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TERMINAL VALUE PROBLEMS OF IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES[1]

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This paper uses cone theory and the monotone iterative technique to investigate the existence of minimal nonnegative solutions of terminal value problems for first order nonlinear impulsive integro-differential equations of mixed type in a Banach space.

Key words: Terminal Value Problem, Impulsive Integro-Differential Equation, Cone Theory, Monotone Iterative Technique.

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1. Introduction

The theory of impulsive differential equations has become an important area of investigation. Initial value problems of such equations have been discussed in detail in recent years (see [3]). In this paper, we shall use cone theory and the monotone iterative technique to investigate the existence of a minimal nonnegative solution of the terminal value problem (TVP) for a first order nonlinear impulsive integro-differential equation of mixed type in a Banach space.

2. Preliminaries

Let E be a real Banach space and P be a cone in E which defines a partial order in E: $x \leq y$ if and only if $y - x \in P$. P is said to be *normal* if there exists a positive constant N such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$, where θ denotes the zero element of E. P is said to be *regular* (or *fully regular*) if $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \leq y$ (or $x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$ with $\sup_n \| x_n \| < \infty$) implies $\| x_n - x \| \rightarrow 0$ as $n \rightarrow \infty$ for some $x \in E$. The full regularity of P implies the regularity of P, and the regularity of P implies the normality of P (see [2], Theorem 1.2.1). Moreover, if E is weakly complete (in particular, reflexive), then the normality of P implies the regularity of

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P (see [1], Theorem 2.2).

Consider the TVP in E:

$$\begin{cases} x' = f(t, x, Tx, Sx), & t \in J, t \neq t_m, \\ \Delta x\,|_{t = t_m} = I_m(x(t_m)), & (m = 1, 2, 3, \ldots), \\ x(\infty) = x^*, & \end{cases} \tag{1}$$

where $J = [0, \infty)$, $f \in C(J \times P \times P \times P, -P)$, $0 < t_1 < \ldots < t_m < \ldots$, $t_m \to \infty$ and $m \to \infty$, $I_m \in C(P, -P)$ $(m = 1, 2, 3, \ldots)$, $x^* \in P$, $x(\infty) = \lim_{t \to \infty} x(t)$, and

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds, \quad (Sx)(t) = \int_0^\infty h(t,s)x(s)ds, \tag{2}$$

$k \in C(D, R_+)$, $D = \{(t,s) \in J \times J : t \geq s\}$, $h \in C(J \times J, R_+)$. $\Delta x\,|_{t = t_m} = x(t_m^+) - x(t_m^-)$ which denotes the jump of $x(t)$ at $t = t_m$. Here $x(t_m^+)$ and $x(t_m^-)$ represent the right- and left-sided limits of $x(t)$ at $t = t_m$, respectively.

Let $PC(J,E) = \{x : x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_m$ and left continuous at $t = t_m$ and $x(t_m^+)$ exists for $m = 1, 2, 3, \ldots$, $BPC(J,E) = \{x \in PC(J,E) : \sup_{t \in J} \| x(t) \| < \infty\}$ and $TPC(J,E) = \{x \in PC(J,E) : x(\infty) = \lim_{t \to \infty} x(t)$ exists$\}$. Evidently, $TPC(J,E) \subset BPC(J,E)$, and $BPC(J,E)$ is a Banach space with norm $\| x \|_B = \sup_{t \in J} \| x(t) \|$. Let $BPC(J,P) = \{x \in BPC(J,E) : x(t) \geq \theta$ for $t \in J\}$, $TPC(J,P) = \{x \in TPC(J,E) : x(t) \geq \theta$ for $t \in J\}$ and $J' = J \backslash \{t_1, \ldots, t_m, \ldots\}$. A map $x \in TPC(J,P) \cap C^1(J', E)$ is called a *non-negative solution* of TVP(1) if it satisfies (1).

3. Main Results

Let us list some conditions.

(H_1) $k^* = \sup_{t \in J} \int_0^t k(t,s)ds < \infty$, $h^* = \sup_{t \in J} \int_0^\infty h(t,s)ds < \infty$ and

$$\lim_{t' \to t} \int_0^\infty | h(t', s) - h(t,s) | ds = 0, \quad t \in J.$$

(H_2) $\| f(t, x, y, z) \| \leq p(t) + q(t)(a \| x \| + b \| y \| + c \| z \|)$, $t \in J$, $x, y, z, \in P$, and

$$\| I_m(x) \| \leq a_m + b_m \| x \|, \quad x \in P (m = 1, 2, 3, \ldots),$$

where $p, q \in C(J, R_+)$ and $a \geq 0$, $b \geq 0$, $a_m \geq 0$, $b_m \geq 0$ $(m = 1, 2, 3, \ldots)$ satisfying

$$p^* = \int_0^\infty p(t)dt < \infty, \; q^* = \int_0^\infty q(t)dt < \infty, \; a^* = \sum_{m=1}^\infty a_m < \infty, \; b^* = \sum_{m=1}^\infty b_m < \infty.$$

(H_3) $f(t, x, y, z)$ is nonincreasing in $x, y, z \in P$ and $I_m(x)$ is nonincreasing in $x \in P$ $(m = 1, 2, 3, \ldots)$, i.e.

$$f(t,x,y,z) \leq f(t,\bar{x},\bar{y},\bar{z}), \ t \in J, \ x \geq \bar{x} \geq \theta, \ y \geq \bar{y} \geq \theta, \ z \geq \bar{z} \geq \theta$$

and

$$I_m(x) \leq I_m(\bar{x}), \ x \geq \bar{x} \geq \theta \ (m = 1,2,3,\ldots).$$

It is easy to see that when (H_1) is satisfied, T and S, defined by (2), are bounded linear operators from $BPC(J,E)$ into $BPC(J,E)$.

Lemma 1: *If conditions (H_1) and (H_2) are satisfied, then for any $x \in BPC(J,P)$, the integral*

$$\int_0^\infty f(t,x(t),(Tx)(t),(Sx)(t))dt \tag{3}$$

and the series

$$\sum_{m=1}^\infty I_m(x(t_m)) \tag{4}$$

are convergent.

Proof: Let $x \in BPC(J,P)$. By virtue of (H_1) and (H_2), it is easy to see that

$$\int_0^\infty \| f(s,x(s),(Tx)(s),(Sx)(s)) \| ds$$

$$\leq \int_0^\infty p(s)ds + (a + bk^* + ch^*) \| x \|_B \int_0^\infty q(s)ds < \infty$$

and

$$\sum_{m=1}^\infty \| I_m(x(t_m)) \| \leq \sum_{m=1}^\infty a_m + \| x \|_B \sum_{m=1}^\infty b_m < \infty,$$

so, integral (3) and series (4) are convergent. □

Lemma 2: *Let conditions (H_1) and (H_2) be satisfied. Then $x \in TPC(J,P) \cap C^1(J',E)$ is a solution of TVP(1) if and only if $x \in BPC(J,P)$ is a solution to the following impulsive integral equation*

$$x(t) = x^* - \int_t^\infty f(s,x(s),(Tx)(s),(Sx)(s))ds - \sum_{t \leq t_m < \infty} I_m(x(t_m)), \ t \in J. \tag{5}$$

Proof: Let $x \in TPC(J,P) \cap C^1(J',E)$ be a solution of TVP(1). We first establish the following formula:

$$x(t) = x(0) + \int_0^t x'(s)ds + \sum_{0 < t_m < t} [x(t_m^+) - x(t_m)], \ t \in J. \tag{6}$$

In fact, let $t_m \leq t \leq t_{m+1}$. Then

$$x(t_1) - x(0) = \int_0^{t_1} x'(x)ds, \ x(t_2) - x(t_1^+) = \int_{t_1}^{t_2} x'(s)ds,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x(t_m) - x(t_{m-1}^+) = \int_{t_{m-1}}^{t_m} x'(s)ds, \quad x(t) - x(t_m^+) = \int_{t_m}^t x'(s)ds.$$

Summing up these equations, we get

$$x(t) - x(0) - \sum_{i=1}^{m} [x(t_i^+) - x(t_i)] = \int_0^t x'(s)ds$$

(i.e., (6) holds). Substituting (1) into (6), we obtain

$$x(t) = x(0) + \int_0^t f(s, x(s), (Tx)(s), (Sx)(s))ds + \sum_{0 < t_m < t} I_m(x(t_m)), \quad t \in J. \quad (7)$$

By Lemma 1, integral (3) and series (4) are convergent, hence, from (1) and (7) we get

$$x^* = x(0) + \int_0^\infty f(s, x(s), (Tx)(s), (Sx)(s))ds + \sum_{m=1}^{\infty} I_m(x(t_m)). \quad (8)$$

Solving $x(0)$ from (8) and substituting it into (7), we find that $x(t)$ satisfies equation (5).

Conversely, if $x \in BPC(J, P)$ is a solution of equation (5), direct differentiation of (5) implies that $x \in C^1(J', E)$ and $x(t)$ satisfies TVP(1). □

Consider operator A defined by

$$(Ax)(t) = x^* - \int_t^\infty f(s, x(s), (Tx)(s), (Sx)(s))ds - \sum_{t \le t_m < \infty} I_m(x(t_m)). \quad (9)$$

Lemma 3: *If conditions* (H_1) *and* (H_2) *are satisfied, then* A *defined by* (9) *is an operator from* $BPC(J, P)$ *into* $BPC(J, P)$.

Proof: Let $x \in BPC(J, P)$. Since $f \in C(J \times P \times P \times P, -P)$, $I_m \in C(P, -P)$ and $x^* \in P$, we see that $(Ax)(t) \ge \theta$ for $t \in J$, and clearly $Ax \in PC(J, P)$. By (H_1) and (H_2), we have

$$\| (Ax)(t) \| \le \| x^* \| + \int_t^\infty p(s)ds + (a + bk^* + ch^*) \| x \|_B \int_t^\infty q(s)ds$$

$$+ \sum_{t \le t_m \le \infty} a_m + \| x \|_B \sum_{t \le t_m < \infty} b_m$$

$$\le \| x^* \| + p^* + a^* + [b^* + (a + bk^* + ch^*)q^*] \| x \|_B, \quad t \in J,$$

and therefore

$$\| Ax \|_B \le \| x^* \| + p^* + a^* + [b^* + (a + bk^* + ch^*)q^*] \| x \|_B. \quad (10)$$

Hence $Ax \in BPC(J, P)$. □

In the following, let $J_0 = [0, t_1]$, $J_m = (t_m, t_{m+1}]$ $(m = 1, 2, 3, \ldots)$.

Theorem 1: *Let cone* P *be fully regular and conditions* (H_1), (H_2), (H_3) *be satisfied. Assume that*

$$r = b^* + (a + bk^* + ch^*)q^* < 1, \quad (11)$$

where constants $k^*, h^*, a, b, c, q^*, b^*$ *are defined by* (H_1) *and* (H_2). *There exists a nondecreasing sequence* $\{x_n\} \subset TPC(J, P) \cap C^1(J', E)$ *which converges on* J *(uniformly in each* J_m, $m = 0, 1, 2, \ldots$*) to the minimal solution* $\overline{x} \in TPC(J, P) \cap C^1(J', E)$ *of TVP(1) in* $TPC(J, P) \cap C^1(J', E)$, *i.e., for any solution* $x \in$

$TPC(J,P)\cap C^1(J',E)$ of TVP(1), we have

$$x(t)\geq \overline{x}(t),\quad t\in J. \tag{12}$$

Moreover,

$$\overline{x}(t)\geq \overline{x}(t')\geq x^*,\quad 0\leq t<t'<\infty, \tag{13}$$

and

$$\|\overline{x}\|_B\leq (1-r)^{-1}(\|x^*\|+p^*+a^*), \tag{14}$$

where r is given by (11) and p^*,a^* are defined by (H_2).

Proof: Let $x_0(t)=\theta$, $x_n(t)=(Ax_{n-1})(t)$ $(n=1,2,3,\ldots)$, i.e.,

$$x_n(t)=x^*-\int_t^\infty f(s,x_{n-1}(s),(Tx_{n-1})(s),(Sx_{n-1})(s))ds$$
$$-\sum_{t\leq t_m<\infty} I_m(x_{n-1}(t_m)),\quad t\in J(n=1,2,3,\ldots). \tag{15}$$

By Lemma 3, $x_n\in BPC(J,P)$ $(n=0,1,2,\ldots)$ and $x_1(t)\geq\theta=x_0(t)$ for $t\in J$, so, (15) and (H_3) imply that

$$\theta=x_0(t)\leq x_1(t)\leq x_2(t)\leq\ldots\leq x_n(t)\leq\ldots,\quad t\in J. \tag{16}$$

On the other hand, from (10) we know

$$\|x_n\|_B=\|Ax_{n-1}\|_B\leq d+r\|x_{n-1}\|_B,\quad (n=1,2,3,\ldots),$$

where $d=\|x^*\|+p^*+a^*$ and r is given by (11), thus

$$\|x_n\|_B\leq d+r(d+r\|x_{n-2}\|_B)\leq d+rd+r^2(d+r\|x_{n-3}\|_B)$$
$$\leq d+rd+\ldots+r^{n-1}d+r^n\|x_0\|_B=d+rd+\ldots+r^{n-1}d=d(1-r^n)(1-r)^{-1}$$
$$\leq d(1-r)^{-1},\quad (n=1,2,3,\ldots). \tag{17}$$

It follows from (16), (17), and the full regularity of P that the following limit exists:

$$\lim_{n\to\infty}x_n(t)=\overline{x}(t),\quad t\in J. \tag{18}$$

Now we have, by (17),

$$\|f(s,x_{n-1}(s),(Tx_{n-1})(s),(Sx_{n-1})(s))\|$$
$$\leq p(s)+(a+bk^*+ch^*)\|x_{n-1}\|_B q(s)$$
$$\leq p(s)+(a+bk^*+ch^*)d(1-r)^{-1}q(s),\quad s\in J\quad (n=1,2,3,\ldots), \tag{19}$$

so, from (15) we know that functions $\{x_{mn}(t)\}$ $(n=0,1,2,\ldots)$ are equicontinuous in $\overline{J}_m$ $(m=0,1,2,\ldots)$, where $\overline{J}_m=[t_m,t_{m+1}]$ and

$$x_{mn}(t)=\begin{cases} x_n(t), & t_m<t\leq t_{m+1};\\ x_n(t_m^+), & t=t_m.\end{cases}$$

Hence, observing (18) and using the Ascoli-Arzela theorem, we see that $\{x_{mn}(t)\}$ $(n=0,1,2,\ldots)$ is compact in $C(\overline{J}_m,E)$ $(m=0,1,2,\ldots)$. and therefore, by diagonal method, $\{x_n(t)\}$ has a subsequence which converges to $\overline{x}(t)$ uniformly in each J_m $(m=0,1,2,\ldots)$. Since P is also normal and $\{x_n(t)\}$ is nondecreaisng, on account of (16), we conclude that the entire sequence $\{x_n(t)\}$ converges to $\overline{x}(t)$ uniformly in each J_m $(m = 0,1,2,\ldots)$, hence, $\bar{x} \in PC(J,P)$. Moreover, from (17) we know that $\bar{x} \in BPC(J,P)$ and $\| \bar{x} \|_B \le d(1-r)^{-1}$, i.e., (14) holds.

From (18) and (19), we see that

$$f(s,x_{n-1}(s),(Tx_{n-1})(s),(Sx_{n-1})(s)) \to f(s,\bar{x}(s),(T\bar{x})(s),(S\bar{x})(s)) \quad \text{as } n\to\infty,\ s \in J, \tag{20}$$

and

$$\| f(s,x_{n-1}(s),(Tx_{n-1})(s),(Sx_{n-1})(s)) - f(s,\bar{x}(s),(T\bar{x})(s),(S\bar{x})(s)) \| \le 2p(s) + 2(a+bk^*+ch^*)d(1-r)^{-1}q(s), \quad s \in J \quad (n = 1,2,3,\ldots). \tag{21}$$

In addition, (17), (18) and (H_2) imply that

$$I_m(x_{n-1}(t_m)) \to I_m(\bar{x}(t_m)) \text{ as } n\to\infty \ (m = 1,2,3,\ldots) \tag{22}$$

and

$$\sum_{m=j}^{\infty} \| I_m(x_{n-1}(t_m)) \| \le \sum_{m=j}^{\infty} a_m + d(1-r)^{-1}\sum_{m=j}^{\infty} b_m \quad (n = 1,2,3,\ldots), \tag{23}$$

$$\sum_{m=j}^{\infty} \| I_m(\bar{x}(t_m)) \| \le \sum_{m=j}^{\infty} a_m + d(1-r)^{-1}\sum_{m=j}^{\infty} b_m. \tag{24}$$

Observing (20)-(24) and taking limits in (15) as $n\to\infty$, we obtain by virtue of the dominated convergence theorem that

$$\bar{x}(t) = x^* - \int_t^{\infty} f(s,\bar{x}(s),(T\bar{x})(s),(S\bar{x})(s))ds - \sum_{t \le t_m < \infty} I_m(\bar{x}(t_m)), \quad t \in J, \tag{25}$$

which by Lemma 2 implies that $\bar{x} \in TPC(J,P) \cap C^1(J',E)$ and $\bar{x}(t)$ is a solution of TVP(1). From (25) we see clearly that (13) holds.

Finally, we prove the minimal property of $\bar{x}(t)$. Let $x \in TPC(J,P) \cap C^1(J',E)$ by any solution of TVP(1). By Lemma 2, $x(t)$ satisfies equation (5). We have $x(t) \ge \theta = x_0(t)$ for $t \in J$. Assume that $x(t) \ge x_{n-1}(t)$ for $t \in J$. Then (15), (5) and (H_3) imply that $x(t) \ge x_n(t)$ for $t \in J$. Hence, by induction, $x(t) \ge x_n(t)$ for $t \in J (n = 0,1,2,\ldots)$, and by taking the limit, we get $x(t) \ge \bar{x}(t)$ for $t \in J$, i.e., (12) holds. The proof is complete. □

Example 1: Consider the TVP of infinite system for scalar nonlinear impulsive integro-differential equations

$$\begin{cases} x_n' = -\dfrac{e^{-t}}{2^{n+3}}(1 + x_n + \sqrt{x_{n+1} + 2x_{2n+1}}) - \dfrac{e^{-2t}}{3^n}\left(\displaystyle\int_0^t e^{-(t+1)s}x_n(s)ds\right)^{1/3} \\ \qquad - \dfrac{e^{-t}}{4^n}\left(\displaystyle\int_0^{\infty} \dfrac{x_{2n}(s)ds}{1+t+s^2}\right)^{1/5}, \quad 0 \le t < \infty, \quad t \ne m, \\ \Delta x_n |_{t=m} = -\dfrac{1}{2^{n+m+2}}[x_n(m) + x_{n+2}(m)], \quad (m = 1,2,3,\ldots), \\ x_n(\infty) = \dfrac{1}{n^2}, \quad (n = 1,2,3,\ldots). \end{cases} \tag{26}$$

Corollary: *TVP(26) has a minimal, nonnegative and continuously differentiable on* $[0,\infty)\setminus \{1,2,3,\ldots\}$ *solution* $\{x_n(t)\}$ $(n = 1,2,3,\ldots)$ *satisfying*

$$\sup_{0 \le t < \infty} \sum_{n=1}^{\infty} x_n(t) < \infty.$$

Proof: Let $E = \ell^1 = \{x = (x_1, \ldots, x_n, \ldots): \sum_{n=1}^{\infty} |x_n| < \infty]$ with norm $\|x\| = \sum_{n=1}^{\infty} |x_n|$ and $P = \{x = (x_1, \ldots, x_n, \ldots) \in \ell^1: x_n \ge 0, \quad n = 1, 2, 3, \ldots\}$. Thus, P is a normal cone in E. Since ℓ^1 is weakly complete, we conclude that P is regular. We now prove that P is fully regular. Let $x_k = (x_{k1}, \ldots, x_{kn}, \ldots) \in \ell^1$ $(k = 1, 2, 3, \ldots)$ satisfy $x_1 \le x_2 \le \ldots \le x_k \le \ldots$ and $M = \sup_k \|x_k\| < \infty$. Then, $x_{1n} \le x_{2n} \le \ldots \le x_{kn} \le \ldots \le M$ $(n = 1, 2, 3, \ldots)$, so, $\lim_{k \to \infty} x_{kn} = y_n$ $(n = 1, 2, 3, \ldots)$ exist. For any positive integer i, we have $\sum_{n=1}^{i} |x_{kn}| \le M$ $(k = 1, 2, 3, \ldots)$, so, by letting $k \to \infty$, we find $\sum_{n=1}^{i} |y_n| \le M$. Since i is arbitrary, it follows that $\sum_{n=1}^{\infty} |y_n| \le M < \infty$, and therefore $y = (y_1, \ldots y_n, \ldots) \in \ell^1$. It is clear that $x_1 \le x_2 \le \ldots \le x_k \le \ldots \le y$, consequently, the regularity of P implies that $\|x_k - x\| \to 0$ as $k \to \infty$ for some $x \in \ell^1$. Hence the full regularity of P is proven.

Now, system (26) can be regarded as a TVP of the form (1), where $k(t,s) = e^{-(t+1)s}$, $h(t,s) = (1 + t + s^2)^{-1}$, $x = (x_1, \ldots, x_n, \ldots)$, $y = (y_1, \ldots, y_n, \ldots)$, $z = (z_1, \ldots, z_n, \ldots)$, $f = (f_1, \ldots, f_n, \ldots)$, in which

$$f_n(t,x,y,z) = -\frac{e^{-t}}{2^{n+3}}\left(1 + x_n + \sqrt{x_{n+1} + 2x_{2n+1}}\right) - \frac{e^{-2t}}{3^n} y_n^{1/3} - \frac{e^{-t}}{4^n} z_{2n}^{1/5},$$

and $t_m = m$, $I_m = (I_{m1}, \ldots, I_{mn}, \ldots)$ with

$$I_{mn}(x) = -\frac{1}{2^{n+m+2}}(x_n + x_{n+2}), \quad (m, n = 1, 2, 3, \ldots),$$

and $x^* = (1, \ldots, \frac{1}{n^2}, \ldots) \in P$. Evidently, $f \in C(J \times P \times P \times P, -P)$ and $I_m \in C(P, -P)$ $(m = 1, 2, 3, \ldots)$. (H_1) is obviously satisfied since

$$k^* = \sup_{t \in J} \int_0^t e^{-(t+1)s} ds = \sup_{t \in J} \frac{1}{t+1}(1 - e^{-(t+1)t}) \le 1,$$

$$h^* = \sup_{t \in J} \int_0^{\infty} \frac{ds}{1 + t + s^2} \le \int_0^{\infty} \frac{ds}{1 + s^2} = \frac{\pi}{2},$$

and

$$\int_0^{\infty} \left| \frac{1}{1 + t' + s^2} - \frac{1}{1 + t + s^2} \right| ds = \int_0^{\infty} \frac{|t' - t|}{(1 + t' + s^2)(1 + t + s^2)} ds \le \frac{\pi}{2} |t' - t| \to 0$$

as $t' \to t$. It is easy to verify the following scalar inequality:

$$u^{\alpha} \le 1 - \alpha + \alpha u, \quad 0 \le u < \infty, \; 0 < \alpha < 1,$$

so, for $t \in J$, $x, y, z \in P$,

$|f_n(t,x,y,z)|$

$$\leq \frac{e^{-t}}{2^{n+3}}(1 + x_n + \frac{1}{2}(x_{n+1} + 2x_{2n+1})) + \frac{e^{-2t}}{3^n}(\frac{2}{3} + \frac{1}{3}y_n) + \frac{e^{-t}}{4^n}(\frac{4}{5} + \frac{1}{5}z_{2n})$$

$$\leq \frac{e^{-t}}{2^{n+3}}(1 + \| x \|) + \frac{e^{-2t}}{3^n}(\frac{2}{3} + \frac{1}{3}\| y \|) + \frac{e^{-t}}{4^n}(\frac{4}{5} + \frac{1}{5}\| z \|),$$

and therefore,

$$\| f(t,x,y,z) \| = \sum_{n=1}^{\infty} | f_n(t,x,y,z) | \leq e^{-t}\left(\sum_{n=1}^{\infty}\frac{1}{2^{n+3}} + \frac{2}{3}\sum_{n=1}^{\infty}\frac{1}{3^n} + \frac{4}{5}\sum_{n=1}^{\infty}\frac{1}{4^n}\right)$$

$$+ e^{-t}\left(\| x \| \sum_{n=1}^{\infty}\frac{1}{2^{n+3}} + \frac{1}{3}\| y \| \sum_{n=1}^{\infty}\frac{1}{3^n} + \frac{1}{5}\| z \| \sum_{n=1}^{\infty}\frac{1}{4^n}\right)$$

$$= \frac{87}{120}e^{-t} + e^{-t}\left(\frac{1}{8}\| x \| + \frac{1}{6}\| y \| + \frac{1}{15}\| z \|\right).$$

In addition, we have, for $x \in P$,

$$| I_{mn}(x) | \leq \frac{1}{2^{n+m+1}}\| x \|,$$

and so

$$\| I_m(x) \| = \sum_{n=1}^{\infty} | I_{mn}(x) | \leq \frac{1}{2^{m+1}}\| x \|.$$

Hence (H_2) is satisfied for $p(t) = (87/120)e^{-t}$, $q(t) = e^{-t}$, $a = 1/8$, $b = 1/6$, $c = 1/15$, $a_m = 0$ and $b_m = 1/2^{m+1}$ $(m = 1,2,3,\ldots)$, and therefore $p^* = 87/120$, $q^* = 1$, $a^* = 0$ and $b^* = 1/2$.

On the other hand, (H_3) is obviously satisfied, and

$$r = b^* + (a + bk^* + ch^*)q^* \leq \frac{1}{2} + \left(\frac{1}{8} + \frac{1}{6} + \frac{\pi}{30}\right) < 1,$$

i.e., (11) holds. Hence the assertion follows from Theorem 1. □

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