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# CONTINUITY PROPERTIES OF SOLUTIONS OF MULTIVALUED EQUATIONS WITH WHITE NOISE PERTURBATION

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In the paper, we consider a set-valued stochastic equation with stochastic perturbation in a Banach space. We prove first the existence theorem and then study continuity properties of solutions.

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## 1. Preliminaries

Problems of existence of solutions to set-valued differential equations were studied by many (see e.g., [3, 8, 9]). In particular, random cases were considered by the author in [11, 12].

In this paper we study the set-valued stochastic equation with white noise drift:

$$\begin{aligned} DX_t &= F(t, X_t)dt + \sigma_t dw_t, t \in I, \\ X_0 &= U \quad P.1, \end{aligned} \tag{$I$}$$

where $F$ and $U$ are given random set-valued mappings with values in the space $K_c(E)$, of all nonempty, compact and convex subsets of the separable Banach space $(E, \| \ \|)$, $I: = [0, T]$; $T > 0$. We assume also that there is a predictable stochastic process $\sigma$ with values in $E$. Finally, $(w_t)_{t \in I}$ denotes a real Wiener process. We interpret the above equation through its integral form as

$$X_t = U + \int_0^t F(s, X_s)ds + \int_0^t \sigma_s dw_s \quad P.1, \quad t \in I. \tag{$II$}$$

Integrals above are Aumann's integral of $F$ and stochastic (Itô) integral of $\sigma$, respectively.

The aim of this work is to study continuity properties of set-valued solutions of

 

$(I)$. First, we recall several notions needed in the sequel. In the space $K_c(E)$ we consider the Hausdorff metric $H$ (see e.g., [5, 7]): $H(A,B) = \max(\bar{H}(A,B), \bar{H}(B,A))$ for $A, B \in K_c(E)$, where $\bar{H}(A,B) = \sup_{a \in A} \inf_{b \in B} \| a - b \|$. By $\| A \|$ we denote the distance $H(A, 0)$. It can be proved that $(K_c(E), H)$ is a Polish metric space.

By $C_I = C(I, K_c(E))$ we denote the space of all $H$-continuous set-valued mappings. In this space we consider metric $\rho$ of uniform convergence:

$$\rho(X,Y) := \sup_{0 \leq t \leq T} H(X(t), Y(t)), \text{ for } X, Y \in C_I.$$

Then we have a Polish metric space.

Let $(\Omega, \mathfrak{F}, \mathfrak{F}_t, P)_{t \in I}$ be a given complete filtered probability space satisfying the usual conditions. We recall the notion of a multivalued $\mathfrak{F}_t$-adapted stochastic process. The family of set-valued mappings $X = (X_t)_{t \in I}$ is said to be a *multivalued $\mathfrak{F}_t$-adapted stochastic process* if for every $t \in I$, the mapping $X_t: \Omega \rightarrow K_c(E)$ is $\mathfrak{F}_t$-measurable, i.e., $\{\omega: X_t \cap V \neq \emptyset\} \in \mathfrak{F}_t$, for every open set $V \subset E$ (see e.g., [7]). It can be noted that $V$ can be chosen as a closed or Borel subset. If the mapping $t \rightarrow X_t(\omega)$ is $H$-continuous with probability one $(P.1)$ then we say it has *continuous paths.* In this case, the set-valued process $X$ can be thought as random element $X: \Omega \rightarrow C_I$. Let $(X_n)$ be a sequence of random elements with values in metric space $(S, \rho)$. Then we say that $X_n$ converges in probability to the random element $X: \Omega \rightarrow S$ $(X_n \xrightarrow{P} X)$, of for every $\epsilon > 0$, it holds true that $P(\rho(X_n, X) \geq \epsilon) \rightarrow 0$, as $n$ tends to infinity. It is known (see e.g., [13]) that $X_n \xrightarrow{P} X$ if and only if every subsequence of $(X_n)$ has a subsequence converging to $X$ with probability one $(P.1)$.

In the theory of differential equations in Banach space the notion of measure of noncompactness plays one of the central roles (see e.g., [1]). Let $B(E)$ denote a family of all nonempty and bounded subsets in $E$.

**Definition 1:** The mapping $\mathcal{N}: B(E) \rightarrow [0, \infty)$, defined by $\mathcal{N}(A) = \inf\{\epsilon > 0: A$ can be covered with a finite number of balls of radii $\leq \epsilon\}$, is called *Hausdorff (ball) measure of noncompactness.*

## 2. A Set-Valued Stochastic Equation and Stochastic Inclusion

We begin with the designation of restrictions imposed on $F, U$ and $\sigma$. Let us assume that $F: I \times \Omega \times K_c(E) \rightarrow K_c(E)$, $U: \Omega \rightarrow K_c(E)$, and $\sigma: I \times \Omega \rightarrow E$ have the following properties:

1) $F$ is an integrably bounded multifunction i.e. there exists a joint measurable function $m: I \times \Omega \rightarrow R_+$ such that $\int_0^T m(s, \omega) ds < \infty$ $P.1$ and $\| F(t, \omega, A) \| \leq m(t, \omega)$ $P.1$, $t$-a.e. $A \in K_c(E)$.
2) $F(t, \omega, \ )$ is $H$-continuous with $P.1$, $t$-a.e.
3) $F(t, \ , A)$ is $\mathfrak{F}_t$-adapted for every $t \in I$, $A \in K_c(E)$.
4) $F(\ , \ , A)$ is measurable for every $A \in K_c(E)$.
5) $U$ is an $\mathfrak{F}_0$-measurable multifunction.
6) $\sigma = (\sigma_t)$ is an $\mathfrak{F}_t$-adapted stochastic process for which $E \int_0^T \| \sigma_s \|^2 ds$ is finite.

Let us notice that under assumptions given above, for every $A \in K_c(E)$, the set-valued process

$$\Phi_t = U + \int_0^t F(s, A)ds + \int_0^t \sigma_s dw_s, t \in I,$$

is $\mathcal{F}_t$-adapted with values in $K_c(E)$. It is also clear that $\Phi$ has continuous "paths".

We also assume the so-called "Kamke condition" imposed on multifunction $F$: for every $A_1, A_2, \ldots \in K_c(E)$ one has

$$\mathcal{N}(\bigcup_{n \geq 1} F(t, A_n)) \leq k(t, \mathcal{N}(\bigcup_{n \geq 1} A_n)) \text{ with } P.1 \quad t \in I \text{ a.e.,} \tag{$*$}$$

where $k: I \times \Omega \times R \to R_+$ satisfies the following conditions:

*a*) $k(t, \ , x)$ is $\mathcal{F}$-measurable for every $(t, x) \in I \times R_+$,

*b*) $k(\ , \omega, \ )$ is a Kamke function (see e.g., [14]) with $P.1$.

**Definition 2:** A multivalued process $X = (X_t)_{t \in I}$ is said to be a *solution of* $(I)$ if it satisfies multivalued stochastic equation $(II)$.

Let us notice that without stochastic perturbation, equation $(II)$ can be written as:

$$D_H X_t = F(t, X_t) \ P.1, \ \ t\text{-a.e.}$$

$$X_0 = U \ P.1,$$

where $D_H$ denotes the Hukuchara derivative operator ([6]) for multifunctions.

Before stating the existence theorem to equation $(II)$ let us recall its special case.

**Theorem 1:** ([11]) *Let $F$ and $U$ be multivalued mappings satisfying conditions* 1)-4) *and* 5), *respectively. Let us also suppose that $F$ satisfies the "Kamke condition." Then the multivalued random differential equation*

$$D_H X_t = F(t, X_t) \textit{ with } P.1 \quad t \in I \textit{ a.e.}$$

$$X_0 = U \quad \textit{with } P.1$$

*has at least one solution.*

**Remark:** In fact, the existence of solutions to the above initial value problem is based on the fact that under these conditions there exists at least one solution to the multivalued equation $X_t = U + \int_0^t F(s, X_s)ds$ and on well-known connection between Aumann's integral of set-valued mapping and its Hukuchara derivative via Radström Embedding Theorem (see e.g. [14]).

**Theorem 2:** *Let $E$ be a Banach space such that its dual $E^*$ is separable. If $F, U$ and $\sigma$ have properties* 1)-6) *and $F$ satisfies the "Kamke condition" then there exists at least one solution of the equation* $(II)$.

**Proof:** Let $\xi_t = \int_0^t \sigma_s dw_s$. Let $\hat{X_t} := X_t - \xi_t$, where $X_t$ is a solution of $(II)$, and $\hat{X_t}(\omega) = \{\hat{x} - \xi_t(\omega); \hat{x} \in X_t(\omega)\}$. The process $\hat{X}$ satisfies the equation

$$\hat{X_t} = U + \int_0^t \hat{F}(s, \hat{X_s})ds \quad P.1, \ t \in I, \tag{$**$}$$

where $\hat{F}(s, \omega, A) = F(s, \omega, A + \xi_s(\omega))$. The set-valued mapping $\hat{F}$ meets properties

1)-4). By properties of measure of noncompactness it also satisfies $(*)$ (cf. [1]). Hence, equation $(II)$ has at least one solution if and only if equation $(**)$ has one. By Theorem 1 (via Remark 1) the proof is completed.

Let us suppose now that $\Gamma: I \times \Omega \times E \to K_c(E)$ is a given set-valued mapping. Let us set $F(t,\omega,A) := \overline{co}\Gamma(t,\omega,A)$, $A \in K_c(E)$, where $\overline{co}B$ denotes the closed convex hull of the set $B$. It is noteworthy to observe the connections between solutions of equation $(II)$, with $F = \overline{co}\Gamma$ and solutions of stochastic inclusion

$$x_t - x_s \in \int_s^t \Gamma(u, x_u)du + \int_s^t \sigma_u dw_u \text{ with } P.1,\ 0 \le s \le t \le T \qquad (II')$$

$$x_0 \in U \text{ with } P.1.$$

We suppose that $\Gamma$ is an integrable bounded multifunction such that:

1′) $\Gamma(t,\omega,\ )$ is $H$-continuous with $P.1$, $t$-a.e.,

2′) $\Gamma(t,\ ,x)$ is $\mathcal{F}_t$-adapted for every $t \in I$, $x \in E$,

3′) $\Gamma(\ ,\ ,x)$ is measurable for every $x \in E$,

4′) $\forall A \subset S_r(U)$: $\mathcal{N}(\Gamma(t,A)) \le k(t, \mathcal{N}(A))$ $P.1$, $t \in I$,

where $S_r(U) = U + rB(0,1)$ and $B(0,1)$ is a closed unit ball in Banach space $E$, centered at zero.

**Theorem 3:** *Suppose that $\Gamma$ satisfies conditions 1′-4′. If a multivalued stochastic process $X = (X_t)_{t \in I}$ is a solution of equation $(II)$ with $F = \overline{co}\Gamma$ then there exists stochastic process $x = (x_t)$ being both a solution to stochastic inclusion $(II')$ and the selection of $X$.*

**Proof:** Similarly, as above, let $\xi_t = \int_0^t \sigma_s dw_s$, $\Gamma^\wedge(t,\omega,x) := \Gamma(t,\omega,x+\xi_t(\omega))$ and $F^\wedge(t,\omega,A) := F(t,\omega,A+\xi_t(\omega))$. Then $F^\wedge = \overline{co}\Gamma^\wedge$. Let us notice than $F^\wedge$ also satisfies 1′-4′. Hence, by Corollary 1 [11], there exists at least one solution of equation

$$X_t^\wedge = U + \int_0^t F^\wedge(s, X_s^\wedge)ds \quad P.1,\ \ t \in I.$$

Taking $X = X^\wedge + \xi$ we get a solution of equation $(II)$, where $F = \overline{co}\Gamma$. Moreover, by Theorem 4 [11] there exists stochastic process, say $x^\wedge = (x_t^\wedge)$, being a selection of $X^\wedge$ such that: $x_t^\wedge - x_s^\wedge \in \int_s^t \Gamma^\wedge(u, x_u^\wedge)du$ with $P.1$, $0 \le s \le t \le T$, and $x_0^\wedge \in U$ $P.1$.

Consequently, there exists stochastic process $x = (x_t)$, as a selection of $X$, such that: $x_t^\wedge = x_t - \xi_t$ with $P.1$. It remains to observe that $x$ is a desired solution of inclusion $(II')$.

## 3. Continuity Properties of Solutions

By $S(I \times \Omega)$ we denote the class of "simple" multivalued processes that can be expressed by: $X = \sum_{i=1}^n I_{D_i} C_i$, where the sets $D_i$, $i = 1,2,\ldots,n$ form a measurable partition of $I \times \Omega$ and $C_i \in K_c(E)$, $i = 1,2,\ldots,n$.

**Lemma 1:** *If $X = (X_t)_{t \in T}$ is a multivalued stochastic process with continuous* *"paths" then there exists a sequence* $\{X_n\} \subseteq S(I \times \Omega)$ *such that* $\forall (t,\omega) \in I \times \Omega$: $\lim_{n\to\infty} H(X(t,\omega), X_n(t,\omega)) = 0$.

**Proof:** It follows directly from the fact that $K_c(E)$ is a separable metric space and Proposition 1.9 [15].

Let $\Lambda$ be a metric space. Let us consider the multivalued mapping $F: I \times \Omega \times K_c(E) \times \Lambda \to K_c(E)$ such that:

A1. For every fixed $A \in K_c(E)$ and $\lambda \in \Lambda$, $F(\ ,\ , A, \lambda)$ is a measurable and integrably bounded multifunction.

A2. The mapping $F(t, \omega, \ , \lambda)$ is with $P.1$ uniformly continuous with respect to $t \in I$ and $\lambda \in \Lambda$.

**Definition 2:** A multifunction $F$ (with properties A1 and A2) is said to be *integrably continuous in probability (icp) at* $\lambda_0 \in \Lambda$ *with respect to a family* $\mathfrak{C} \subset K_c(E)$ ($\mathfrak{C} \neq \emptyset$) if

$$\forall C \in \mathfrak{C}, \forall t \in I: \quad \int_0^t F(s, C, \lambda) ds \xrightarrow{P} \int_0^t F(s, C, \lambda_0) ds$$

for $\lambda \to \lambda_0$.

The results presented below give characterizations of icp multifunctions. We use them to obtain the main theorem.

**Lemma 2:** *If* $F$ *is an icp multifunction at* $\lambda_0$ *with respect to* $\mathfrak{C}$ *then for every* $C \in \mathfrak{C}$ *one has:* $\int_0^t F(s, C, \lambda_n) ds \to \int_0^t F(s, C, \lambda_0) ds$ $P.1$ *uniformly in* $t \in I$, *for some sequence* $(\lambda_n)$ *convergent to* $\lambda_0$.

**Proof:** Let $D$ be a set of rationals in $I$, $D = \{t_1, t_2, \ldots\}$ and let $(\lambda_n)$ be an arbitrary sequence of elements of $\Lambda$ that converges to $\lambda_0$. Fix $C \in \mathfrak{C}$. Then for $t_1 \in D$, there exist a sequence $(\lambda_n(t_1))_n$, convergent to $\lambda_0$ and set $\Omega(t_1) \subset \Omega$, $P(\Omega(t_1)) = 1$, such that

$$\forall \omega \in \Omega(t_1): H\Big(\int_0^{t_1} F(s, \omega, C, \lambda_n(t_1)) ds, \int_0^{t_1} F(s, \omega, C, \lambda_0) ds\Big) \to 0, \text{ for } n \to \infty.$$

Similarly, for $t_2 \in D$ we can find a sequence $(\lambda_n(t_2))_n$ being a subsequence of $(\lambda_n(t_1))_n$ and $\Omega(t_2) \subset \Omega$, $P(\Omega(t_2)) = 1$ for which a similar convergence holds. Continuing this selection process we obtain the infinite table

$$\begin{array}{ccccc}
\lambda_1(t_1) & \lambda_2(t_1) & \ldots & \lambda_n(t_1) & \ldots \\
\lambda_1(t_2) & \lambda_2(t_2) & \ldots & \lambda_n(t_2) & \ldots \\
\vdots & \vdots & & \vdots & \\
\lambda_1(t_n) & \lambda_2(t_n) & \ldots & \lambda_n(t_n) & \ldots \\
\vdots & \vdots & & \ldots &
\end{array} \tag{1}$$

By diagonal selection we can find a sequence $(\lambda_n)_n$ being a subsequence of each row of table (1) that converges to $\lambda_0$. Let $\Omega_0 = \bigcap \{\Omega(t_n); n \geq 1\}$. Then $P(\Omega_0) = 1$. Moreover,

$$\forall \omega \in \Omega_0, \forall t \in D: H\Big(\int_0^t F(s, \omega, C, \lambda_n) ds, \int_0^t F(s, \omega, C, \lambda_0) ds\Big) \to 0, \ n \to \infty.$$

Since the set-valued process $J_t = \int_0^t F(s,C,\lambda)ds,\ t \in I$ has with $P.1$ uniformly continuous "paths", we can find $\Omega_0'$, $P(\Omega_0') = 1$ such that

$$\forall \omega \in \Omega_0' \colon\ \sup_{t \in I} H(\int_0^t F(s,\omega,C,\lambda_n)ds, \int_0^t F(s,\omega,C,\lambda_0)ds) \to 0, \text{ if } n \to \infty.$$

This completes the proof.

By $\mathbb{B}_I$ we denote the $\sigma$-field of Borel subsets of $I$.

**Lemma 3:** *A multifunction $F$ is icp at $\lambda_0$ with respect to family $\mathfrak{C}$ if and only if:* $\forall C \in \mathfrak{C},\ \forall (\lambda_n) \subset \Lambda \colon \lambda_n \to \lambda_0, \exists (\lambda_n') \subset (\lambda_n)$:

$$H(\int_B F(s,C,\lambda_n')ds, \int_B F(s,C,\lambda_0 ds) \to 0 \quad P.1 \tag{2}$$

*as $n \to \infty$, for every $B \in \mathbb{B}_I$.*

**Proof:** Fix $C \in \mathfrak{C}$ and let $(\lambda_n)$ be an arbitrary sequence convergent to $\lambda_0$. Then by Lemma 2, we can find its subsequence $(\lambda_n')$ and $\Omega_0 \colon P(\Omega_0) = 1$ such that for every $\omega \in \Omega_0$ and $0 \leq s < t \leq T$,

$$\int_s^t F(u,\omega,C,\lambda_n')du \to \int_s^t F(u,\omega,C,\lambda_0)du, \text{ as } n \to \infty. \tag{3}$$

Let $\mathfrak{I} \colon = \{[s,t) \colon 0 \leq s < t \leq T\}$ and

$$\mathcal{A} \colon = \{\bigcup_{i=1}^n R_i \colon R_i \in \mathfrak{I}, R_i \cap R_j = \emptyset, i \neq j, i,j = 1,2,\ldots,n, n \geq 1\}.$$

Since $\sigma(\mathfrak{I}) = \sigma(\mathcal{A}) = \mathbb{B}_I$ and $\mathcal{A}$ is a ring of subsets of $I$, then for every $\epsilon > 0$ and $B \in \mathbb{B}_I$, there exists $A \in \mathcal{A}$ such that $| B \Delta A | < \epsilon$ (c.f. e.g., Th. 11.4 [2]), where $|\ |$ is Lebesgue measure and $B \Delta A \colon = (B \backslash A) \cup (A \backslash B)$. By integrably boundness of $F$ we get:

$$H(\int_B F(s,C,\lambda_n')ds, \int_B F(s,C,\lambda_0)ds) \leq H(\int_A F(s,C,\lambda_n')ds, \int_A F(s,C,\lambda_0)ds)$$
$$+ \int_{B \Delta A} m(s,\omega)ds, \text{ for every } A \in \mathcal{A}.$$

Then by (3), $\limsup_{n \to \infty} H(\int_B F(s,C,\lambda_n')ds, \int_B F(s,C,\lambda_0)ds) \leq \int_{B \Delta A} m(s,\omega)ds$.

Taking $A$ sufficiently close to $B$ we claim (2). The converse is obvious.

**Lemma 4:** *A multifunction $F$ is icp at $\lambda_0$ with respect to $K_c(E)$ if and only if $F$ is icp at $\lambda_0$ with respect to $S(I \times \Omega)$.*

**Proof:** Let us assume that $F$ is icp with respect to $K_c(E)$. Let $X \in S(I \times \Omega)$. Then there exist $C_1, C_2, \ldots, C_r \in K_c(E)$ and a measurable partition $\{D_1, D_2, \ldots, D_r\}$ of space $I \times \Omega$ such that $X = \sum_{i=1}^r I_{D_i} C_i$. Take $C_1$ and $(\lambda_n)$ to be an arbitrary sequence convergent to $\lambda_0$. Next let $(\lambda_{n_k})$ be any subsequence of $(\lambda_n)$. By Lemma 3 there exists a sequence $(\lambda_{n,1}')$ being a subsequence of $(\lambda_{n_k})$ and a subset $\Omega_{0,1} \subseteq \Omega$;

$P(\Omega_{0,1}) = 1$ such that:

$$\forall \omega \in \Omega_{0,1}, \forall B \in \mathbb{B}_I: \lim_{n\to\infty} H(\int_B F(s,\omega,C_1,\lambda'_{n,1})ds, \int_B F(s,\omega,C_1,\lambda_0)ds) = 0.$$

Similarly, for $C_2$ we can extract a subsequence $(\lambda'_{n,2})$ from $(\lambda'_{n,1})$ and $\Omega_{0,2} \subseteq \Omega$ $P(\Omega_{0,2}) = 1$, with the desired property, and so on. Thus we obtain a sequence $(\lambda'_{n,r})$ which is a subsequence of $(\lambda'_{n,i})$, $i = 1,2,\ldots,r-1$ and $\Omega_{0,r}$, $P(\Omega_{0,r}) = 1$, such that

$$\forall \omega \in \Omega_{0,r}, \forall B \in \mathbb{B}_I: \lim_{n\to\infty}(\int_B F(s,\omega,C_r,\lambda'_{n,r})ds, \int_B F(s,\omega,C_r,\lambda_0)ds) = 0.$$

Let $\Omega_0 = \bigcap_{1 \le i \le r} \Omega_{0,1}$. For any $A \in \mathbb{B}_I \otimes \mathfrak{F}$ and $\omega \in \Omega$, we define the set $(A)_\omega: = \{t \in I: (t,\omega) \in A\}$. Then $(A)_\omega \in \mathbb{B}_I$. Let $\omega \in \Omega_0$. Then $X(\cdot,\omega) = \sum_{i=1}^{r} I_{(D_i)_\omega}(\cdot)\ C_1$ and $\{(D_i)_\omega: i = 1,2,\ldots,r\}$ is measurable partition of $I$. Hence, the following inequality holds:

$$H(\int_0^t F(s,\omega,X_s,\lambda'_{n,r})ds, \int_0^t F(s,\omega,X_s,\lambda_0)ds)$$

$$\le \sum_{i=1}^{r} H(\int_{(D_i)_\omega \cap [0,t]} F(s,\omega,C_i,\lambda'_{n,r})ds, \int_{(D_i)_\omega \cap [0,t]} F(s,\omega,C_i,\lambda_0 ds)).$$

It remains to observe that each term of the above sum converges to zero as $n$ tends to infinity.

The converse statement is obvious. It is enough to take $X: = I_{I\times\Omega}C$, for $C \in K_c(E)$. This completes the proof.

By $X^\lambda$ we denote a multivalued process being the solution of the equation

$$X_t = U + \int_0^t F(s,X_s,\lambda)ds + \int_0^t \sigma_s dw_s\ P.1,\ t \in I,\ \lambda \in A. \qquad (III)$$

**Theorem 3:** *Let us assume that $F$ is an icp set-valued mapping at $\lambda_0 \in A$ with respect to $K_c(E)$. Then,*

*i)* *if $X^\lambda \xrightarrow{P} X^{\lambda_0}$ then $\forall t \in I$: $\int_0^t F(s,X_s^\lambda,\lambda)ds \xrightarrow{P} \int_0^t F(s,X_s^{\lambda_0},\lambda_0)ds$, $\lambda \to \lambda_0$,*

*ii)* *if for every $A_1, A_2, \ldots \in K_c(E)$ and $(\lambda_n); \lambda_n \to \lambda_0$ we have*

$$\mathcal{N}\left\{\bigcup_{n\ge 1} F(t,A_n,\lambda_n)\right\} \le k\left(t, \mathcal{N}\left\{\bigcup_{n\ge 1} A_n\right\}\right) \text{ with } P.1\ t \in I \text{ a.e., then } X^\lambda \xrightarrow{P} X^{\lambda_0}.$$

**Proof:** $(i)$ Let $(\lambda_n)$ be an arbitrary sequence convergent to $\lambda_0$. Then its every subsequence contains a further subsequence, say, $(\lambda'_n)$, such that $X^{\lambda'_n} \to X^{\lambda_0}$ with $P.1$ in $C_I$. Take $\omega$ from an appropriate set (for which this convergence holds). By condition A2, for any $\epsilon > 0$, there exists $\delta > 0$ such that $H(F(t,C,\lambda'_n), F(t,D,\lambda'_n)) < \epsilon/4T$, for $n \in N$, $C, D \in K_c(E)$ whenever $H(C,D) < \delta$.

Let $V_0$ be an open neighborhood for $\lambda_0$ such that

$$\text{if } \lambda_n' \in V_0 \text{ then } \sup_{t \in I} H(X_t^{\lambda_n'}, X^{\lambda_0}) < \delta. \tag{4}$$

Let $(X_k^{\lambda_0})_k$ be a sequence of simple multifunctions (Lemma 1) convergent to $X^{\lambda_0}$ for every $t \in I$ and $\omega \in \Omega$. Then for every $t \in I$ and $\lambda \in \Lambda$, we have:

$$\lim_k H(F(t,\omega,X_k^{\lambda_0}(t,\omega),\lambda), F(t,\omega,X_t^{\lambda_0}(\omega),\lambda) = 0 \quad P.1.$$

Next by the Lebesgue Dominated Convergence Theorem (via integrably boundedness of $F$) we obtain that

$$\int_0^T H(F(s,X_k^{\lambda_0}(s),\lambda), F(s,X_s^{\lambda_0},\lambda))ds \to 0 \ P.1$$

for every $\lambda \in \Lambda$. Hence by (4), after standard calculation we see that

$$H(\int_0^t F(s,\omega,X_s^{\lambda_n}(\omega),\lambda_n')ds, \int_0^t F(s,\omega,X_s^{\lambda_0}(\omega),\lambda_0)ds) \le (3/4)\epsilon$$
$$+ H(\int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega),\lambda_n')ds, \int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega)\lambda_0)ds),$$

for $t \in I$, $k$ sufficiently large and $\omega$ taken from an appropriate set of probability one.

By Lemma 4, multifunction $F$ is icp at $\lambda_0$ with respect to $S(I \times \Omega)$. Hence there exists a sequence $(\lambda_n'')$ being a subsequence of $(\lambda_n')$, a subset of $\Omega$ of measure one such that for every $\epsilon > 0$ and appropriate $\omega$ we can find an open neighborhood $V_1$ of $\lambda_0$ with

$$H(\int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega),\lambda_n'')ds, \int_0^t F(s,\omega,X_k^{\lambda_0}(s,\omega),\lambda_0)ds) < \epsilon/4,$$

for $t \in I$ and $\lambda_n'' \in V_1$. Therefore, taking $n''$ sufficiently large and $\lambda_n'' \in V_0 \cap V_1$ we have:

$$H(\int_0^t F(s,\omega,X_s^{\lambda_n''}(\omega),\lambda_n'')ds, \int_0^t F(s,\omega,X_s^{\lambda_0}(\omega),\lambda_0)ds) < \epsilon$$

for $t \in I$. This completes the proof of part $(i)$.

Proof of part $(ii)$.

Let $(\lambda_n)$ be a sequence convergent to $\lambda_0$. Consider its arbitrary subsequence, denoted for simplicity by the same symbol. We define the multivalued mapping $\Pi: \Omega \to 2^{C_I}$ by

$$\Pi(\omega) = \{X \in C_I : X^{\lambda_n'} \to X \text{ in } C_I \text{ for some sequence } (\lambda_n'), (\lambda_n') \subseteq (\lambda_n)\}.$$

By the assumption of the integrably boundness of $F$ it follows that

$$\forall n \in N, \forall 0 \le s \le t \le T : H(X_t^{\lambda_n}, X_s^{\lambda_n}) \le \int_s^t m(u)du \text{ with } P.1.$$

Thus the sequence $(X^{\lambda_n})$ is equicontinuous in $C_I$ with $P.1$. Similarly, (compare [14]) by assumption $(iii)$, it can be proved that $\bigcup_{n \geq 1} \{X_t^{\lambda_n}\}$ is a relatively compact subset of $E$, for every $t \in I$ with $P.1$. Thus, by Ascoli Theorem we claim that the sequence $(X^{\lambda_n})$ is relatively compact (with $P.1$). Hence the multifunction $\Pi \neq \emptyset$ $P.1$ and has closed values. Moreover, we claim that $\Pi$ is measurable. To see this, let $\Omega_0{:} = \{\omega{:}\, \Pi(\omega)$ is closed subset of $C_I\}$. For $X \in C_I$ we consider a mapping $\Omega_0 \ni \omega \rightarrow \mathrm{Dist}(X, \Pi(\omega))$, where $\mathrm{Dist}(X, \Pi(\omega)) = \inf_{Y \in \Pi(\omega)} \rho(X, Y)$. Fix $r > 0$.

Then $\{\omega{:}\, \mathrm{Dist}(X, \Pi(\omega)) < r\} = \{\omega{:}\, \exists Y \in \Pi(\omega){:}\, Y \in B_r(X)\}$, where $B_r(X){:} = \{Y \in C_I{:}\, \rho(X,Y) < r\}$. Let $\{t_k\}$ be a sequence of rationals in $I$. Then we get:

$$\{\omega{:}\, \mathrm{dist}(X, \Pi(\omega)) < r\} = \{\omega{:}\, \Pi(\omega) \cap B_r(X) \neq \emptyset\}$$

$$= \bigcup_{m \geq 1} \bigcap_{l \geq 1} \bigcup_{j \geq l} \bigcap_{k \geq 1} \{\omega{:}\, X_{t_k}^{\lambda_{n_j}}(\omega) \cap B_{r - 1/m}(X) \neq \emptyset\}.$$

Since $X_{t_k}^{\lambda_{n_j}}$ is an $\mathcal{F}_{t_k}$-measurable multifunction then the last set above belongs to $\sigma$-field $\mathcal{F}$, which yields the $\mathcal{F}$-measurability of $\Pi$ (see, e.g. [4]). Thus, by Kuratowski and Ryll-Nardzewski Selection Theorem [10], there exists a measurable selection $\hat{X}$ of $\Pi$; $\hat{X} \in \Pi$ $P.1$. The definition of $\Pi$ implies then that $X^{\lambda'_n} \rightarrow \hat{X}$ $P.1$ in $C_I$, for some sequence $(\lambda'_n)$ tending to $\lambda_0$ and this yields convergence in probability in $C_I$. Finally, we claim that $\hat{X}$ is a solution of $(III)$. Indeed, let us notice that

$$H(\hat{X}_t, U + \int_0^t F(s, \hat{X}_s, \lambda_0)ds + \int \sigma_s dw_s)$$

$$\leq H(\hat{X}_t, X_t^{\lambda'_n}) + H(\int_0^t F(s, X_s^{\lambda'_n}, \lambda'_n)ds, \int_0^t F(s, \hat{X}_s, \lambda_0)ds),$$

with $P.1$ and for $t \in I$.

Since the first term above converges to zero then by $(i)$ the second term converges to zero as well. This completes the proof.

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