde{S}(\lambda+s)\Big)^{n+n_0-1}\int_0^\infty e^{-sx}dS_n(t+x)\end{aligned}$$

for $N = 1$. Hence after routine calculation we have

$$\begin{aligned}\widetilde{W}(s) &= \pi_0 + \lambda\overline{x}\big[(p_0x_1+p_1)\cdot\mathbf{I} \\ &+\Big(P(\widetilde{S}(\lambda+s))-p_0(1-X(\widetilde{S}(\lambda+s)))\Big)\cdot\mathbf{II} \qquad (7.13)\\ &+\Big(P(\widetilde{S}(s))-p_0(1-X(\widetilde{S}(s)))\Big)\cdot\mathbf{III}\Big],\end{aligned}$$

where

$$\mathbf{I} = \frac{\widetilde{S}_1(s)-\widetilde{S}(s)}{\widetilde{S}(\lambda+s)}\left[\frac{\widetilde{S}_1(\omega)-\widetilde{S}(\omega)-\widetilde{S}_1(\lambda+s)+\widetilde{S}(\lambda+s)}{s+\lambda X(\widetilde{S}(\lambda+s))} - \frac{\widetilde{S}_1(\lambda)-\widetilde{S}(\lambda+s)}{s}\right]$$

$$+\frac{\widetilde{S}_1(\eta)-\widetilde{S}(\eta)-\widetilde{S}_1(s)+\widetilde{S}(s)}{s-\lambda(1-X(\widetilde{S}(s)))},$$

$$\mathbf{II}=\frac{\widetilde{S}_1(s)-\widetilde{S}(s)}{\widetilde{S}(\lambda+s)^2}\frac{\widetilde{S}(\omega)-\widetilde{S}(\lambda+s)}{s+\lambda X(\widetilde{S}(\lambda+s))},$$

$$\mathbf{III}=\frac{\widetilde{S}(\eta)-\widetilde{S}(s)}{s-\lambda(1-X(\widetilde{S}(s)))}\frac{1}{\widetilde{S}(s)}.$$

Here $\omega=\lambda(1-X(\widetilde{S}(\lambda+s)))$ and $\eta=\lambda(1-X(\widetilde{S}(s)))$ as for $N=0$. We note from (3.3) and (2.1) that for $N=1$

$$\begin{aligned}P(z)-p_0(1-X(z)) &= \frac{z}{z-\widetilde{B}(z,0)}\Big[(p_0x_1+p_1)(\widetilde{A}_1(z,0)-\widetilde{B}(z,0))-p_0(1-X(z))\Big] \\ &= \frac{z}{z-\widetilde{S}(\xi)}\Big[(p_0x_1+p_1)\Big(\widetilde{S}_1(\xi)-\widetilde{S}(\xi)\Big)-p_0(1-X(z))\Big],\end{aligned} \tag{7.14}$$

where $\xi=\lambda(1-X(z))$. Thus we have from (7.13) and (7.14) that

$$\widetilde{W}(s)=\pi_0+\lambda\bar{x}\,(p_0x_1+p_1)\cdot\mathbf{IV}+\lambda\bar{x}\,p_0\cdot\mathbf{V},$$

where

$$\mathbf{IV}=\frac{\widetilde{S}_1(s)-\widetilde{S}(s)}{\widetilde{S}(\lambda+s)}\left[\frac{\widetilde{S}(\lambda+s)-\widetilde{S}_1(\lambda+s)}{s+\lambda X(\widetilde{S}(\lambda+s))}-\frac{\widetilde{S}_1(\lambda)-\widetilde{S}_1(\lambda+s)}{s}\right]$$

$$+\frac{\widetilde{S}(s)-\widetilde{S}_1(s)}{s-\lambda(1-X(\widetilde{S}(s)))},$$

$$\mathbf{V}=\frac{\widetilde{S}_1(s)-\widetilde{S}(s)}{\widetilde{S}(\lambda+s)}\frac{1-X(\widetilde{S}(\lambda+s))}{s+\lambda X(\widetilde{S}(\lambda+s))}+\frac{1-X(\widetilde{S}(s))}{s-\lambda(1-X(\widetilde{S}(s)))}.$$

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