Journal of Applied Mathematics and Stochastic Analysis, **10**:1 (1997), 21-45.

THE FAILURE RATE IN RELIABILITY. NUMERICAL TREATMENT[1]

CHRISTIANE COCOZZA-THIVENT
Université de Marne la Vallée, Equipe d'Analyse et de Mathématiques Appliquées, 2 rue de la Butte Verte
93166 Noisy le Grand Cedex, France

VLADIMIR KALASHNIKOV
Institute of Systems Analysis, 9, Prospect 60 let Oktyabrya
117312, Moscow Russia

(Received April, 1996; Revised December, 1996)

This paper contains approximations and bounds of failure rate functions of redundant systems. We illustrate properties of these bounds and their accuracy.

Key words: Redundant System, Markov Model, Semi-Markov Process, Failure Rate, Accuracy Estimate.

AMS subject classifications: 60K10, 60K15, 60K20.

1. Introduction

In this paper, we consider various examples to gain a more precise understanding of the constructions and estimates developed in the first part of this work [3]. These examples are chosen because of their importance in reliability area. Our purpose is two-fold: (1) to illustrate real accuracy of the proposed bounds and depict their domain; (2) to show the simplicity of corresponding calculations for their immediate applications. We suppose (perhaps, self-sufficiently) that the reader is familiar with the results of [3]. Moreover, we will often refer to some relations and assertions from [3]. In this case, if we refer to Corollary 5.4 proved in [3], then we will write Corollary I.5.4. Similarly, if we refer to equation (5.1) from [3], then we will write (I.5.1). Of course, similar agreement remains true for other references from [3] as well. In addition, we preserve some basic notations from [3].

Recall that the principal object of our investigation is a random process (η_s) describing the dynamics of a system; its state space E is partitioned into two subsets: $E = \mathcal{M} \cup \mathcal{P}$, where $\mathcal{M}$ is treated as a subset of "good" (or operating) states and $\mathcal{P}$ as

[1]This work was partly supported by Russian Foundation of Fundamental Research (grant 95-01-00023), International Science Foundation and Russian Government (grant J76100), EEC (grant INTAS-93-893), and University of Marne-La-Vallée.

a subset of "bad" (or failed) states. Let $\sigma = \inf\{s: \eta_s \in \mathcal{P}\}$ be the first break-down time of the system. The *failure rate function* for process (η_s) is defined as

$$\lambda(t) = \lim_{\Delta \to 0} \frac{\mathbf{P}(\sigma \le t + \Delta \mid \sigma > t)}{\Delta} \tag{1.1}$$

and the *Vesely failure rate* as

$$\lambda_V(t) = \lim_{\Delta \to 0} \frac{\mathbf{P}(\eta_{t+\Delta} \in \mathcal{P} \mid \eta_t \in \mathcal{M})}{\Delta}. \tag{1.2}$$

In Proposition I.3.3 we introduced a quantity λ^0 which can be regarded as the failure intensity provided that the states of (η_s) follow the stationary distribution of an auxiliary process (η_s^0) (see (I.3.9)). We are interested in limiting values $\lambda(\infty)$ and $\lambda_V(\infty)$ of the above functions. More precisely, we are looking for accuracy estimates in approximations of $\lambda(\infty)$ by $\lambda_V(\infty)$ and by λ^0. It turned out that, in many cases $\lambda(\infty) \le \lambda^0$ (see Proposition I.3.3). Hence, the following accuracy estimates are of interest:

$$\rho_V = \frac{\mid \lambda_V(\infty) - \lambda(\infty) \mid}{\lambda(\infty)}, \quad \rho^0 = \frac{\lambda^0 - \lambda(\infty)}{\lambda(\infty)}. \tag{1.3}$$

For the case of special interest, when (η_s) is a Markov process with a finite state space, Corollary I.5.4 gives accuracy estimates in terms of the following quantities

$$\epsilon = \delta \max_{\eta \ne 0} \alpha(\eta), \tag{1.4}$$

$$\epsilon_0 = \frac{\alpha(0)}{\mid A^0(0,0) \mid} + \epsilon, \tag{1.5}$$

$$\underline{\delta}_0 = \frac{1}{\mid A^0(0,0) \mid} + \sum_{\eta \ne 0} \frac{A^0(0,\eta)}{A^0(0,0)A^0(\eta,\eta)}, \tag{1.6}$$

$$\underline{\beta}_D = \frac{1}{\mid A(0,0) \mid} + \sum_{\eta \in \mathcal{M}, \eta \ne 0} \frac{A(0,\eta)}{A(0,0)A(\eta,\eta)}, \tag{1.7}$$

where $A(i,j)$ is the intensity for Markov process (η_s) to jump from state i to state j and $A^0(i,j) = A(i,j)$ for all $i, j \in \mathcal{M}$. Denote upper bounds of ρ_V and $\lambda(\infty)$ by

$$r_V = e^{\epsilon}(\epsilon + \frac{\delta \epsilon_0}{\underline{\beta}_D}), \tag{1.8}$$

$$\lambda_{sup} = (1 + r_V)\lambda_V(\infty), \tag{1.9}$$

respectively (see Corollary I.5.4). Following Section I.5.1, let

$$r^0 = \frac{\delta \mathbf{E}\xi / \underline{\delta}_0 + \mathbf{E}\xi^2 / 2\mathbf{E}\xi}{1 - \mathbf{E}\xi^2/2\mathbf{E}\xi} \text{ (if } 1 - \mathbf{E}\xi^2/2\mathbf{E}\xi > 0) \tag{1.10}$$

be an upper bound of ρ^0 in the case where $\mathbf{E}\xi$ and $\mathbf{E}\xi^2$ can be calculated explicitly (see (I.5.7)), and let

$$\lambda^0_{sup} = \frac{\epsilon_0}{\underline{\delta}_0} \tag{1.11}$$

be an upper bound of λ^0 (see (I.5.1)).

The paper is organized as follows. In Section 2, we consider a set of simple redundant systems consisting of three components under Markov assumptions and various types of redundancy. Section 3 is devoted to so-called k-out-of-n systems (also under Markov assumptions) consisting of n components such that the system is operating if and only if at least k components are operating. A generalization of this type of systems is considered in Section 4 where Markov systems with independent components are studied and where we do not impose specific restrictions on set $\mathcal{M}$ except for the system coherence. In Section 5, we relax the Markov assumption and consider semi-Markov models. All considerations are demonstrated by corresponding numerical results.

2. Simple Examples

In this group of examples, we show how to estimate the parameters involved in Corollary I.5.4 and to obtain the accuracies of the bounds derived for $\lambda(\infty)$. In all examples, we deal with a system consisting of three components, designated by $C1$, $C2$, and $C3$. Further restrictions will be imposed in the upcoming subsections.

2.1 Passive redundancy

Assume that components $C1$ and $C3$ can be in two states: "operating" denoted by 1 (resp. 3) and "failed" denoted by $\overline{1}$ (resp. $\overline{3}$). Component $C2$ is redundant with respect to $C1$. Normally, $C2$ is waiting (state 2_w), but when $C1$ fails, $C2$ tries to replace the failed component. The replacement occurs with probability $1-\gamma$, $0 \leq \gamma \leq 1$, and then $C2$ enters the operating state denoted by 2; otherwise, $C2$ enters the failed state $\overline{2}$. Upon repairing $C1$, component $C2$ (if it is not failed) returns to the waiting state. The failure rate of Ci is λ_i and its repair rate is μ_i. Component $C2$ cannot fail while it is waiting.

The dynamics of the system can be described by a Markov process with 8 states:

$$e_0 = (12_w3),\ e_1 = (\overline{1}23),\ e_2 = (1\overline{2}3),\ e_3 = (12_w\overline{3}),\ e_4 = (1\overline{2}\overline{3}),\ e_5 = (\overline{1}2\overline{3}),$$
$$e_6 = (\overline{1}\overline{2}3),\ e_7 = (\overline{1}\overline{2}\overline{3}).$$

Transition rates of the first four states are given as follows:

$$A(e_0,e_1) = (1-\gamma)\lambda_1, A(e_0,e_3) = \lambda_3, A(e_0,e_6) = \gamma\lambda_1,$$
$$A(e_1,e_0) = \mu_1, A(e_1,e_5) = \lambda_3, A(e_1,e_6) = \lambda_2,$$
$$A(e_2,e_0) = \mu_2, A(e_2,e_4) = \lambda_3, A(e_2,e_6) = \lambda_1,$$
$$A(e_3,e_0) = \mu_3, A(e_3,e_5) = (1-\gamma)\lambda_1, A(e_3,e_7) = \gamma\lambda_1.$$

The rest rates can be written easily. Let us consider three different modes for the subset of operating states:

$$\mathcal{M}_1 = \{e_0,e_1,e_2\},\quad \mathcal{M}_2 = \{e_0,e_1,e_2,e_3,e_4\},\quad \mathcal{M}_3 = \{e_0,e_1,e_2,e_3,e_5,e_6\}.$$

Mode $\mathcal{M}_1$ corresponds to the situation where $C3$ is in series with the other components $C1$ and $C2$, in the sense that the system is operating if $C3$ is operating and either $C1$ or $C2$ is operating. In mode $\mathcal{M}_2$, the system is operating if $C1$ is operating or if $C2$ and $C3$ are both operating. Mode $\mathcal{M}_3$ has no clear meaning, in general; it is introduced just to test our methods and to compare this case with the case of independent components.

Let the system be in the "perfect" state $e_0 = 0$ at time 0.

Mode $\mathcal{M}_1$. Matrix A^0 has the form

$$A^0 = \begin{pmatrix} -(1-\gamma)\lambda_1 & (1-\gamma)\lambda_1 & 0 \\ \mu_1 & -\mu_1 & 0 \\ \mu_2 & 0 & -\mu_2 \end{pmatrix}$$

and non-zero values of the intensity function are listed below:

$\alpha(e_0) = A(e_0;\mathcal{P}) = \gamma\lambda_1 + \lambda_3, \alpha(e_1) = A(e_1;\mathcal{P}) = \lambda_2 + \lambda_3, \alpha(e_2) = A(e_2;\mathcal{P}) = \lambda_1 + \lambda_3.$

Let

$$\epsilon_\eta = \mathbf{E}_\eta \int_0^{\tau_0} \alpha(\eta_u^0)du, \quad \delta_\eta = \mathbf{E}_\eta \tau_0. \tag{2.1}$$

Then

$$\epsilon_{e_1} = \frac{\lambda_2+\lambda_3}{\mu_1}, \ \epsilon_{e_2} = \frac{\lambda_1+\lambda_3}{\mu_2}, \ \epsilon_0 = \epsilon_{e_0} = \frac{\gamma\lambda_1+\lambda_3}{(1-\gamma)\lambda_1} + \epsilon_{e_1},$$

$$\delta_{e_1} = \frac{1}{\mu_1}, \ \delta_{e_2} = \frac{1}{\mu_2}, \ \delta_{e_0} = \frac{1}{(1-\gamma)\lambda_1} + \frac{1}{\mu_1},$$

and

$$\mathbf{E}(\sigma \wedge \sigma_0) = \frac{1}{\lambda_1+\lambda_3} + \frac{(1-\gamma)\lambda_1}{\lambda_1+\lambda_3}\,\frac{1}{\mu_1+\lambda_2+\lambda_3}.$$

So,

$$\epsilon = \max(\epsilon_{e_1}, \epsilon_{e_2}), \ \epsilon_0 = \epsilon_{e_0}, \ \delta = \max(\delta_{e_1}, \delta_{e_2}), \ \underline{\delta}_0 = \delta_{e_0}, \ \underline{\beta}_D = \mathbf{E}(\sigma \wedge \sigma_0),$$

and, by (I.5.1)

$$\lambda_0 = \frac{\epsilon_{e_0}}{\delta_{e_0}}.$$

In order to find the accuracy of relation (I.5.7), we have to calculate moments of the r.v.ξ. It can be done by using the equality

$$\xi = (\gamma\lambda_1 + \lambda_3)W_0 + (\lambda_2 + \lambda_3)W_1,$$

where W_0 and W_1 are two independent r.v.'s exponentially distributed with parameters $(1-\gamma)\lambda_1$ and μ_1, respectively. This yields

$$r^0 = \frac{\delta\epsilon_{e_0}/\delta_{e_0} + \mathbf{E}\xi^2/2\epsilon_{e_0}}{1 - \mathbf{E}\xi^2/2\epsilon_{e_0}} \quad \text{if } \mathbf{E}\xi^2 > 2\epsilon_{e_0},$$

$$\mathbf{E}\xi^2 = \frac{(\gamma\lambda_1+\lambda_3)^2}{(1-\gamma)^2\lambda_1^2} + \frac{(\lambda_2+\lambda_3)^2}{\mu_1^2} + \epsilon_{e_0}^2.$$

Mode $\mathcal{M}_2$. Matrix A^0 has the form

$$A^0 = \begin{pmatrix} -((1-\gamma)\lambda_1+\lambda_3) & (1-\gamma)\lambda_1 & 0 & \lambda_3 & 0 \\ \mu_1 & -\mu_1 & 0 & 0 & 0 \\ \mu_2 & 0 & -(\mu_2+\lambda_3) & 0 & \lambda_3 \\ \mu_3 & 0 & 0 & -\mu_3 & 0 \\ 0 & 0 & \mu_3 & \mu_2 & -(\mu_2+\mu_3) \end{pmatrix}$$

and non-zero values of the intensity function are

$$\alpha(e_0) = \gamma\lambda_1, \quad \alpha(e_1) = \lambda_2 + \lambda_3, \quad \alpha(e_2) = \alpha(e_3) = \alpha(e_4) = \lambda_1.$$

Let ϵ_η and δ_η be as in (2.1). In order to estimate them we can use (1.4) through (1.7) and Lemma I.5.5. However, a better result can be achieved if we estimate δ_{e_1} and ϵ_{e_1} separately from δ_{e_i} and ϵ_{e_i} for $i \geq 2$. In fact,

$$\delta_{e_1} = \frac{1}{\mu_1}, \qquad \epsilon_{e_1} = (\lambda_2 + \lambda_3)\delta_{e_1}.$$

Quantities δ_{e_i}, $i \geq 2$, can easily be found as the solution of three linear algebraic equations. But we prefer to use Lemma I.5.5 to show how it works and to prove its accuracy.

Take the test function with

$$V(e_0) = 0, \; V(e_2) = V(e_3) = a, \; V(e_4) = a + b.$$

Conditions (I.5.26) yield

$$-\mu_2 a + \lambda_3 b \leq -1, \quad -\mu_3 a \leq -1, \quad -(\mu_2 + \mu_3)b \leq -1.$$

Let us choose

$$b = \frac{1}{\mu_2 + \mu_3}, \quad a = \max\left(\frac{1}{\mu_3}, \frac{1}{\mu_2}\left(1 + \frac{\lambda_3}{\mu_2 + \mu_3}\right)\right),$$

$$\delta = \max(\delta_{e_1}, a + b), \quad \epsilon = \max(\epsilon_{e_1}, \lambda_1(a + b)),$$

and

$$\epsilon_0 = \frac{\gamma\lambda_1}{(1-\gamma)\lambda_1 + \lambda_3} + \frac{(1-\gamma)\lambda_1\epsilon_{e_1}}{(1-\gamma)\lambda_1 + \lambda_3} + \frac{\lambda_3}{(1-\gamma)\lambda_1 + \lambda_3}\frac{\lambda_1}{\mu_3}.$$

Parameters $\underline{\delta}_0$ and $\underline{\beta}_D$ are given in (1.6) and (1.7).

Mode $\mathcal{M}_3$. Matrix A^0 has the form

$$A^0 = \begin{pmatrix} -(\lambda_1+\lambda_3) & (1-\gamma)\lambda_1 & 0 & \lambda_3 & 0 & \gamma\lambda_1 \\ \mu_1 & -(\mu_1+\lambda_2+\lambda_3) & 0 & 0 & \lambda_3 & \lambda_2 \\ \mu_2 & 0 & -(\mu_2+\lambda_1) & 0 & 0 & \lambda_1 \\ \mu_3 & 0 & 0 & -(\mu_3+(1-\gamma)\lambda_1) & (1-\gamma)\lambda_1 & 0 \\ 0 & \mu_3 & 0 & \mu_1 & -(\mu_1+\mu_3) & 0 \\ 0 & \mu_2 & \mu_1 & 0 & 0 & -(\mu_1+\mu_2) \end{pmatrix}$$

and non-zero values of the intensity function are

$$\alpha(e_2) = \lambda_3, \quad \alpha(e_3) = \gamma\lambda_1, \; \alpha(e_5) = \lambda_2, \quad \alpha(e_6) = \lambda_3.$$

For estimating, let us employ (1.4) through (1.7) and Lemma I.5.5. Take the test function with

$$V(e_0) = 0, \; V(e_1) = \frac{1}{\mu_1}\left(1 + \frac{\lambda_2 + \lambda_3}{\underline{\mu}}\right), \; V(e_i) = V(e_1) + \frac{1}{\underline{\mu}}(i \in \{2,3,5,6\}), \qquad (2.2)$$

where

$$\underline{\mu} = \min(\mu_2, \mu_3).$$

Then

$$\delta = \left(1 + \frac{\lambda_2 + \lambda_3}{\underline{\mu}}\right)\frac{1}{\mu_1} + \frac{1}{\underline{\mu}}.$$

Parameters $\underline{\delta}_0$ and $\underline{\beta}_D$ can be estimated with the help of (1.6) and (1.7).

2.2 Independent Components

Assume now that the three components are independent.

Mode $\mathcal{M}_1$. All the characteristics can be calculated explicitly:

$$\delta = \max\left(\frac{1}{\mu_1}, \frac{1}{\mu_2}\right), \quad \epsilon = \max\left(\frac{\lambda_2 + \lambda_3}{\mu_1}, \frac{\lambda_1 + \lambda_3}{\mu_2}\right),$$

$$\epsilon_0 = \frac{1}{\lambda_1 + \lambda_2}\left(\lambda_3 + \frac{\lambda_1(\lambda_2 + \lambda_3)}{\mu_1} + \frac{\lambda_2(\lambda_1 + \lambda_3)}{\mu_2}\right),$$

$$\mathbf{E}_0\tau_0 = \frac{1}{\lambda_1 + \lambda_2}\left(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}\right) \quad \mathbf{E}_0(\sigma \wedge \sigma_0) = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}\left(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2}\right).$$

Mode $\mathcal{M}_2$. We estimate δ similarly for the passive redundance case. Take $\delta = \max(1/\mu_1, a + b)$ with

$$b = \frac{1}{\mu_2 + \mu_3}, \quad a = \max\left(\frac{1}{\mu_2}\left(1 + \frac{\lambda_3}{\mu_2 + \mu_3}\right), \frac{1}{\mu_3}\left(1 + \frac{\lambda_2}{\mu_2 + \mu_3}\right)\right),$$

$$\epsilon = \epsilon_0 = \max\left(\frac{\lambda_2 + \lambda_3}{\mu_1}, \ \delta\max(\lambda_1 + \lambda_3, \lambda_1 + \lambda_2)\right),$$

Formulas (1.6) and (1.7) yield

$$\underline{\delta}_0 = \underline{\beta}_D = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}\left(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} + \frac{\lambda_3}{\mu_3}\right).$$

Mode $\mathcal{M}_3$. The steps are the same as in the passive redundance case. Let us take the following test function

$$V(e_0) = 0, \ \ V(e_1) = V(e_2) = V(e_3) = a, \ \ V(e_6) = V(e_7) = a + b,$$

where

$$a = \max\left(\frac{1}{\mu_1}\left(1 + \frac{\lambda_2 + \lambda_3}{\mu_1 + \underline{\mu}}\right), \frac{1}{\underline{\mu}}\left(1 + \frac{\lambda_1}{\mu_1 + \underline{\mu}}\right)\right), \quad b = \frac{1}{\mu_1 + \underline{\mu}}, \ \underline{\mu} = \min(\mu_1, \mu_2).$$

Then

$$\delta = a + b, \ \ \epsilon = \epsilon_0 = \delta\max(\lambda_2, \lambda_3).$$

Quantities $\underline{\delta}_0$ and $\underline{\beta}_D$ can be estimated with the help of formulas (1.6) and (1.7). The result is the same as in the case $\mathcal{M}_2$.

2.3 Numerical Results

The Vesely failure rate is supposed to approximate the true failure rate to a good accuracy if "failure rates of components are small in comparison with their repair rates". This is examined in the numerical calculations. Consider the following five groups of figures.

	λ_1	μ_1	λ_2	μ_2	λ_3	μ_3	γ
1.*a*	10	10^4	10	10^4	1	$5 \cdot 10^3$	0
1.*b*	10	10^4	10	10^4	1	$5 \cdot 10^3$	0.75
2.*a*	10	10^4	100	$2 \cdot 10^5$	1	$5 \cdot 10^3$	0.05
2.*b*	10	10^4	100	$2 \cdot 10^5$	1	$5 \cdot 10^3$	0.2
3.*a*	10	10^3	10	$2 \cdot 10^3$	1	$5 \cdot 10^2$	0.05
3.*b*	10	10^3	1	$5 \cdot 10^2$	10	10^3	0.05
4.*a*	10	10^4	100	$2 \cdot 10^5$	0.02	100	0.05
4.*b*	0.02	100	100	$2 \cdot 10^5$	10	10^4	0.05
5.*a*	10	10^4	10^{-3}	1	1	$5 \cdot 10^3$	0.05
5.*b*	1	$5 \cdot 10^3$	10	10^4	10^{-3}	1	0.05

In groups 1, 2, 4, and 5 $\lambda_i/\mu_i \leq 10^{-3}$ for all i; $\lambda_i/\mu_i \leq 10^{-2}$ in group 3. In addition, $\lambda_i/\mu_j \leq 2 \cdot 10^{-3}$ in group 1 and $\lambda_i/\mu_j \leq 2 \cdot 10^{-2}$ in groups 2 and 3 for all i and j; in group 4 and 5, there exists a pair i, j such that $\lambda_i = \mu_j$ and $\lambda_i = 10\mu_j$ correspondingly. The results are contained in the following table where *pas.red.* means *passive redundancy*, and *ind.* means *independent case*. See (1.3) to (1.11) for other notations.

data	mode	$\lambda(\infty)$	$\lambda_V(\infty)$ / λ^0	ρ_V / ρ^0	r_V / r^0	λ_{sup} / λ^0_{sup}
1.*a*	$\mathcal{M}_1$	1.0100	1.0100	$9.86 \cdot 10^{-6}$	$1.21 \cdot 10^{-3}$	1.0112
pas.			1.0100	$9.86 \cdot 10^{-6}$	$1.11 \cdot 10^{-1}$	1.0100
	$\mathcal{M}_2$	$1.2968 \cdot 10^{-2}$	$1.2984 \cdot 10^{-2}$	$1.20 \cdot 10^{-3}$	$2.68 \cdot 10^{-3}$	$1.3019 \cdot 10^{-2}$
			$1.2984 \cdot 10^{-2}$	$1.23 \cdot 10^{-3}$	$1.24 \cdot 10^{-3}$	$1.2984 \cdot 10^{-2}$
	$\mathcal{M}_3$	$2.9941 \cdot 10^{-6}$	$2.9949 \cdot 10^{-6}$	$2.56 \cdot 10^{-4}$	$3.02 \cdot 10^{-3}$	$3.0040 \cdot 10^{-6}$
			$2.9956 \cdot 10^{-5}$	$4.78 \cdot 10^{-4}$		$3.2985 \cdot 10^{-2}$
1.*a*	$\mathcal{M}_1$	1.0199	1.0200	$1.95 \cdot 10^{-5}$	$1.21 \cdot 10^{-3}$	1.0212
ind.			same	same		1.0200
	$\mathcal{M}_2$	$2.2926 \cdot 10^{-2}$	$2.2952 \cdot 10^{-2}$	$1.13 \cdot 10^{-3}$	$5.39 \cdot 10^{-3}$	$2.3075 \cdot 10^{-2}$
			same	same		$1.1181 \cdot 10^{-1}$
	$\mathcal{M}_3$	$2.2923 \cdot 10^{-3}$	$2.9964 \cdot 10^{-3}$	$1.36 \cdot 10^{-3}$	$2.69 \cdot 10^{-3}$	$3.0045 \cdot 10^{-3}$
			same	same		$5.5905 \cdot 10^{-2}$

data	mode	$\lambda(\infty)$	$\lambda_V(\infty)$ λ^0	ρ_V ρ^0	r_V r^0	λ_{sup} λ^0_{sup}
1.*b* *pas.*	$\mathcal{M}_1$	8.5006	8.5025 8.5006	$2.20 \cdot 10^{-4}$ $1.84 \cdot 10^{-8}$	$4.85 \cdot 10^{-3}$ – – –	8.5437 8.5006
	$\mathcal{M}_2$	7.5014	7.5036 7.5014	$3.00 \cdot 10^{-4}$ $7.40 \cdot 10^{-8}$	$8.98 \cdot 10^{-3}$ – – –	7.5710 7.5014
	$\mathcal{M}_3$	$2.2459 \cdot 10^{-3}$	$2.2483 \cdot 10^{-3}$ $2.2482 \cdot 10^{-3}$	$1.09 \cdot 10^{-3}$ $1.03 \cdot 10^{-3}$	$3.02 \cdot 10^{-3}$	$2.2551 \cdot 10^{-3}$ $3.2997 \cdot 10^{-2}$
1.*b* *ind.*	$\mathcal{M}_1$	1.0199	1.0200 same	$1.95 \cdot 10^{-5}$ same	$1.21 \cdot 10^{-3}$	1.0212 1.0200
	$\mathcal{M}_2$	$2.2926 \cdot 10^{-2}$	$2.2952 \cdot 10^{-2}$ same	$1.13 \cdot 10^{-3}$ same	$5.39 \cdot 10^{-3}$	$2.3075 \cdot 10^{-2}$ $1.1181 \cdot 10^{-1}$
	$\mathcal{M}_3$	$2.2923 \cdot 10^{-3}$	$2.9964 \cdot 10^{-3}$ same	$1.36 \cdot 10^{-3}$ same	$2.69 \cdot 10^{-3}$	$3.0045 \cdot 10^{-3}$ $5.5905 \cdot 10^{-2}$
2.*a* *pas.*	$\mathcal{M}_1$	1.5935	1.5991 1.5944	$3.52 \cdot 10^{-3}$ $5.83 \cdot 10^{-4}$	$1.04 \cdot 10^{-2}$ $1.89 \cdot 10^{-1}$	1.6157 1.5944
	$\mathcal{M}_2$	$5.9631 \cdot 10^{-1}$	$6.0199 \cdot 10^{-1}$ $5.9726 \cdot 10^{-1}$	$9.52 \cdot 10^{-3}$ $1.59 \cdot 10^{-3}$	$1.03 \cdot 10^{-2}$ $5.19 \cdot 10^{-2}$	$6.0821 \cdot 10^{-1}$ $5.9726 \cdot 10^{-1}$
	$\mathcal{M}_3$	$1.2203 \cdot 10^{-4}$	$1.2279 \cdot 10^{-4}$ $1.2217 \cdot 10^{-4}$	$6.27 \cdot 10^{-3}$ $1.15 \cdot 10^{-3}$	$3.12 \cdot 10^{-2}$	$1.2663 \cdot 10^{-4}$ $3.3184 \cdot 10^{-3}$
2.*a* *ind.*	$\mathcal{M}_1$	1.1039	1.1048 same	$8.94 \cdot 10^{-4}$ same	$1.03 \cdot 10^{-2}$	1.1162 1.1048
	$\mathcal{M}_2$	$1.0681 \cdot 10^{-1}$	$1.0782 \cdot 10^{-1}$ same	$9.46 \cdot 10^{-3}$ same	$2.36 \cdot 10^{-2}$	$1.1036 \cdot 10^{-1}$ $2.4985 \cdot 10^{-1}$
	$\mathcal{M}_3$	$2.0094 \cdot 10^{-2}$	$2.0486 \cdot 10^{-2}$ same	$1.95 \cdot 10^{-2}$ same	$2.82 \cdot 10^{-2}$	$2.1064 \cdot 10^{-2}$ 2.9565
2.*b* *pas.*	$\mathcal{M}_1$	3.0776	3.0969 3.0783	$6.29 \cdot 10^{-3}$ $2.47 \cdot 10^{-4}$	$1.06 \cdot 10^{-2}$ $6.01 \cdot 10^{-1}$	3.1298 3.0783
	$\mathcal{M}_2$	2.0799	2.0995 2.0807	$9.40 \cdot 10^{-3}$ $3.74 \cdot 10^{-4}$	$1.07 \cdot 10^{-2}$ $2.87 \cdot 10^{-1}$	2.1220 2.0807
	$\mathcal{M}_3$	$4.2702 \cdot 10^{-4}$	$4.2978 \cdot 10^{-4}$ $4.2730 \cdot 10^{-4}$	$6.47 \cdot 10^{-3}$ $6.62 \cdot 10^{-4}$	$3.12 \cdot 10^{-2}$	$4.4320 \cdot 10^{-4}$ $3.3189 \cdot 10^{-1}$

data	mode	$\lambda(\infty)$	$\lambda_V(\infty)$ λ^0	ρ_V ρ^0	r_V r^0	λ_{sup} λ^0_{sup}
2.*b* *ind.*	$\mathcal{M}_1$	1.1039	1.1048 same	$8.94 \cdot 10^{-4}$ same	$1.03 \cdot 10^{-2}$	1.1162 1.1048
	$\mathcal{M}_2$	$1.0681 \cdot 10^{-1}$	$1.0782 \cdot 10^{-1}$ same	$9.46 \cdot 10^{-3}$ same	$2.36 \cdot 10^{-2}$	$1.1036 \cdot 10^{-1}$ $2.4985 \cdot 10^{-1}$
	$\mathcal{M}_3$	$2.0094 \cdot 10^{-2}$	$2.0486 \cdot 10^{-2}$ same	$1.95 \cdot 10^{-2}$ same	$2.82 \cdot 10^{-2}$	$2.1064 \cdot 10^{-2}$ 2.9565
3.*a* *pas.*	$\mathcal{M}_1$	1.5886	1.5941 1.5894	$3.45 \cdot 10^{-3}$ $5.20 \cdot 10^{-4}$	$1.30 \cdot 10^{-2}$ $1.90 \cdot 10^{-1}$	1.6148 1.5894
	$\mathcal{M}_2$	$6.1606 \cdot 10^{-1}$	$6.2224 \cdot 10^{-1}$ $6.1740 \cdot 10^{-1}$	$1.00 \cdot 10^{-2}$ $2.18 \cdot 10^{-3}$	$2.92 \cdot 10^{-2}$ $5.43 \cdot 10^{-2}$	$6.4039 \cdot 10^{-1}$ $6.1740 \cdot 10^{-1}$
	$\mathcal{M}_3$	$1.7637 \cdot 10^{-3}$	$1.7693 \cdot 10^{-3}$ $1.7668 \cdot 10^{-3}$	$3.18 \cdot 10^{-3}$ $1.75 \cdot 10^{-3}$	$3.22 \cdot 10^{-2}$	$1.8262 \cdot 10^{-3}$ $3.2861 \cdot 10^{-1}$
3.*a* *ind.*	$\mathcal{M}_1$	1.1942	1.1961 same	$1.56 \cdot 10^{-3}$ same	$1.24 \cdot 10^{-2}$	1.2109 1.1961
	$\mathcal{M}_2$	$2.2282 \cdot 10^{-1}$	$2.2524 \cdot 10^{-1}$ same	$1.09 \cdot 10^{-2}$ same	$5.97 \cdot 10^{-2}$	$2.3868 \cdot 10^{-1}$ 1.1014
	$\mathcal{M}_3$	$2.9250 \cdot 10^{-2}$	$2.9644 \cdot 10^{-2}$ same	$1.35 \cdot 10^{-2}$ same	$2.90 \cdot 10^{-2}$	$3.0505 \cdot 10^{-2}$ $5.5068 \cdot 10^{-1}$
3.*b* *pas.*	$\mathcal{M}_1$	10.505	10.511 10.505	$6.12 \cdot 10^{-4}$ $2.20 \cdot 10^{-7}$	$8.77 \cdot 10^{-2}$ — — —	11.433 10.505
	$\mathcal{M}_2$	$6.8920 \cdot 10^{-1}$	$6.9889 \cdot 10^{-1}$ $6.9103 \cdot 10^{-1}$	$1.41 \cdot 10^{-2}$ $2.65 \cdot 10^{-3}$	$2.95 \cdot 10^{-2}$ $3.12 \cdot 10^{-2}$	$7.1949 \cdot 10^{-1}$ 6.9103
	$\mathcal{M}_3$	$1.4781 \cdot 10^{-2}$	$1.4979 \cdot 10^{-2}$ $1.4978 \cdot 10^{-2}$	$1.34 \cdot 10^{-2}$ $1.34 \cdot 10^{-2}$	$3.30 \cdot 10^{-2}$	$1.5473 \cdot 10^{-2}$ $5.9276 \cdot 10^{-1}$
3.*b* *ind.*	$\mathcal{M}_1$	10.029	10.030 same	$3.93 \cdot 10^{-5}$ same	$8.15 \cdot 10^{-2}$	13.014 10.030
	$\mathcal{M}_2$	$2.2282 \cdot 10^{-1}$	$2.2524 \cdot 10^{-1}$ same	$1.09 \cdot 10^{-2}$ same	$5.97 \cdot 10^{-2}$	$2.3868 \cdot 10^{-1}$ 1.1014
	$\mathcal{M}_3$	$2.9250 \cdot 10^{-2}$	$2.9644 \cdot 10^{-2}$ same	$1.35 \cdot 10^{-2}$ same	$2.90 \cdot 10^{-2}$	$3.0505 \cdot 10^{-1}$ $5.5068 \cdot 10^{-1}$

data	mode	$\lambda(\infty)$	$\lambda_V(\infty)$ λ^0	ρ_V ρ^0	r_V r^0	λ_{sup} λ^0_{sup}
4.*a* *pas.*	$\mathcal{M}_1$	$6.1351 \cdot 10^{-1}$	$6.1912 \cdot 10^{-1}$ $6.1444 \cdot 10^{-1}$	$9.15 \cdot 10^{-3}$ $1.51 \cdot 10^{-3}$	$1.02 \cdot 10^{-2}$ $5.97 \cdot 10^{-2}$	$6.2541 \cdot 10^{-1}$ $6.1444 \cdot 10^{-1}$
	$\mathcal{M}_2$	$5.9524 \cdot 10^{-1}$	$6.0102 \cdot 10^{-1}$ $5.9633 \cdot 10^{-1}$	$9.70 \cdot 10^{-3}$ $1.83 \cdot 10^{-3}$	$1.17 \cdot 10^{-1}$ $6.39 \cdot 10^{-2}$	$6.7165 \cdot 10^{-1}$ $5.9633 \cdot 10^{-1}$
	$\mathcal{M}_3$	$1.1805 \cdot 10^{-4}$	$1.1986 \cdot 10^{-4}$ $1.1894 \cdot 10^{-4}$	$1.53 \cdot 10^{-2}$ $7.49 \cdot 10^{-3}$	3.12 2.99	$4.9351 \cdot 10^{-4}$ 10.209
4.*a* *ind.*	$\mathcal{M}_1$	$1.2386 \cdot 10^{-1}$	$1.2484 \cdot 10^{-1}$ same	$7.97 \cdot 10^{-3}$ same	$1.01 \cdot 10^{-2}$	$1.2611 \cdot 10^{-1}$ $1.2484 \cdot 10^{-1}$
	$\mathcal{M}_2$	$1.0567 \cdot 10^{-1}$	$1.0684 \cdot 10^{-1}$ same	$1.10 \cdot 10^{-2}$ same	6.95	$8.4962 \cdot 10^{-1}$ 120.94
	$\mathcal{M}_3$	$1.0004 \cdot 10^{-2}$	$1.9996 \cdot 10^{-2}$ same	$9.99 \cdot 10^{-1}$ same	5.86	$1.3722 \cdot 10^{-1}$ 111.03
4.*b* *pas.*	$\mathcal{M}_1$	10.010	10.021 10.020	$1.05 \cdot 10^{-3}$ $9.49 \cdot 10^{-4}$	162.07 – – –	1634.1 10.020
	$\mathcal{M}_2$	$1.0961 \cdot 10^{-2}$	$2.2981 \cdot 10^{-2}$ 2.1894	1.10 $9.98 \cdot 10^{-1}$	3.31 – – –	$9.8938 \cdot 10^{-2}$ $2.1894 \cdot 10^{-2}$
	$\mathcal{M}_3$	$2.1785 \cdot 10^{-5}$	$2.2013 \cdot 10^{-5}$ $2.2002 \cdot 10^{-5}$	$1.05 \cdot 10^{-2}$ $9.95 \cdot 10^{-3}$	3.12 2.99	$9.0778 \cdot 10^{-5}$ 10.219
4.*b* *ind.*	$\mathcal{M}_1$	10.010	10.020 same	$9.98 \cdot 10^{-4}$ same	3.64	46.45 10.220
	$\mathcal{M}_2$	$1.0490 \cdot 10^{-2}$	$2.1993 \cdot 10^{-2}$ same	1.10 same	6.93	$1.7449 \cdot 10^{-1}$ 120.82
	$\mathcal{M}_3$	$1.0396 \cdot 10^{-1}$	$1.0484 \cdot 10^{-1}$ same	$9.50 \cdot 10^{-3}$ same	6.01	$7.3489 \cdot 10^{-1}$ 112.12
5.*a* *pas.*	$\mathcal{M}_1$	1.4995	4.6620 1.4995	2.109 $1.57 \cdot 10^{-8}$	$7.63 \cdot 10^{5}$ 1.97	$3.5 \cdot 10^{6}$ 1.4995
	$\mathcal{M}_2$	$5.0237 \cdot 10^{-1}$	3.6639 $5.0237 \cdot 10^{-1}$	6.29 $7.21 \cdot 10^{-6}$	$2.33 \cdot 10^{5}$ $3.18 \cdot 10^{-1}$	$8.53 \cdot 10^{5}$ $5.2837 \cdot 10^{-2}$
	$\mathcal{M}_3$	$2.1909 \cdot 10^{-1}$	$3.3314 \cdot 10^{-1}$ $3.3314 \cdot 10^{-1}$	$5.21 \cdot 10^{-1}$ $5.21 \cdot 10^{-1}$	32.60 32.58	11.195 10.989

data	mode	$\lambda(\infty)$	$\lambda_V(\infty)$ λ^0	ρ_V ρ^0	r_V r^0	λ_{sup} λ^0_{sup}
5.*a*	$\mathcal{M}_1$	1.0009	1.010	$9.06 \cdot 10^{-3}$	$7.25 \cdot 10^5$	$7.32 \cdot 10^5$
ind.			same	same		1.010
	$\mathcal{M}_2$	$3.9015 \cdot 10^{-3}$	$1.2974 \cdot 10^{-2}$	2.33	$1.03 \cdot 10^5$	$6.63 \cdot 10^3$
			same	same		107.9
	$\mathcal{M}_3$	$4.9997 \cdot 10^{-4}$	$9.9900 \cdot 10^{-4}$	$9.98 \cdot 10^{-1}$	32.7	$3.3627 \cdot 10^{-2}$
			same	same		10.99
5.*b*	$\mathcal{M}_1$	$5.2886 \cdot 10^{-2}$	$5.2957 \cdot 10^{-2}$	$1.33 \cdot 10^{-3}$	$2.02 \cdot 10^{-3}$	$5.3064 \cdot 10^{-2}$
pas.			$5.2890 \cdot 10^{-2}$	$7.10 \cdot 10^{-5}$	$5.68 \cdot 10^{-2}$	$5.2890 \cdot 10^{-2}$
	$\mathcal{M}_2$	$5.2373 \cdot 10^{-2}$	$5.2904 \cdot 10^{-2}$	$1.01 \cdot 10^{-2}$	2.87	$2.0474 \cdot 10^{-1}$
			$5.2837 \cdot 10^{-2}$	$8.86 \cdot 10^{-3}$	$1.34 \cdot 10^{-1}$	$5.2887 \cdot 10^{-2}$
	$\mathcal{M}_3$	$4.9287 \cdot 10^{-5}$	$5.1843 \cdot 10^{-5}$	$5.19 \cdot 10^{-2}$	$4.52 \cdot 10^5$	23.46
			$5.1843 \cdot 10^{-5}$	$5.19 \cdot 10^{-2}$	$4.41 \cdot 10^5$	10.003
5.*b*	$\mathcal{M}_1$	$3.9923 \cdot 10^{-3}$	$3.9964 \cdot 10^{-3}$	$1.02 \cdot 10^{-3}$	$2.01 \cdot 10^{-3}$	$4.0044 \cdot 10^{-3}$
ind.			same	same		$3.9964 \cdot 10^{-3}$
	$\mathcal{M}_2$	$3.4915 \cdot 10^{-3}$	$3.9924 \cdot 10^{-3}$	$1.43 \cdot 10^{-1}$	$8.00 \cdot 10^6$	$3.19 \cdot 10^4$
			same	same		108.8
	$\mathcal{M}_3$	$9.0917 \cdot 10^{-4}$	$9.9810 \cdot 10^{-3}$	9.99	$2.65 \cdot 10^6$	$2.65 \cdot 10^4$
			same	same		109.8

In the above table, some values of r^0 are absent. This means that we could not estimate them for different reasons. It is seen that estimate λ^0 is always better than $\lambda_V(\infty)$ and, since it is an upper bound of $\lambda(\infty)$, it is worthy to be used instead of $\lambda(\infty)$ when it can be calculated easily. The results on the Vesely failure rate accuracy are quite expected. The accuracy is very good for group 1, good for groups 2 and 3, bad for group 4, and very bad for group 5. The difference between passive redundancy and independent case is negligible. Parameter γ does not affect the accuracy. We never have $r_V < 10^{-3}$, even if ρ_V is small. Nevertheless, estimate λ_{sup} is good in the first three groups, very bad in group 4 and awful in group 5. Generally, δ_0 is not a good estimate for $\mathbf{E}S$, except for the case where $\mathbf{E}S$ can be calculated explicitly (passive redundancy, variants $\mathcal{M}_1$ and $\mathcal{M}_2$). Consequently, λ^0_{sup} is not a good approximation of $\lambda(\infty)$.

3. The *k*-out-of-*n* Systems

Let us consider a system with n independent components $C1$ to Cn. The system is operating if and only if at least k components are operating ($1 \leq k < n$). The failure rate of component Ci is λ_i, its repair rate is μ_i. If all components are identical, then we can lump the states with the same number of operating components resulting in a birth-and-death process describing the dynamics of such a system. In this case, the system can be examined easily. If the components are not identical, it is still possible to find a pessimistic bound of the reliability in the following way. Let

$\lambda_{(1)} \geq \lambda_{(2)} \geq \ldots \geq \lambda_{(n)}$ (resp. $\mu_{(1)} \leq \mu_{(2)} \leq \ldots \leq \mu_{(n)}$) be failure rates (resp. repair rates) arranged in an decreasing order (resp. ascending order), and let

$$a_i = \sum_{j=1}^{n-i} \lambda_{(j)}, \quad 0 \leq i \leq n-1,$$

$$b_i = \sum_{j=1}^{i} \mu_{(j)}, \quad 1 \leq i \leq n.$$

Consider a birth-and-death process with transition rates

$$\bar{A}(i, i+1) = a_i, \quad 0 \leq i \leq n-1,$$

$$\bar{A}(i, i-1) = b_i, \quad 1 \leq i \leq n.$$

For this process, state i means that exactly i components failed. It can be shown (see Cocozza-Thivent and Roussignol [2]) that the system described by the birth-and-death process (with the subset of "good" states $\bar{\mathcal{M}} = \{0, 1, \ldots, n-k\}$) has a worse reliability performance than the initial one. But the pessimistic estimations using the birth-and-death process may be not very accurate. We will see that, in this case, it is better to use the Vesely failure rate approximation and to give bounds for the relative error using the birth-and-death process.

3.1 Principle

Let $(\bar{\eta}_s^0)$ be a birth-and-death process with transition rates

$$\bar{A}^0(i, i+1) = a_i, \quad 0 \leq i < n-k, \tag{3.1}$$

$$\bar{A}^0(i, i-1) = b_i, \quad 1 \leq i \leq n-k, \tag{3.2}$$

and

$$\bar{\tau}_0 = \inf\{t : \bar{\eta}_t^0 = 0, \bar{\eta}_{t-}^0 \neq \bar{\eta}_t^0\}$$

be its first return time to state 0. It can be shown, using test functions or Cocozza-Thivent and Roussignol [2], that $\mathbf{E}_\eta \tau_0 \leq \mathbf{E}_{n-k} \bar{\tau}_0 (\eta \in \mathcal{M})$ and $\mathbf{E}S \leq \mathbf{E}_0 \bar{\tau}_0$. Let us take

$$\delta = \mathbf{E}_{n-k} \bar{\tau}_0, \quad \bar{\delta}_0 = \mathbf{E}_0 \bar{\tau}_0.$$

These quantities can easily be found as solutions of the following linear system:

$$-(a_1 + b_1)x_1 + a_1 x_2 = -1,$$

$$b_i x_{i-1} - (a_i + b_i)x_i + a_i x_{i+1} = -1, \quad 2 \leq i \leq n-k-1, \tag{3.3}$$

$$b_{n-k} x_{n-k-1} - b_{n-k} x_{n-k} = -1,$$

where $x_i = \mathbf{E}_i \bar{\tau}_0$. This yields

$$\delta = x_{n-k}, \quad \bar{\delta}_0 = \frac{1}{a_0} + x_1.$$

Let α be the intensity function of the process. Then $\alpha(\eta) \neq 0$ if η is a state with exactly k operating components. In this case,

$$\alpha(\eta) \leq \lambda_{(1)} + \ldots + \lambda_{(k)} = a_{n-k}.$$

Therefore,

$$\mathbf{E}_\eta \int_0^{\tau_0} \alpha(\eta_u^0)du \leq a_{n-k}\mathbf{E}_\eta \int_0^{\tau_0} I(\eta_u^0 = n-k)du \leq a_{n-k}\mathbf{E} \int_0^{\overline{\tau}_0} I(\overline{\eta}_u^0 = n-k)du.$$

The last inequality is very intuitive and can be proved using Cocozza-Thivent and Roussignol [2]. In fact,

$$\mathbf{E}_\eta \int_0^{\tau_0} \alpha(\eta_u^0)du \leq \frac{a_{n-k}}{qb_{n-k}}, \tag{3.4}$$

where q is the probability that the process $(\overline{\eta}_s^0)$ starting from state $n-k-1$ reaches state 0 earlier than state $n-k$. If y_i is the probability that this process starting from state i reaches state 0 earlier than state $n-k$, then

$$\begin{aligned} -(a_1+b_1)y_1 + a_1y_2 &= -b_1, \\ b_iy_{i-1} - (a_i+b_i)y_i + a_iy_{i+1} &= 0, \quad 2 \leq i \leq n-k-2 \\ b_{n-k-1}y_{n-k-2} - (a_{n-k-1}+b_{n-k-1})y_{n-k-1} &= 0, \end{aligned} \tag{3.5}$$

and $q = y_{n-k-1}$. Using (3.4), we obtain

$$\epsilon = \frac{a_{n-k}}{b_{n-k}}\frac{1}{q} = \frac{a_{n-k}}{y_{n-k-1}b_{n-k}}$$

(if $k = n-1$ take $y_1 = 0$). Since $\alpha(\eta) = 0$ if $\eta \neq n-k$,

$$\mathbf{E}_0 \int_0^{\tau_0} \alpha(\eta_u^0)du = (1-y_1)\mathbf{E}_{n-k} \int_0^{\tau_0} \alpha(\eta_u^0)du \leq (1-y_1)\epsilon,$$

and hence

$$\epsilon_0 = (1-y_1)\epsilon.$$

Now, let us find $\underline{\beta}_D = \mathbf{E}_0(\sigma \wedge \sigma_0)$ by solving the linear system

$$\begin{aligned} -(a_1+b_1)z_1 + a_1z_2 &= -1, \\ b_iz_{i-1} - (a_i+b_i)z_i + a_iz_{i+1} &= -1, \quad 2 \leq i \leq n-k-1 \\ b_{n-k}z_{n-k-1} - (a_{n-k}+b_{n-k})z_{n-k} &= -1, \end{aligned} \tag{3.6}$$

where z_i is the mean first passage time for the (initial) birth-and-death process (starting from state i) to subset $\{0, n-k+1]$. It follows that

$$\underline{\beta}_D = \frac{1}{a_0} + z_1.$$

3.2 Numerical results

In all these examples $n = 6$. Consider the following four groups of data.

case	k	λ_1 / μ_1	λ_2 / μ_2	λ_3 / μ_3	λ_4 / μ_4	λ_5 / μ_5	λ_6 / μ_6
1.a	3	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}	10^{-3}
		1	1	1	1	1	1
1.b	3	10^{-3}	$5\cdot10^{-3}$	10^{-2}	$2\cdot10^{-3}$	$8\cdot10^{-4}$	10^{-3}
		1	5	10	2	$8\cdot10^{-1}$	1
1.c	2	10^{-3}	$5\cdot10^{-3}$	10^{-2}	$2\cdot10^{-3}$	$8\cdot10^{-4}$	10^{-3}
		1	5	10	2	$8\cdot10^{-1}$	1
1.d	3	10^{-3}	$5\cdot10^{-3}$	10^{-2}	$2\cdot10^{-3}$	$8\cdot10^{-4}$	10^{-3}
		1	1	1	1	1	1
2.a	3	10^{-2}	10^{-2}	10^{-2}	10^{-2}	10^{-2}	10^{-2}
		1	1	1	1	1	1
2.b	2	10^{-2}	10^{-2}	10^{-2}	10^{-2}	10^{-2}	10^{-2}
		1	1	1	1	1	1
2.c	3	10^{-2}	$5\cdot10^{-2}$	10^{-1}	$2\cdot10^{-2}$	$8\cdot10^{-3}$	10^{-2}
		1	1	1	1	1	1
2.d	3	10^{-2}	$5\cdot10^{-2}$	10^{-1}	$2\cdot10^{-2}$	$8\cdot10^{-3}$	10^{-2}
		1	5	10	2	$8\cdot10^{-1}$	1
3.a	3	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}
		1	1	1	1	1	1
3.b	2	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}	10^{-1}
		1	1	1	1	1	1
4	3	10^{-1}	$5\cdot10^{-1}$	1	$2\cdot10^{-1}$	$8\cdot10^{-2}$	10^{-1}
		1	1	1	1	1	1

In group 1, $\lambda_i/\mu_i \leq 1.25\cdot10^{-3}$ and $\lambda_i/\mu_j \leq 1.25\cdot10^{-2}$ for all i and j. In group 2, $\lambda_i/\mu_i \leq 1.25\cdot10^{-2}$ and $\lambda_i/\mu_j \leq 1.25\cdot10^{-1}$for all i and j. In group 3, $\lambda_i/\mu_j = 10^{-1}$ for all i and j. In group 4, there exists i such that $\lambda_i/\mu_i = 1$.

Since the components are independent, the underlying Markov process is reversible and $\lambda_V(\infty) = \lambda^0$ is an upper bound for $\lambda(\infty)$; therefore, the calculation of λ_{sup} is useless. Let us introduce quantity $\lambda^1(\infty)$ which is the asymptotic failure rate of the birth-and-death process with transition rates a_i and b_i. This coincides with the true asymptotic failure rate if $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$ for all i, j; in all other cases this is an upper bound of $\lambda(\infty)$. Notice that if $\lambda_i = \lambda_j$ and $\mu_i = \mu_j$ for all i and j, then ϵ_0 and $\overline{\delta}_0$ are the exact values of $\mathbf{E}\xi$ and $\mathbf{E}S$, respectively, and, therefore, $\lambda^0 = \lambda^0_{sup}$. The results are listed in the following table.

data	$\lambda(\infty)$	$\lambda_V(\infty)=\lambda^0$	ρ_V r_V	$\lambda^1(\infty)$	λ^0_{sup}
1.a	$5.9582\cdot10^{-11}$	$5.9641\cdot10^{-11}$	$1.00\cdot10^{-3}$ $1.00\cdot10^{-3}$	$5.9582\cdot10^{-11}$	$5.9641\cdot10^{-11}$
1.b	$1.9636\cdot10^{-10}$	$1.9682\cdot10^{-10}$	$2.31\cdot10^{-3}$ $6.17\cdot10^{-3}$	$2.7684\cdot10^{-8}$	$2.7854\cdot10^{-8}$
1.c	$9.8144\cdot10^{-14}$	$9.8409\cdot10^{-14}$	$2.70\cdot10^{-3}$ $3.15\cdot10^{-3}$	$8.6771\cdot10^{-11}$	$8.7044\cdot10^{-11}$
1.d	$1.9623\cdot10^{-9}$	$1.9672\cdot10^{-9}$	$2.47\cdot10^{-3}$ $5.75\cdot10^{-3}$	$1.8703\cdot10^{-8}$	$1.8810\cdot10^{-8}$
2.a	$5.5952\cdot10^{-7}$	$5.6523\cdot10^{-7}$	$1.02\cdot10^{-2}$ $1.03\cdot10^{-2}$	$5.5952\cdot10^{-7}$	$5.6523\cdot10^{-7}$
2.b	$2.8119\cdot10^{-9}$	$2.8261\cdot10^{-9}$	$5.05\cdot10^{-3}$ $5.08\cdot10^{-3}$	$2.8261\cdot10^{-9}$	$2.8261\cdot10^{-9}$
2.c	$1.6147\cdot10^{-5}$	$1.6563\cdot10^{-5}$	$2.58\cdot10^{-2}$ $6.71\cdot10^{-2}$	$1.4831\cdot10^{-4}$	$1.5753\cdot10^{-4}$
2.d	$1.8219\cdot10^{-6}$	$1.8652\cdot10^{-6}$	$2.38\cdot10^{-2}$ $7.36\cdot10^{-2}$	$2.0977\cdot10^{-4}$	$2.2388\cdot10^{-4}$
3.a	$3.0335\cdot10^{-3}$	$3.3898\cdot10^{-3}$	$1.17\cdot10^{-1}$ $1.57\cdot10^{-1}$	$3.0335\cdot10^{-3}$	$3.3898\cdot10^{-3}$
3.b	$1.6040\cdot10^{-4}$	$1.6935\cdot10^{-4}$	$5.58\cdot10^{-2}$ $6.02\cdot10^{-2}$	$1.6935\cdot10^{-4}$	$1.6935\cdot10^{-4}$
4	$3.3541\cdot10^{-2}$	$4.3139\cdot10^{-2}$	$2.86\cdot10^{-1}$ 30.5	$2.0175\cdot10^{-1}$	$3.2033\cdot10^{-1}$

The upper bound $\lambda_V(\infty)$ is better than $\lambda^1(\infty)$ and $\lambda_V(\infty)$ is a good approximation of $\lambda(\infty)$ in groups 1 to 3 (very good in groups 1 and 2). In group 4, the approximation is not tight.

If $\lambda_i=\lambda_j$ and $\mu_i=\mu_j$ for all i and j, then a good way to find $\lambda_V(\infty)$ is to compute $\lambda^0=\epsilon_0/\overline{\delta}_0$ with

$$\epsilon_0=\frac{1-y_1}{y_{n-k-1}}\frac{a_{n-k}}{b_{n-k}},\quad \overline{\delta}_0=\frac{1}{a_0}+x_1.$$

4. Systems with Independent Components

4.1 Principle

Consider a system with independent components and *general working space* which means that we do not impose specific restrictions on $\mathcal{M}$ (such as in the case of k-out-of-n systems) except for the coherence of the system. But the dynamics of the system is still described by a Markov process with finite states space and transition rates A.

Let all the components be operating at time 0.

Denote by $\mathcal{C}_i$ the set of the states with exactly $n-1$ operating components. Let

$$a_i = \max_{\eta \in \mathcal{C}_i} \sum_{\xi \in \mathcal{C}_{i+1} \cap \mathcal{M}} A(\eta, \xi), \quad b_i = \min_{\eta \in \mathcal{C}_i \cap \mathcal{M}} \sum_{\xi \in \mathcal{C}_{i-1}} A(\eta, \xi), \tag{4.1}$$

$$\underline{n}_c = \max\{i: \mathcal{C}_i \subseteq \mathcal{M}\}, \quad \overline{n}_c = \max\{i: \mathcal{C}_i \cap \mathcal{M} \neq \emptyset\}, \tag{4.2}$$

where the number $\underline{n}_c + 1$ is the order of the smallest minimal cutset.

In this section, when referring to relations (3.1) *to* (3.6), *the coefficients* a_i *and* b_i *are assumed to be those given in* (4.1).

Let

$$\delta = x_{\overline{n}_c},$$

where x_i's are solutions of system (3.3) with $n-k = \overline{n}_c$ and

$$\overline{\alpha} = \max_{\eta \notin \mathcal{C}_0} \alpha(\eta) = \max_{\eta \in \mathcal{M}, \eta \notin \mathcal{C}_0} A(\eta; \mathcal{P}).$$

Let $\overline{\eta}_t^0$ be a birth-and-death process with transition rates given by relations (3.1) and (3.2). The same arguments as for the k-out-of-n system give

$$\mathbf{E}_\eta \int_0^{\tau_0} \alpha(\eta_u^0)du \leq \overline{\alpha}\, \mathbf{E}_\eta \int_0^{\tau_0} I(\eta_u^0 \in \cup_{\underline{n}_c \leq i \leq \overline{n}_c} \mathcal{C}_i \cap \mathcal{M})du$$

$$\leq \overline{\alpha}\, \mathbf{E}_{\overline{n}_c} \int_0^{\overline{\tau}_0} I(\underline{\eta}_c \leq \overline{\eta}_u^0 \leq \overline{n}_c)du.$$

Let t_i, $1 \leq i \leq \overline{n}_c - \underline{n}_c + 1$, be the mean time for process $\overline{\eta}_t^0$ starting at $\underline{n}_c + i - 1$ to leave interval $[\underline{n}_c, \overline{n}_c]$. Then

$$\mathbf{E}_{\overline{n}_c} \int_0^{\overline{\tau}_0} I(\underline{n}_c \leq \overline{\eta}_u^0 \leq \overline{n}_c)du = t_{\overline{n}_c - \underline{n}_c + 1} + \frac{1-q'}{q'} t_1,$$

where

$$q' = y_{\underline{n}_c - 1}, \quad (q' = 1 \text{ if } \underline{n}_c = 1),$$

and y_i's satisfy linear system (3.5) with $n-k = \underline{n}_c$. Quantities t_i's are solutions of the following system

$$-(a_{\underline{n}_c} + b_{\underline{n}_c})t_1 + a_{\underline{n}_c} t_2 = -1,$$

$$b_{\underline{n}_c + i - 1} t_{i-1} - (a_{\underline{n}_c + i - 1} + b_{\underline{n}_c + i - 1})t_i + a_{\underline{n}_c + i - 1} t_{i+1} = -1, \; 2 \leq i \leq \overline{n}_c - \underline{n}_c,$$

$$b_{\overline{n}_c} t_{\overline{n}_c - \underline{n}_c} - b_{\overline{n}_c} t_{\overline{n}_c - \underline{n}_c + 1} = -1.$$

This yields

$$\epsilon = \overline{\alpha}\left(t_{\overline{n}_c - \underline{n}_c + 1} + \frac{1-q'}{q'} t_1\right), \quad \epsilon_0 = \frac{A(\mathcal{C}_0; \mathcal{P})}{a_0} + \frac{1}{q'}\, t_1 \overline{\alpha}\,(1 - y_1)$$

($y_1 = 0$ if $\underline{n}_c = 1$). Let us notice that $A(\mathcal{C}_0; \mathcal{P}) \neq 0$ if and only if $\underline{n}_c = 1$. Finally, we take

$$\underline{\delta}_0 = \frac{1}{a_0} + x_1,$$

and

$$\underline{\beta}_D = \frac{1}{a_0} + z_1$$

where z_i are solutions of the system (3.6) with $n-k = \underline{n}_c$ and $a_{\underline{n}_c} = \max_{\eta \in \mathcal{C}_{\underline{n}_c}} \sum_{x \in \mathcal{C}_{\underline{n}_c + 1}} A(\eta, \xi)$.

4.2 Systems with common mode

Let us return to the previous example but add there a so-called *common mode*: the system behavior is just the same as described in Subsection 4.1 but it is subjected to additional events called the *common mode*. If a common mode occurs, then the ith component (if it is operating) can fail with probability p_i independently of other components. Let the occurrence rate of the common mode be Λ. Define $p = \max_i p_i$, and denote by $\bar{\alpha}$ the maximum value of the intensity process *calculated without the common mode factor*. Let $\underline{n}_c$ and $\bar{n}_c$ be the quantities defined in (4.2).

Process $(\bar{\eta}_t^0)$ is no longer a birth-and-death process. Let $\bar{A}^0$ be the transition rates matrix of the process $(\bar{\eta}_t^0)$ without the common mode. The true transition rates matrix $\bar{B}^0$ of process $(\bar{\eta}_t^0)$ can be expressed in terms of $\bar{A}^0$ as follows:

$$\bar{B}^0(i,i-1) = \bar{A}^0(i,i-1),$$

$$\bar{B}^0(i,i+1) = \bar{A}^0(i,i+1) + \Lambda(n-i)p(1-p)^{n-i-1},$$

$$\bar{B}^0(i,i+m) = \Lambda C^m_{n-i} p^m (1-p)^{n-i-m}, \quad 2 \le m \le \bar{n}_c - i.$$

Define $\bar{B}^0(i,i) = -\sum_{j: j \neq i} \bar{B}^0(i,j)$ and take $\delta = x_{\bar{n}_c}$, where x_i's satisfy the linear equations

$$x_0 = 0, \quad \sum_j \bar{B}^0(i,j)x_j = -1 \quad (i \ge 1).$$

Let

$$q'' = y_{\underline{n}_c - 1},$$

where y_i's are the solutions of the system

$$y_0 = 1, \quad \sum_j \bar{B}^0(i,j)y_j = 0 \quad (1 \le i \le \underline{n}_c - 1), \quad y_i = 0 (\underline{n}_c \le i \le \bar{n}_c).$$

We now define t_i's as the solutions of the system

$$t_0 = 0, \quad \sum_j \bar{B}^0(i + \underline{n}_c - 1, j)t_j = -1 \quad (1 \le i \le \bar{n}_c - \underline{n}_c + 1).$$

Then

$$\epsilon = \Lambda p \delta + \frac{\bar{\alpha}\, t_{\bar{n}_c - \underline{n}_c + 1}}{q''}.$$

Quantities ϵ_0, δ_0, and $\underline{\beta}_D$ can be estimated with the help (1.5), (1.6), and (1.7).

4.3 Numerical results

In practice, the set of failed states is often expressed in terms of so-called *minimal cutsets*. A cutset is a collection of components such that their simultaneous failures imply the failure of the system. A cutset is called minimal if it does not contain smaller cutsets. Clearly, the knowledge of all minimal cutsets enables us to construct all failed states.

In the following examples, we consider a system with $n = 5$ components $C1, \ldots, C5$, and four families of minimal cutsets:

- $\mathcal{C}_1 = \{(C1,C2),(C2,C4,C5),(C1,C3,C5)\}$,
- $\mathcal{C}_2 = \{(C2,C4,C5),(C1,C3,C5)\}$,
- $\mathcal{C}_3 = \{(C1,C3,C5),(C2,C3,C4,C5)\}$,
- $\mathcal{C}_4 = \{(C2,C3,C4,C5)\}$.

data	$\mathcal{C}$	λ_1 μ_1 p_1	λ_2 μ_2 p_2	λ_3 μ_3 p_3	λ_4 μ_4 p_4	λ_5 μ_5 p_5	Λ μ_6
1	$\mathcal{C}_1$	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	0
2	$\mathcal{C}_2$	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	0
3	$\mathcal{C}_3$	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	0
4	$\mathcal{C}_4$	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	10^{-3} 1 0	0
5	$\mathcal{C}_1$	10^{-2} 1 0	10^{-2} 1 0	10^{-2} 1 0	10^{-2} 1 0	10^{-2} 1 0	0
6	$\mathcal{C}_4$	10^{-2} 1 0	10^{-2} 1 0	10^{-2} 1 0	10^{-2} 1 0	10^{-2} 1 0	0
7	$\mathcal{C}_1$	10^{-1} 1 0	10^{-1} 1 0	10^{-1} 1 0	10^{-1} 1 0	10^{-1} 1 0	0
8	$\mathcal{C}_4$	10^{-1} 1 0	10^{-1} 1 0	10^{-1} 1 0	10^{-1} 1 0	10^{-1} 1 0	0
9	$\mathcal{C}_1$	10^{-3} 1 0.1	10^{-3} 1 0.1	10^{-3} 1 0.1	10^{-3} 1 0.1	10^{-3} 1 0.1	10^{-5}
10	$\mathcal{C}_1$	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-5}
11	$\mathcal{C}_1$	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-5}
12	$\mathcal{C}_1$	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-4}

data	$\mathcal{C}$	λ_1 μ_1 p_1	λ_2 μ_2 p_2	λ_3 μ_3 p_3	λ_4 μ_4 p_4	λ_5 μ_5 p_5	Λ
13	$\mathcal{C}_1$	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-3} 1 1	10^{-3}
14	$\mathcal{C}_1$	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3}
15	$\mathcal{C}_1$	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-3} 1 0.5	10^{-2}
16	$\mathcal{C}_1$	10^{-3} 1 0.5	$5\cdot10^{-3}$ 5 0.5	10^{-2} 10 0.5	$2\cdot10^{-3}$ 2 0.5	$8\cdot10^{-4}$ 0.8 0.5	10^{-2}
17	$\mathcal{C}_1$	10^{-3} 1 0.5	$5\cdot10^{-3}$ 5 0.5	10^{-2} 10 0.5	$2\cdot10^{-3}$ 2 0.5	$8\cdot10^{-4}$ 0.8 0.5	10^{-5}
18	$\mathcal{C}_1$	10^{-3} 1 0.5	$5\cdot10^{-3}$ 5 0.5	$3\cdot10^{-3}$ 1 0.5	$2\cdot10^{-3}$ 2 0.5	$8\cdot10^{-4}$ 0.8 0.5	10^{-2}
19	$\mathcal{C}_1$	10^{-3} 1 0.5	$5\cdot10^{-3}$ 5 0.5	$3\cdot10^{-3}$ 1 0.5	$2\cdot10^{-3}$ 2 0.5	$8\cdot10^{-4}$ 0.8 0.5	10^{-5}
20	$\mathcal{C}_1$	10^{-3} 1 0.5	$5\cdot10^{-3}$ 5 1	$3\cdot10^{-3}$ 1 0	$2\cdot10^{-3}$ 2 0.25	$8\cdot10^{-4}$ 0.8 0.75	10^{-2}
21	$\mathcal{C}_1$	10^{-3} 1 0.5	$5\cdot10^{-3}$ 5 1	$3\cdot10^{-3}$ 1 0	$2\cdot10^{-3}$ 2 0.25	$8\cdot10^{-4}$ 0.8 0.75	10^{-5}
22	$\mathcal{C}_1$	10^{-2} 1 0.5	$5\cdot10^{-2}$ 5 1	$3\cdot10^{-2}$ 1 0	$2\cdot10^{-2}$ 2 0.25	$8\cdot10^{-3}$ 0.8 0.75	10^{-2}
23	$\mathcal{C}_1$	10^{-2} 1 0.5	$5\cdot10^{-2}$ 5 1	$3\cdot10^{-2}$ 1 0	$2\cdot10^{-2}$ 2 0.25	$8\cdot10^{-3}$ 0.8 0.75	10^{-5}
24	$\mathcal{C}_4$	10^{-2} 1 0.5	$5\cdot10^{-2}$ 5 1	$3\cdot10^{-2}$ 1 0	$2\cdot10^{-2}$ 2 0.25	$8\cdot10^{-3}$ 0.8 0.75	10^{-2}

data	$\mathcal{C}$	λ_1 μ_1 p_1	λ_2 μ_2 p_2	λ_3 μ_3 p_3	λ_4 μ_4 p_4	λ_5 μ_5 p_5	Λ
25	$\mathcal{C}_4$	10^{-2} 1 0.5	$5 \cdot 10^{-2}$ 5 1	$3 \cdot 10^{-2}$ 1 0	$2 \cdot 10^{-2}$ 2 0.25	$8 \cdot 10^{-3}$ 0.8 0.75	10^{-5}

The following table contains the results of our calculations. If a system consists of independent components (see lines 1 to 8 in the table above and below), then we estimate r_V and λ^0_{sup} by formulas from paragraph 4.1. In the absence of common mode, $\lambda_V(\infty) = \lambda^0$ is an upper bound for $\lambda(\infty)$ and therefore, no need to fill the column λ_{sup}. In the case of common mode, we use formulas from paragraph 4.2.

data	$\lambda(\infty)$	$\lambda_V(\infty)$	ρ_V r_V	λ_{sup}	λ^0 λ^0_{sup}
1	$2.0000 \cdot 10^{-6}$	$2.0020 \cdot 10^{-6}$	$9.99 \cdot 10^{-4}$ $3.70 \cdot 10^{-3}$		$2.0020 \cdot 10^{-6}$ $9.9701 \cdot 10^{-6}$
2	$5.9790 \cdot 10^{-9}$	$5.9820 \cdot 10^{-9}$	$5.02 \cdot 10^{-4}$ $2.18 \cdot 10^{-3}$		$5.9820 \cdot 10^{-9}$ $1.9920 \cdot 10^{-8}$
3	$2.9935 \cdot 10^{-9}$	$2.9950 \cdot 10^{-9}$	$5.01 \cdot 10^{-4}$ $2.18 \cdot 10^{-3}$		$2.9950 \cdot 10^{-9}$ $1.9920 \cdot 10^{-8}$
4	$3.9819 \cdot 10^{-12}$	$3.9840 \cdot 10^{-12}$	$5.30 \cdot 10^{-4}$ $5.84 \cdot 10^{-4}$		$3.9840 \cdot 10^{-12}$ $9.9552 \cdot 10^{-12}$
5	$1.9986 \cdot 10^{-4}$	$2.0183 \cdot 10^{-4}$	$9.85 \cdot 10^{-3}$ $4.05 \cdot 10^{-2}$		$2.0183 \cdot 10^{-4}$ $9.7069 \cdot 10^{-4}$
6	$3.8310 \cdot 10^{-8}$	$3.8499 \cdot 10^{-8}$	$3.37 \cdot 10^{-3}$ $5.94 \cdot 10^{-3}$		$3.8499 \cdot 10^{-8}$ $9.5714 \cdot 10^{-8}$
7	$1.9094 \cdot 10^{-2}$	$2.0690 \cdot 10^{-2}$	$8.35 \cdot 10^{-2}$ $9.29 \cdot 10^{-1}$		$2.0690 \cdot 10^{-2}$ $7.5776 \cdot 10^{-2}$
8	$2.6356 \cdot 10^{-4}$	$2.7322 \cdot 10^{-4}$	$3.37 \cdot 10^{-2}$ $7.28 \cdot 10^{-2}$		$2.7322 \cdot 10^{-4}$ $7.0000 \cdot 10^{-4}$
9	$2.1223 \cdot 10^{-6}$	$2.1244 \cdot 10^{-6}$	$1.00 \cdot 10^{-3}$ $3.72 \cdot 10^{-3}$	$2.1323 \cdot 10^{-6}$	$2.1243 \cdot 10^{-6}$ $1.8408 \cdot 10^{-5}$
10	$5.7632 \cdot 10^{-6}$	$5.7699 \cdot 10^{-6}$	$1.16 \cdot 10^{-3}$ $3.74 \cdot 10^{-3}$	$5.7915 \cdot 10^{-6}$	$5.7655 \cdot 10^{-6}$ $2.2066 \cdot 10^{-5}$
11	$1.2000 \cdot 10^{-5}$	$1.2015 \cdot 10^{-5}$	$1.28 \cdot 10^{-3}$ $3.76 \cdot 10^{-3}$	$1.2060 \cdot 10^{-5}$	$1.2002 \cdot 10^{-5}$ $2.8309 \cdot 10^{-5}$

data	$\lambda(\infty)$	$\lambda_V(\infty)$	ρ_V r_V	λ_{sup}	λ^0 λ^0_{sup}
12	$1.0200 \cdot 10^{-4}$	$1.0214 \cdot 10^{-4}$	$1.32 \cdot 10^{-3}$ $4.09 \cdot 10^{-3}$	$1.0255 \cdot 10^{-4}$	$1.0200 \cdot 10^{-4}$ $1.1868 \cdot 10^{-4}$
13	$1.0020 \cdot 10^{-3}$	$1.0033 \cdot 10^{-3}$	$1.33 \cdot 10^{-3}$ $7.80 \cdot 10^{-3}$	$1.0112 \cdot 10^{-3}$	$1.0020 \cdot 10^{-3}$ $1.0224 \cdot 10^{-3}$
14	$3.7848 \cdot 10^{-4}$	$3.7907 \cdot 10^{-4}$	$1.56 \cdot 10^{-3}$ $5.40 \cdot 10^{-3}$	$3.8112 \cdot 10^{-4}$	$3.7848 \cdot 10^{-4}$ $3.9856 \cdot 10^{-4}$
15	$3.7812 \cdot 10^{-3}$	$3.2978 \cdot 10^{-3}$	$4.40 \cdot 10^{-3}$ $2.27 \cdot 10^{-2}$	$3.8841 \cdot 10^{-3}$	$3.7813 \cdot 10^{-3}$ $3.8588 \cdot 10^{-3}$
16	$3.7887 \cdot 10^{-3}$	$3.8221 \cdot 10^{-3}$	$8.83 \cdot 10^{-3}$ $5.55 \cdot 10^{-2}$	$4.0342 \cdot 10^{-3}$	$3.7889 \cdot 10^{-3}$ $4.7521 \cdot 10^{-3}$
17	$9.7547 \cdot 10^{-6}$	$9.8024 \cdot 10^{-6}$	$4.89 \cdot 10^{-3}$ $3.35 \cdot 10^{-2}$	$1.0131 \cdot 10^{-5}$	$9.7799 \cdot 10^{-6}$ $5.8858 \cdot 10^{-4}$
18	$3.7900 \cdot 10^{-3}$	$3.8163 \cdot 10^{-3}$	$6.95 \cdot 10^{-3}$ $3.87 \cdot 10^{-2}$	$3.9640 \cdot 10^{-3}$	$3.7902 \cdot 10^{-3}$ $4.1912 \cdot 10^{-3}$
19	$9.7508 \cdot 10^{-6}$	$9.7911 \cdot 10^{-6}$	$4.14 \cdot 10^{-3}$ $1.73 \cdot 10^{-2}$	$9.9608 \cdot 10^{-6}$	$9.7759 \cdot 10^{-6}$ $1.9870 \cdot 10^{-4}$
20	$5.9540 \cdot 10^{-3}$	$6.0009 \cdot 10^{-3}$	$7.88 \cdot 10^{-3}$ $5.80 \cdot 10^{-2}$	$6.3491 \cdot 10^{-3}$	$5.9540 \cdot 10^{-3}$ $6.4811 \cdot 10^{-3}$
21	$1.1926 \cdot 10^{-5}$	$1.1978 \cdot 10^{-5}$	$4.39 \cdot 10^{-3}$ $1.73 \cdot 10^{-2}$	$1.2186 \cdot 10^{-5}$	$1.1951 \cdot 10^{-5}$ $2.0096 \cdot 10^{-4}$
22	$6.6055 \cdot 10^{-3}$	$6.9191 \cdot 10^{-3}$	$4.75 \cdot 10^{-2}$ $3.12 \cdot 10^{-1}$	$9.0785 \cdot 10^{-3}$	$6.6331 \cdot 10^{-3}$ $2.7801 \cdot 10^{-2}$
23	$5.8590 \cdot 10^{-4}$	$6.0980 \cdot 10^{-4}$	$4.08 \cdot 10^{-2}$ $2.54 \cdot 10^{-1}$	$7.6474 \cdot 10^{-4}$	$6.0953 \cdot 10^{-4}$ $1.9121 \cdot 10^{-2}$
24	$6.5064 \cdot 10^{-5}$	$6.5859 \cdot 10^{-5}$	$1.22 \cdot 10^{-2}$ $7.56 \cdot 10^{-2}$	$7.0837 \cdot 10^{-5}$	$6.5213 \cdot 10^{-5}$ $6.6217 \cdot 10^{-3}$
25	$3.1126 \cdot 10^{-7}$	$3.1401 \cdot 10^{-7}$	$8.82 \cdot 10^{-3}$ $3.99 \cdot 10^{-2}$	$3.2655 \cdot 10^{-7}$	$3.1347 \cdot 10^{-7}$ $3.3694 \cdot 10^{-3}$

One can see that $\lambda_V(\infty)$ approximates $\lambda(\infty)$ to a good accuracy and that r_V is an accurate estimate of ρ_V in all cases except for 7, 22, and 23. In all cases, λ^0 is a better bound than the Vesely failure rate $\lambda_V(\infty)$. The smaller ratios of failure rates to repair rates are and the larger the set $\mathcal{M}$ of good states is, the better are the approximations. When failure rates are small in comparison with repair rates, the approximation is good, since the first regeneration time is small and processes $(\eta_t^0), (\eta_t^1)$ and (η_t'') are close to each other. The reason of the influence of the size of set $\mathcal{M}$ will be explained in the following section (see Remark 5.1).

Remark 4.1: All systems considered in previous examples are *NBU* (see Cocozza-Thivent and Roussignol [1]) and therefore,

$$R(t) \geq e^{-\lambda(\infty)t}.$$

Because of this, our estimates of $\lambda(\infty)$ lead to pessimistic bounds of the system reliability.

5. Semi-Markov Process

5.1 An alternative renewal process

We have already seen (Remark I.3.13) that the Vesely failure rate of a semi-Markov process is equal to $1/MUT$. We now consider an *alternative renewal process*, which can be regarded as a semi-Markov process with two states: 1 (operating state) and 0 (failed state). Denote by F the distribution function of the sojourn time in state 1, by $m = \int_0^\infty (1 - F(t))dt$ its mean value, and by $h(t) = \frac{F'(t)}{1 - F(t)}$ its hazard rate function. Assume that there exists the limit $h(\infty) = \lim_{t \to \infty} h(t) < \infty$. Then the asymptotic failure rate and the asymptotic Vesely failure rate functions for such a system are equal to

$$\lambda(\infty) = h(\infty), \quad \text{and } \lambda_V(\infty) = \frac{1}{m}, \text{respectively.} \tag{5.1}$$

If $\lambda_V(\infty)$ is regarded as an estimate of $\lambda(\infty)$, then the accuracy of this estimate depends (in general) on distribution function F. Indeed, in the exponential case, h is constant and $\lambda(\infty) = \lambda_V(\infty) \equiv \lambda(t)$. But if, say, F is a gamma-distribution with the density

$$F'(t) = \frac{1}{\Gamma(a)b^a} t^{a-1} e^{-x/b},$$

then

$$\lambda(\infty) = \frac{1}{b}, \quad \lambda_V(\infty) = \frac{1}{ab}, \quad \frac{\lambda_V(\infty) - \lambda(\infty)}{\lambda(\infty)} = \frac{1}{a} - 1$$

and the accuracy depends on how far is the gamma-distribution from an exponential distribution.

Although we cannot use our method because process (η_t^0) is not regenerative, this does not matter, since we obtain accuracy estimates directly from (5.1).

Remark 5.1: Let us return for a while to the Markovian case. When failure rates are small in comparison with repair rates and when set $\mathcal{M}$ is large, then the probability for Markov process (η_t) to return to the "perfect" state before entering into $\mathcal{P}$ is high. Because of this, the sojourn times distributions in set $\mathcal{M}$ are close to exponential distributions (see Keilson [4]), and it is intuitively reasonable to consider that the sequence of successive sojourn times of process (η_s) in sets $\mathcal{M}$ and $\mathcal{P}$ is approximately an alternative renewal process with an exponential distribution function F of interrenewal times. This can explain why the approximation of $\lambda(\infty)$ by $\lambda_V(\infty)$ (and even of $\lambda(t)$ by $\lambda_V(t)$) is good in this case.

5.2 A simple example of a semi-Markov process

Let a facility have four operating states e_1, e_2, e_3, e_4 and two failed states e_5 and e_6. The initial state is e_1. When leaving state e_1, the facility can reach one of the three following states with probabilities p_1, q_1, and r_1, respectively, $(p_1 + q_1 + r_1 = 1)$:

- state e_2; this can be treated as if a minor failure would have occurred which was detected and the facility had been repaired;
- state e_3; this means that a minor failure has occurred but it was not detected;
- state e_5; the system fails.

After this, the system behaves as follows. Upon repairing a minor failure, the facility comes back to state e_1. From state e_3, the facility can go either to operating state e_4 (if safety system has worked) with probability p_2 or to failed state e_6 with probability $1-p_2$. State e_4 leads to state e_2.

Let us describe the facility's behavior by a semi-Markov process with the transition rate functions including only non-zero terms as follows

$$A(e_1,e_2,s)=p_1\lambda_1(s),\quad A(e_1,e_3,s)=q_1\lambda_1(s),\quad A(e_1,e_5,s)=r_1\lambda_1(s),$$
$$A(e_3,e_4,s)=p_2\lambda_2(s),\quad A(e_3,e_6,s)=(1-p_2)\lambda_2(s),$$
$$A(e_5,e_1,s)=\mu_1(s),\quad A(e_2,e_1,s)=\mu_2(s), A(e_4,e_2,s)=\mu_3(s),\quad A(e_6,e_2)=\mu_4(s),$$

and apply Proposition I.5.2 to Markov process $\eta'_t=(\eta_t,y_t)$, where y_t is the elapsed time of the process in its current state at time t. Using Proposition I.2.5, one can see that non-zero terms of transition rate functions of semi-Markov process (η^0_s) have the form

$$A^0(e_1,e_2,s)=p_1\lambda_1(s),\quad A^0(e_1,e_3,s)=q_1\lambda_1(s),$$
$$A^0(e_3,e_4,s)=p_2\lambda_2(s),\quad A^0(e_2,e_1,s)=\mu_2(s),\quad A^0(e_4,e_2,s)=\mu_3(s).$$

Process $(\eta'^0_s)=((\eta^0_s,y^0_s))$ is Markovian and

$$\alpha(\eta^0_t,y^0_t)=A^0(\eta^0_t,\mathfrak{P},y^0_t)=\begin{cases} r_1\lambda_1(y^0_t), & \text{if } \eta^0_t=e_1,\\ (1-p_2)\lambda_2(y^0_t), & \text{if } \eta^0_t=e_3.\end{cases}$$

Let W_1,W_2,W_3 and W_4 be r.v.'s with hazard rates $(p_1+q_1)\lambda_1$, $\mu_2, p_2\lambda_2$, and μ_3 respectively. Evidently the sojourn time of the process (η^0_s) in state e_i has the same distribution as W_i. Process (η'^0_s) is regenerative with regeneration state e_1 and

$$\mathbf{E}\xi=\mathbf{E}\left(\int_0^{W_1} r_1\lambda_1(s)ds+\frac{q_1}{p_1+q_1}\int_0^{W_3}(1-p_2)\lambda_2(s)ds\right)$$
$$=\int_0^\infty r_1\lambda_1(s)\mathbf{P}(W_1>s)ds+\frac{q_1}{(p_1+q_1)}(1-p_2)\int_0^\infty \lambda_2(s)\mathbf{P}(W_3>s)ds$$
$$=\frac{r_1}{p_1+q_1}+\frac{q_1(1-p_2)}{(p_1+q_1)p_2}.$$

Quite similarly,

$$\max(\mathbf{E}\xi^1,\mathbf{E}\xi'')\le \mathbf{E}\int_0^{W_3}(1-p_2)\lambda_2(s)ds=\frac{1-p_2}{p_2}.$$

It can be easily seen that

$$\max(\mathbf{E}S^1_0,\mathbf{E}S''_0)\le \mathbf{E}(W_2+W_3+W_4),$$
$$\mathbf{E}S=\mathbf{E}W_1+\mathbf{E}W_2+\frac{q_1}{p_1+q_1}\mathbf{E}(W_3+W_4).$$

Recall that σ is the first entrance time of process (η_s^0) into $\mathcal{P} = \{e_5; e_6\}$. Let σ_0 be the recurrence time to state e_1. Then

$$\beta_D = \mathbf{E}\int_0^S \exp\left(-\int_0^t \alpha(\eta_u'^0)du\right)dt = \mathbf{E}(\sigma \wedge \sigma_0)$$

$$= \mathbf{E}W_1 + \frac{q_1}{p_1+q_1}\mathbf{E}W_3 + \frac{q_1 p_2}{p_1+q_1}\mathbf{E}(W_4+W_2).$$

Proposition I.5.2 yields

$$\rho_V \le e^{\epsilon}\left(\epsilon + \frac{\delta\epsilon_0}{\beta_D}\right), \quad \lambda_0 = \frac{\epsilon_0}{\beta'_D},$$

where

$$\epsilon = \frac{r_1}{p_1+q_1} + \frac{q_1(1-p_2)}{(p_1+q_1)p_2}, \quad \epsilon_0 = \frac{1-p_2}{p_2}, \tag{5.2}$$

$$\delta = \mathbf{E}(W_2+W_3+W_4), \quad \beta_D = \mathbf{E}W_1 + \frac{q_1}{p_1+q_1}\mathbf{E}W_3 + \frac{q_1}{p_1+q_1}p_2\mathbf{E}(W_4+W_2),$$

$$\beta'_D = \mathbf{E}W_1 + \mathbf{E}W_2 + \frac{q_1}{p_1+q_1}\mathbf{E}(W_3+W_4).$$

5.3 Numerical results

Let us consider the example described in Subsection 5.2. With m_i denoting the mean sojourn time in state e_i $(1 \le i \le 6)$, we set

$$m_1 = 1, \quad m_2 = \frac{1}{200}, \quad m_3 = 1, \quad m_4 = \frac{1}{100}, \quad m_5 = \frac{1}{10}, \quad m_6 = 1$$

and suppose that W_i (the sojourn time of the process (η_t) in state e_i) has the Erlang distribution with parameter k_i (and mean m_i) (that is, W_i can be treated as a sum of k_i i.i.d.r.v.'s with common exponential distribution having parameter k_i/m_i).

Case 1. Let

$$p_1 = 0.5, \quad q_1 = 0.4, \quad r_1 = 0.1, \quad p_2 = 0.99.$$

Then

$$\lambda_V(\infty) = 7.3840 \cdot 10^{-2}, \qquad r_V = 9.19 \cdot 10^{-2},$$

$$\lambda_{sup} = 8.0624 \cdot 10^{-2}, \quad \lambda_0 = 7.9511 \cdot 10^{-2},$$

and

- for $k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1, k_5 = 1, k_6 = 1$
  $$\lambda(\infty) = 7.2432 \cdot 10^{-2}, \qquad \rho_V = 1.94 \cdot 10^{-2};$$
- for $k_1 = 1, k_2 = 5, k_3 = 1, k_4 = 10, k_5 = 7, k_6 = 6$
  $$\lambda(\infty) = 7.2432 \cdot 10^{-2}, \qquad \rho_V = 1.94 \cdot 10^{-2};$$
- for $k_1 = 3, k_2 = 10, k_3 = 12, k_4 = 50, k_5 = 10, k_6 = 15$
  $$\lambda(\infty) = 7.4459 \cdot 10^{-2}, \qquad \rho_V = 8.32 \cdot 10^{-3};$$

Case 2. Let

$$p_1 = 0.1, \quad q_1 = 0.5, \quad r_1 = 0.4, \quad p_2 = 0.99.$$

Then

$$\lambda_V(\infty) = 2,6858 \cdot 10^{-1}, \qquad r_V = 3.85 \cdot 10^{-1},$$

$$\lambda_{sup} = 3.7204 \cdot 10^{-1}, \quad \lambda_0 = 3.6557 \cdot 10^{-1},$$

and

- for $k_1 = 1,\ k_2 = 1, k_3 = 1, k_4 = 1, k_5 = 1, k_6 = 1$
$$\lambda(\infty) = 2.4331 \cdot 10^{-1}, \rho_V = 1.04 \cdot 10^{-1};$$
- for $k_1 = 1, k_2 = 5, k_3 = 1, k_4 = 10, k_5 = 7, k_6 = 6$
$$\lambda(\infty) = 2.4332 \cdot 10^{-1}, \quad \rho_V = 1.04 \cdot 10^{-1};$$
- for $k_1 = 3, k_2 = 10, k_3 = 12, k_4 = 50, k_5 = 10, k_6 = 15$
$$\lambda(\infty) = 2.7024 \cdot 10^{-1}, \quad \rho_V = 6.15 \cdot 10^{-1}.$$

Case 3. Let

$$p_1 = 0.1, \ q_1 = 0.5, \ r_1 = 0.4, \quad p_2 = 0.99.$$

Then

$$\lambda_V(\infty) = 2.9851 \cdot 10^{-1}, \quad r_V = 5.91 \cdot 10^{-1},$$

$$\lambda_{sup} = 4.7494 \cdot 10^{-1}, \quad \lambda_0 = 4.1115 \cdot 10^{-1},$$

and

- for $k_1 = 1, k_2 = 1, k_3 = 1, k_4 = 1, k_5 = 1, k_6 = 1$
$$\lambda(\infty) = 2.7586 \cdot 10^{-1}, \quad \rho_V = 8.21 \cdot 10^{-2};$$
- for $k_1 = 1, k_2 = 5, k_3 = 1, k_4 = 10, k_5 = 7, k_6 = 6$
$$\lambda(\infty) = 2.7586 \cdot 10^{-1}, \quad \rho_V = 8.21 \cdot 10^{-2};$$
- for $k_1 = 3,\ k_2 = 10,\ k_3 = 12,\ k_4 = 50,\ k_5 = 10,\ k_6 = 15$
$$\lambda(\infty) = 3.1128 \cdot 10^{-1}, \quad \rho_V = 4.10 \cdot 10^{-2}.$$

Comments: The results displayed above show that the influence of the order of the Erlang distributions on reliability characteristics is minor. But our estimate is very sensitive to the value of p_2. From other examples, one can see that the sensitivity with respect to p_1, q_1, and r_1 is also high. This can be seen from equation (5.2). In our examples, $\lambda(\infty) \leq \lambda_V(\infty)$ for small values of k_i's, but not for large values. In addition, the bounding by λ_0 gives better results than by λ_{sup}.

References

[1] Cocozza-Thivent, C. and Roussignol, M., *Comparaison des lois stationnaire et quasi-stationnaire d'un processus de Markov et applications à la fiabilité*, Séminaire de Probabilités XXX, Lecture Notes in Mathematics **1626**, Springer (1996) 24-39.

[2] Cocozza-Thivent, C. and Roussignol, M., Techniques de couplage en fiabilité, *Ann. I.H.P.* **31**:1 (1995), 119-141.

[3] Cocozza-Thivent, C. and Kalashnikov, V., Failure rate in reliability: approximations and bounds, *J. Appl. Math. Stoch. Anal.* **9**:4 (1996), 497-530.

[4] Keilson, J., *Markov Chain Models - Rarity and Exponentiality*, Springer-Verlag, New York 1979.