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EXISTENCE OF PERIODIC SOLUTION FOR FIRST ORDER NONLINEAR NEUTRAL DELAY EQUATIONS

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In this paper by using the coincidence degree theory, sufficient conditions are given for the existence of periodic solutions of the first order nonlinear neutral delay differential equation.

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1. Introduction

In [3], Kuang and Feldstein proposed to study the existence of a periodic solution for the first order periodic neutral delay equation. In particular, Gopalsamy, He and Wen [2] studied the existence of periodic solutions of the first order neutral delay logistic equation. In this paper, we discuss the following nonlinear neutral delay equation:

$$[x(t)+cx(t-\tau)]'+g(t,x(t-\sigma))=p(t), \tag{1}$$

where τ,σ and c are constants, and $\tau\geq 0$, $\sigma\geq 0$, $|c|<1$; $g\in C(R^2,R)$, $g(t,x)$ is a function with period $T(>0)$ for t, and $g(t,x)$ is nondecreasing for x in $[0,+\infty)$; $p\in C(R,R)$, $p(t,T)=p(t)$ for $t\in R$ and $\int_0^T p(t)dt=0$. Using coincidence degree theory developed by Mawhin [1], we establish a theorem of the existence of periodic solutions with period T of Equation (1).

2. Main Result

The following result provides sufficient conditions for the existence of periodic solution of Equation (1).

Theorem: *Assume that there exist constants $D>0$ and $M>0$ such that*

$$xg(t,x) > 0 \text{ for } t \in R \text{ and } |x| \geq D, \tag{2}$$

$$g(t,x) \geq -M \text{ for } t \in R \text{ and } x \leq -D, \tag{3}$$

and

$$|g(t,x)| \leq g(t, |x|) \text{ for } (t,x) \in R^2. \tag{4}$$

Then there exists a periodic solution with period T of Equation (1).

In order to prove the above theorem, we introduce the following preliminaries.

Let X and Z be two Banach spaces. Consider an operator equation,

$$Lx = \lambda Nx,$$

where $L\colon \mathrm{Dom}L \cap X \to Z$ is a linear operator and $\lambda \in [0,1]$ is a parameter. Let P and Q denote two projectors,

$$P\colon \mathrm{Dom}L \cap X \to \mathrm{Ker}L \text{ and } Q\colon Z \to Z/\mathrm{Im}L.$$

We will use the following result of Mawhin [1].

Lemma 1: *Let X and Y be two Banach spaces and L be a Fredholm mapping with index null. Assume that Ω is open bounded in X and $N\colon \bar{\Omega} \to Z$ is L-compact on $\bar{\Omega}$. Furthermore, suppose that*

(*a*) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \in \mathrm{Dom}L$,*

$$Lx \neq \lambda Nx;$$

(*b*) *for each $x \in \partial\Omega \cap \mathrm{Ker}L$,*

$$QNx \neq 0,$$

and

$$\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0,$$

then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \mathrm{Dom}L$.

To prove Lemma 2, we make the following preparations. Set

$$X\colon = \{x \in C^1(R,R) \mid x(t+T) = x(t)\}$$

and define the norm on X as $\|x\| = \max_{t \in [0,T]} \{|x(t)|, |x'(t)|\}$. Similarly, set

$$Z\colon = \{z \in C(R,R) \mid z(t+T) = z(t)\}$$

and define the norm on Z as $\|z\|_0 = \max_{t \in [0,T]} |z(t)|$. Then both $(X, \|\cdot\|)$ and $(Z, \|\cdot\|_0)$ are Banach spaces. Define respectively the operators L and N as

$$L\colon X \to Z, \quad x(t) \mapsto x'(t), \tag{5}$$

and

$$N: X \rightarrow Z,\ x(t) \mapsto -cx'(t-\tau) - g(t, x(t-\sigma)) + p(t). \tag{6}$$

We know that $\mathrm{Ker}L = R$. Define, respectively, the projective operators P and Q as

$$P: X \rightarrow \mathrm{Ker}L,\quad x \mapsto Px = \frac{1}{T}\int_0^T x(t)dt, \tag{7}$$

and

$$Q: Z \rightarrow Z/\mathrm{Im}L,\quad z \mapsto Qz = \frac{1}{T}\int_0^T z(t)dt. \tag{8}$$

Hence, we have $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q$. Consider the equation

$$x'(t) + \lambda cx'(t-\tau) + \lambda g(t, x(t-\sigma)) = \lambda p(t), \tag{9}$$

where $\lambda \in (0,1)$ is a parameter.

Lemma 2: *Suppose that conditions (2)-(4) are satisfied. If $x(t)$ is any periodic solution with period T of Equation (9), then there exist positive constants $D_j (j = 0,1)$ independent of λ and such that*

$$|x(t)| \leq D_0 \text{ and } |x'(t)| \leq D_1,\ \ t \in [0,T]. \tag{10}$$

Proof: Suppose that $x(t)$ is a periodic solution with period T of Equation (9). By integrating (9) from 0 to T, we find

$$\int_0^T g(t, x(t-\sigma))dt = 0. \tag{11}$$

Set

$$E_1 = \{t \in [0,T] \mid x(t-\sigma) > D\},\quad E_2 = [0,T] \backslash E_1.$$

Since $g \in C(R^2, R)$ and $g(t,x)$ is a function with period T for t, we know that

$$\sup_{(t,x) \in R \times [-D,D]} |g(t,x)| = \max_{(t,x) \in [0,T] \times [-D,D]} |g(t,x)| < \infty.$$

From (2) and (3) we see that

$$\int_{E_2} |g(t,x(t-\sigma))| dt \leq T \max\{M, \sup_{(t,x) \in R \times [-D,D]} |g(t,x)|\}. \tag{12}$$

Using (2) and (11), we have

$$\int_{E_1} |g(t, x(t-\sigma))| dt = \int_{E_1} g(t, x(t-\sigma))dt$$

$$= -\int_{E_2} g(t,x(t-\sigma))dt \tag{13}$$

$$\leq \int_{E_2} |\, g(t,x(t-\sigma))\,|\, dt.$$

By (12) and (13), we have

$$\int_0^T |\, g(t,x(t-\sigma))\,|\, dt \leq 2T \max\{M, \sup_{(t,x)\in R\times[-D,D]} |\, g(t,x)\,|\,\}.$$

Thus

$$\int_0^T |\, g(t,x(t-\sigma))\,|\, dt \leq K_0, \tag{14}$$

where K_0 is a positive constant. Since $x'(t)$ is a periodic function with period T, it follows from (9) that

$$\int_0^T |\, x'(t)\,|\, dt \leq \lambda\,|\, c\,| \int_0^T |\, x'(t-\sigma)\,|\, dt + \lambda \int_0^T |\, g(t,x(t-\sigma))\,|\, dt + \lambda \int_0^T |\, p(t)\,|\, dt$$

$$\leq |\, c\,| \int_0^T |\, x'(t)\,|\, dt + \int_0^T |\, g(t,x(t-\sigma))\,|\, dt + T \max_{t\in[0,T]} |\, p(t)\,|\,. \tag{15}$$

From (14) and (15) we see that

$$\int_0^T |\, x'(t)\,|\, dt \leq |\, c\,| \int_0^T |\, x'(t)\,|\, dt + K_1, \tag{16}$$

where $K_1 = K_0 + T \max_{t\in[0,T]} |\, p(t)\,|$. It follows from (16) that

$$\int_0^T |\, x'(t)\,|\, dt \leq K_2, \tag{17}$$

where $K_2 = K_1/(1-|\, c\,|)$. By (2) and (11), there exists $t_1 \in [0,T]$ such that $|\, x(t_1-\sigma)\,| < D$. Taking $t_1 - \sigma = nT + t_2$, where n is an integer and $t_2 \in [0,T]$, we have

$$|\, x(t_2)\,| < D. \tag{18}$$

Then, by (17) and (18), we conclude that for any $t \in [0,T]$,

$$|x(t)| = |x(t_2) + \int_{t_2}^{t} x'(s)ds| \le |x(t_2)| + \int_0^T |x'(t)|dt$$
$$\le |x(t_2)| + K_2 < D_0, \tag{19}$$

where $D_0 = D + K_2$. Since $x(t)$ is a periodic function with period T, from (19) we see that $|x(t-\sigma)| < D_0$ for $t \in [0,T]$. Note that $g(t,x)$ is nondecreasing for x in $[0, +\infty)$. Hence we have for any $t \in [0,T]$,

$$|g(t,x(t-\sigma))| \le g(t, |x(t-\sigma)|) \le g(t,D_0). \tag{20}$$

Note that if $g(t,D_0)$ is a periodic continuous function, then there exists a positive constant K_3, for any $t \in [0,T]$, such that

$$g(t,D_0) \le |g(t,D_0)| \le K_3. \tag{21}$$

From (9), (20), (21) and note that $x'(t)$ is a periodic function with period T, we conclude that for any $t \in [0,T]$

$$\begin{aligned}|x'(t)| &\le \lambda|c|\,|x'(t-\tau)| + \lambda|g(t,x(t-\sigma))| + \lambda|p(t)| \\ &\le |c| \max_{t\in[0,T]}|x'(t)| + g(t,x(t-\sigma)) + \max_{t\in[0,T]}|p(t)| \\ &\le |c| \max_{t\in[0,T]}|x'(t)| + g(t,D_0) + \max_{t\in[0,T]}|p(t)| \\ &\le |c| \max_{t\in[0,T]}|x'(t)| + K_3 + \max_{t\in[0,T]}|p(t)|.\end{aligned} \tag{22}$$

Letting $K_4 = K_3 + \max_{t\in[0,T]}|p(t)|$, for any $t \in [0,T]$ we have

$$|x'(t)| \le |c| \max_{t\in[0,T]}|x'(t)| + K_4. \tag{23}$$

By (23) we obtain

$$\max_{t\in[0,T]}|x'(t)| \le |c| \max_{t\in[0,T]}|x'(t)| + K_4.$$

Thus,

$$\max_{t\in[0,T]}|x'(t)| \le D_1, \tag{24}$$

where $D_1 = K_4/(1-|c|)$. The proof of Lemma 2 is complete.

Proof of the Theorem: Suppose that $x(t)$ is any periodic solution with period T of Equation (9). By Lemma 2, there exist positive constants D_j $(j = 0,1)$, which are independent of λ, such that

$$|x(t)| \le D_0 \text{ and } |x'(t)| \le D_1,\ t \in [0,T].$$

Let $D_2 = \max\{D_0, D_1, D\} + 1$, and

$$\Omega: = \{x \in X \mid \| x \| < D_2\}.$$

In view of (2), we see that

$$-\frac{1}{T}\int_0^T g(t, -D_2)dt > 0 \text{ and } -\frac{1}{T}\int_0^T g(t, D_2)dt < 0. \tag{25}$$

By (5)-(7) and (8), we know that L is the Fredholm operator with index null and N is L-compact on $\bar{\Omega}$ (see [1]). In terms of evaluation of a bound of periodic solutions in Lemma 2, we know that for any $x \in \partial\Omega \cap \mathrm{Dom}L$ and $\lambda \in (0,1)$, $Lx \neq \lambda Nx$. Since for any $x \in \partial\Omega \cap \mathrm{Ker}L$, $x = D_2(>D)$ or $x = -D_2$, in view of (25) and $\int_0^T p(t)dt = 0$, we have

$$QNx = \frac{1}{T}\int_0^T [-cx'(t-\tau) - g(t, x(t-\sigma)) + p(t)]dt$$

$$= -\frac{1}{T}\int_0^T g(t, \pm D_2)dt \neq 0,$$

which shows that

$$\deg\{QN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0.$$

By Lemma 1, there exists a periodic solution with period T of Equation (1). The proof is complete.

Example: Consider the equation

$$[x(t) - \frac{1}{3}x(t-\pi)]' + e^{\sin t}x(t-\pi)e^{x(t-\pi)} = \frac{4}{3}\cos t - \sin t. \tag{26}$$

It is easy to verify that for Equation 926), all the conditions of the theorem are satisfied with $D > 0$ and $M = 3$. Thus Equation (26) has a periodic solution with period 2π. We see that $x(t) = \sin t$ is such a periodic solution of Equation (26).

References

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