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Recursive Estimation of Inventory Quality Classes Using Sampling

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Abstract. In this paper we propose a new discrete time discrete state inventory model for perishable items of a single product. Items in stock are assumed to belong to one of a finite number of quality classes that are ordered in such a way that Class 1 contains the best quality and the last class contains the pre-perishable quality. By the end of each epoch, items in each inventory class either stay in the same class or lose quality and move to a lower class. The movement between classes is not observed. Samples are drawn from the inventory and based on the observations of these samples, optimal estimates for the number of items in each quality classes are derived.

Keywords: Hidden Markov Models, Measure change techniques, Inventory Control, Perishable Items.

1. Introduction

In this paper, we propose a new discrete time, discrete state inventory model for perishable items of a single product.

The vector $X = (X_n^1, X_n^2, \ldots, X_n^K)$ represents the state of the inventory by the end of epoch n, $n \in \mathbb{N}$ and X_n^i, $i = 1, \ldots, K$ stands for the inventory level of quality i. New arriving items are assigned to quality 1. Further, by the end of each epoch n, due to uncontrollable factors, some items in Class i may move to Class $(i+1)$ signaling the fact that the items have moved to a lower quality class, $i = 1, \ldots, K$. Items in Class K either perish or remain in the same class.

It is implicit in the model that classes are ordered in such a way that Class 1 contains the best quality of items, Class 2 the second best, etc.

We assume that the inventory is large and that all items are held in one place together making it impractical to count the number of items X_n^i within each class. We also assume that at each epoch n, the demand that is not fulfilled is lost.

The assumption of lost sale is strictly not needed but is kept to make the

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mathematics less messy.

It is certainly desirable for most practitioners and managers of stock to have an idea about the level of stock of each quality class. This obviously is helpful in setting up plans for future targets. On way of overcoming the difficulty of counting the entire stock is simply to take a sample of the stock, then count the number of items in each class in this sample. Based on the results of this sampling scheme an estimate of the level of stock of each quality class is proposed. By doing this, we in fact have put our problem within the general framework of *Filtering Theory*. Here, we have the main model (the inventory model) where items are either sold or stay in the same class or move to a lower class. The movement between classes is too expensive to be observed and so a less expensive alternative is to observe a sample of the stock on hand. Then, an estimate of the state of the inventory conditional on the observed processes is proposed.

Filtering theory is popular among engineers: Anderson and Moore [3], and does not seem to have had a big impact in Operations Research. This paper along with others: see Aggoun, Benkherouf and Tadj [1] and [2], we hope will open up ways to use powerful tools of stochastic analysis in yet unexplored questions in Operations Research.

Our approach in tackling the proposed problem hinges on the so called *Change of Measures Techniques*. This basically means that the real world probability measure on which the inventory model was introduced is transformed by a technical artifice to another probability measure where various technical derivations are made easy. Then, another reversed transformation will recover the original model.

As mentioned at the outset, this paper deals with a product experiencing changes in quality over its life span. Products experiencing perishability are numerous. To name a few, food stuff, blood samples, drugs, electronic components etc. For more details about the development of inventory models with deteriorating items see the review of Raafat [13].

It is worth noting from the review of Raafat that there are very limited number of stochastic models as opposed to deterministic ones, apart from the work of Nahmias [12]. Kaspi and Perry [8] and [9], Benssoussan, Nissen and Tapiero [6] and more recently Aggoun, Benkherouf and Tadj [1] and [2].

The paper is organized as follows. The next section introduces the mathematical model with the required set up. Section 3 deals with the problem of estimating the number of items in each quality class. Section 4 contains details of a parameters estimation problem related to the model. The paper concludes with some general remarks.

2. The Mathematical Model

Before introducing formally the mathematical model for the inventory system with quality classes we need to fix some notations.
Let X be a nonnegative integer-valued random variable. Then for any $\alpha \in (0,1)$ define the operator "$\circ$" by:

$$\alpha \circ X = \sum_{j=1}^{X} Z_j, \tag{1}$$

where Z_j is a sequence of i.i.d random variables, independent of X, such that:

$$P(Z_j = 1) = 1 - P(Z_j = 0) = \alpha.$$

Recall that the $X_n = (X_n^1, X_n^2, \ldots, X_n^K)$ is a $\mathbb{Z}_+^K-$ valued random variable representing the level of inventory by the end of epoch n, $n \in \mathbb{N}$, where X_n^i stands for the inventory level of quality i, $i = 1, \ldots, K$, and X_0 is the initial inventory. Also, let $D_n = (D_n^1, \ldots, D_n^K)$ be a vector defined on $\mathbb{Z}_+^K$ with distribution ϕ representing the demand for the items at time n. New arriving items are assigned to Class 1.
Now, each item in Class i at time $(n-1)$ is assumed to remain in the same class with probability α^i. Otherwise, it moves to Class $(i+1)$ with probability $(1-\alpha^i)$ where i is just an index and $0 < \alpha^i < 1$.
Let X_0 be the initial inventory which is supposed to be known. Then, it follows from the above assumptions that the dynamics describing the inventory movement have the representation:

$$\begin{aligned}
X_n^1 &= \alpha^1 \circ X_{n-1}^1 + U_n - D_n^1, \\
X_n^2 &= \alpha^2 \circ X_{n-1}^2 + (1-\alpha^1) \circ X_{n-1}^1 - D_n^2, \\
X_n^3 &= \alpha^3 \circ X_{n-1}^3 + (1-\alpha^2) \circ X_{n-1}^2 - D_n^3, \\
&\vdots \\
X_n^K &= \alpha^K \circ X_{n-1}^K + (1-\alpha^{K-1}) \circ X_{n-1}^{K-1} - D_n^K
\end{aligned} \tag{2}$$

Here, put $\alpha^i \circ X_{n-1}^i = 0$ for all $X_{n-1}^i \leq 0$ where the operator "$\circ$" is defined by (1). Also, set $X_n^i = 0$ whenever $X_n^i \leq 0$. We also remark that it is implicit in the model that items arriving from the previous period go first through the classification process before they get affected by demand. The variable U_n is a $\mathbb{Z}_+$-valued sequence, representing the replenishment quantity at time n, which is either deterministic or predictable, that is, function of whatever information we have available at time $n-1$. Also,

note that it is implicit in (2) that new arriving items go to Class 1.
Also, note from (2) that there is no restriction on the inventory space available. The case where there is limited storage space can be handled with obvious changes.

Remark. The operator "$\circ$" defined in (2.1) was used by Al-Osh and Alzaid [4] and McKenzie [11] for modeling integer-valued time series. For more details about this and similar idea consult the book of MacDonald and Zucchnini [10].

As mentioned at the outset, the company holding the stock with the plant equations (2) has all the items in one place mixed together and it is desirable to know the partition of the stock among the K classes. sampling with replacement, a random sample of size M from the inventory is selected of which the outcome is denoted by:

$$Y_n = (Y_n^1, \dots, Y_n^K) \in \mathbb{Z}_+^K. \tag{3}$$

Let

$$\mathcal{F}_n = \sigma\{X_k^i, D_k^i, U_k, 1 \le i \le K \ \ , k \le n\}, \tag{4}$$

and

$$\mathcal{Y}_n = \sigma\{Y_k^i, 1 \le i \le K \ \ , k \le n\}, \tag{5}$$

be the complete filtrations generated by the inventory model and the outcome of the sampling process up to epoch n, $n = 0, \dots$. Also, let

$$L_n = \sum_{i=1}^K X_n^i$$

be the inventory level at time n.
Now, assume that the ($\mathbb{Z}_+^K$-valued) random variables Y_n, $n = 1, 2, \dots,$ have probability distributions

$$\begin{aligned} f_{Y/X}^{(n)}(y^1, \dots, y^K) &= P\left[Y_n^1 = y^1, \dots, Y_n^K = y^K \mid X_n^1 = x^1, \dots, X_n^K = x^K\right] \\ &= M! \prod_{i=1}^K \left(\frac{x^i}{L_n}\right)^{y^i} \frac{1}{y^i!}. \end{aligned} \tag{6}$$

3. Recursive Estimators

The main result of this paper is the derivation of recursive estimators for the distribution of the vector X_n, representing the level of inventory at

time n, based on the information obtained from (5).
We shall construct a new probability measure $\overline{P}$ under which the processes $\{X_n\}$ and $\{Y_n\}$ are independent. This fact shall greatly simplify the derivation of our results. For more information regarding change of measure techniques for discrete time processes: see the book by Elliott, Aggoun and Moore [7].
Under the new probability measure $\overline{P}$, to be defined below, we shall have

1. For each $n \geq 1$, $Y_n = (Y_n^1, ..., Y_n^K)$ has a multinomial distribution such that

$$P\left[Y_n^1 = y^1, \dots, Y_n^K = y^K\right] = \frac{M!}{y^1!...y^K!}\prod_{i=1}^{K} K^{-y^i} = \frac{M!}{y^1!...y^K!}K^{-M}. \tag{7}$$

2. For each $n \geq 1$, $X_n = (X_n^1, ..., X_n^K)$ has probability distribution ϕ.

The crucial step here is the construction of a suitable $P-$ martingale which will provide us with the Radon-Nikodym derivative of $\overline{P}$ with respect to P . To do that define

$$\begin{aligned}
\gamma_0 &= 1, \\
\gamma_k &= \frac{\phi(X_k^1, \dots, X_k^K)}{R_k} K^{-M} \prod_{i=1}^{K} \left(\frac{X_k^i}{L_k}\right)^{-Y_k^i}.
\end{aligned} \tag{8}$$

Here

$$\begin{aligned}
&R_k(U_k, X_{k-1}^1, \dots, X_{k-1}^K, X_k^1, \dots, X_k^K) \\
&= \phi\Big(\alpha^1 \circ X_{k-1}^1 + U_k - X_k^1, \alpha^2 \circ X_{k-1}^2 + (1-\alpha^1) \circ X_{k-1}^1 \\
&- X_k^2, \dots, \alpha^K \circ X_{k-1}^K + (1-\alpha^{K-1}) \circ X_{k-1}^{K-1} - X_k^K\Big),
\end{aligned} \tag{9}$$

and let

$$\Gamma_n = \prod_{k=1}^{n} \gamma_k. \tag{10}$$

Write

$$\mathcal{G}_n = \mathcal{F}_n \vee \mathcal{Y}_n.$$

Let $\overline{E}$ denotes the expectation under probability measure $\overline{P}$.

Lemma 1

$$E[\gamma_k \mid \mathcal{G}_{k-1}] = 1.$$

Proof. Let $\dfrac{X_n^i}{L_n} \triangleq p_n^i$, and $\phi(\alpha^1 \circ X_{k-1}^1 + U_k - D_k^1, \ldots, \alpha^K \circ X_{k-1}^K + (1 - \alpha^{K-1}) \circ X_{k-1}^{K-1} - D_k^K) \triangleq \Phi(U_k, X_{k-1}^1, \ldots, X_{k-1}^K, D_k^1, \ldots, D_k^K)$. In view of (8) we have

$$
\begin{aligned}
E[\gamma_k \mid \mathcal{G}_{k-1}] &= K^{-M} E\Bigg[\frac{\phi(X_k^1, \ldots, X_k^K)}{R_k} \\
&\times E\Bigg[\prod_{i=1}^{K} (p_k^i)^{-Y_k^i} \Bigg| X_k^1, \ldots, X_k^K, \mathcal{G}_{k-1}\Bigg] \Bigg| \mathcal{G}_{k-1}\Bigg] \\
&= \frac{1}{K^M} E\Bigg[\frac{\phi(X_k^1, \ldots, X_k^K)}{R_k} \sum_{y_k^1, \ldots, y_k^K} M! \prod_{i=1}^{K} (p_k^i)^{y^i} \frac{1}{y^i!} \\
&\times (p_n^1)^{-y^1} \ldots (p_n^K)^{-y^K} \Bigg| \mathcal{G}_{k-1}\Bigg] \\
&= \frac{1}{K^M} E\Bigg[\frac{\phi(X_k^1, \ldots, X_k^K)}{R_k} \sum_{y_k^1, \ldots, y_k^K} M! \prod_{i=1}^{K} \frac{1}{y^i!} \Bigg| \mathcal{G}_{k-1}\Bigg] \\
&= E\Bigg[\frac{\phi(X_k^1, \ldots, X_k^K)}{R_k} \Bigg| \mathcal{G}_{k-1}\Bigg] \\
&= E\Bigg[\frac{\Phi(U_k, X_{k-1}^1, \ldots, X_{k-1}^K, D_k^1, \ldots, D_k^K)}{\phi(D_k^1, \ldots, D_k^K)} \Bigg| \mathcal{G}_{k-1}\Bigg] \\
&= E\Bigg[E\Bigg[\frac{\Phi(U_k, X_{k-1}^1, \ldots, X_{k-1}^K, D_k^1, \ldots, D_k^K)}{\phi(D_k^1, \ldots, D_k^K)} \Bigg| \mathcal{G}_{k-1}, \alpha^1 \circ X_{k-1}^1 \\
&+ U_k, \ldots, \alpha^K \circ X_{k-1}^K, \ldots, (1 - \alpha^{K-1}) \circ X_{k-1}^{K-1}]\Bigg] \Bigg| \mathcal{G}_{k-1}\Bigg] \\
&= E\Bigg[\sum_{d^1, \ldots, d^K} \frac{\Phi(U_k, X_{k-1}^1, \ldots, X_{k-1}^K, d^1, \ldots, d^K)}{\phi(d^1, \ldots, d^K)} \\
&\times \phi(d^1, \ldots, d^K) \Bigg| \mathcal{G}_{k-1}\Bigg] = 1
\end{aligned}
$$

which completes the proof.

Lemma 2 *The sequence* $\{\Gamma_n\}_{n \in N}$ *is an-* $(\mathcal{G}_n, P)$ *martingale.*

Proof. Using Lemma 1 and the fact that Γ_{n-1} is $\mathcal{G}_{n-1}$-measurable

$$\begin{aligned} E\left[\Gamma_n \mid \mathcal{G}_{n-1}\right] &= \Gamma_{n-1} E[\gamma_n \mid \mathcal{G}_{n-1}] \\ &= \Gamma_{n-1}, \end{aligned}$$

by Lemma 1, which finishes the proof.
The following two theorems are preliminary results that are needed in our estimation problem.

Theorem 1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with the filtration $\{\mathcal{G}_n\}$. Write $\mathcal{F}_\infty = \vee \mathcal{G}_n \subset \mathcal{F}$. Let $\overline{P}$ be another probability measure on $(\Omega, \mathcal{F})$ which is absolutely continuous with respect to P. Suppose that Γ_n are the corresponding Randon-Nikodym derivatives when both are restricted to $\mathcal{F}_n$ for each n . Then Γ_n converges to an integrable random variable with probability 1.*

The proof of the above Theorem uses Lemma 2 and Martingale convergence Theorem: see Shiryayev [14].
Using Theorem 1 and Kolmogorov's Extension Theorem, see [14], we set:

$$E\left[\frac{d\overline{P}}{dP} | \mathcal{G}_n\right] = \Gamma_n,$$

that is, for $G \in \mathcal{G}_n$,

$$\overline{P}(G) = \int_G \Gamma_n dP.$$

Theorem 2 *(Abstract Bayes Theorem) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G} \subset \mathcal{F}$ is a sub$-\sigma-$ field. Suppose $\overline{P}$ is another probability measure absolutely continuous with respect to P and with Radon-Nikodym derivative $\dfrac{d\overline{P}}{dP} = \Gamma$. Then if f is any integrable $\mathcal{F}$-measurable random variable*

$$\overline{E}[f \mid \mathcal{G}] = \frac{E[\Gamma f \mid \mathcal{G}]}{E[\Gamma \mid \mathcal{G}]}.$$

For the proof of Theorem 2: see [7] (page 23).

Lemma 3 *Under probability measure $\overline{P}$, Y_n has probability distribution given by (7).*

Proof. let f be a test function. Using Theorem 2, Lemma 1 and repeated conditioning as in the proof of Lemma 1

$$\overline{E}\left[f(Y_n) \mid \mathcal{G}_{n-1}\right] = \frac{E\left[f(Y_n)\Gamma_n|\mathcal{G}_{n-1}\right]}{E\left[\Gamma_n \mid \mathcal{G}_{n-1}\right]} = \frac{\Gamma_{n-1}}{\Gamma_{n-1}} \frac{E\left[f(Y_n)\gamma_n|\mathcal{G}_{n-1}\right]}{E\left[\gamma_n \mid \mathcal{G}_{n-1}\right]}$$

$$= E\left[f(Y_n) \frac{\phi(X_n^1, \ldots, X_n^K)}{R_n} K^{-M} \prod_{i=1}^{K} \left(\frac{X_n^i}{L_n}\right)^{-Y_n^i} \middle| \mathcal{G}_{n-1}\right]$$

$$= E\left[E\left[\frac{\phi(X_n^1, \ldots, X_n^K)}{R_n} \sum_{y^1,\ldots,y^K} f(y^1, \ldots, y^K) K^{-M}\right.\right.$$

$$\left.\left.\times \frac{M!}{\prod_{i=1}^{K} y^i!} \prod_{i=1}^{K} \left(\frac{X_n^i}{L_n}\right)^{y_n^i} \prod_{i=1}^{K} \left(\frac{X_n^i}{L_n}\right)^{-y_n^i} \middle| \mathcal{G}_{n-1}, X_n^1, \ldots, X_n^K\right] \middle| \mathcal{G}_{n-1}\right]$$

$$= \sum_{y^1,\ldots,y^K} f(y^1, \ldots, y^K) K^{-M} \frac{M!}{\prod_{i=1}^{K} y^i!} E\left[\frac{\phi(X_n^1, \ldots, X_n^K)}{R_n} \middle| \mathcal{G}_{n-1}\right]$$

$$= \sum_{y^1,\ldots,y^K} f(y^1, \ldots, y^K) K^{-M} \frac{M!}{\prod_{i=1}^{K} y^i!}.$$

The term $E\left[\frac{\phi(X_n^1,\ldots,X_n^K)}{R_n} \middle| \mathcal{G}_{n-1}\right] = 1$, by the proof of lemma 1. This completes the poof.

Write

$$p_n(x^1, \ldots, x^K) = E[I(X_n^1 = x^1, \ldots, X_n^K = x^K) \mid \mathcal{Y}_n].$$

Using Theorem 2

$$p_n(x^1, \ldots, x^K) = \frac{\overline{E}[I(X_n^1 = x^1, \ldots, X_n^K = x^K)\Gamma^{-1}{}_n \mid \mathcal{Y}_n]}{\overline{E}[\Gamma^{-1}{}_n \mid \mathcal{Y}_n]}$$

$$:= \frac{q_n(x^1, \ldots, x^K)}{\sum_{k^1,\ldots,k^K} q(k^1, \ldots, k^K)}$$

where $E[.]$ and $\overline{E}[.]$ are the expectations under P and $\overline{P}$ respectively and $I(.)$ is the indicator of the set $(.)$.

Expression $q_n(x^1, \ldots, x^K)$ is an unnormalized conditional probability distribution of $(X_n^1 = x^1, \ldots, X_n^K = x^K)$ given the observations up to time n .

Theorem 3 *Let $\pi_0(x^1, \ldots, x^K)$ be the initial probability distribution of $(X_0^1, \ldots, X_0^K)$. Then, the unnormalized conditional probability distribution of $(X_n^1, \ldots, X_n^K)$ given $\mathcal{Y}_n$ is given by the recursion:*

$$q_n(x^1, \ldots, x^K)$$
$$= K^M \prod_{i=1}^{K} \left(\frac{x_i}{L_n}\right)^{Y_n^i} \sum_{z^1,\ldots,z^K} R_n(U_n, z^1, \ldots, z^K, x^1, \ldots, x^K) q_{n-1}(z^1, \ldots, z^K), \tag{11}$$

where R_n is given by (9).

Proof. Let $f : \mathbb{Z}_+^K \to \mathbb{R}$ be a Borel test function. Then we have

$$\overline{E}[f(X_n^1, \ldots, X_n^K)\Gamma^{-1}{}_n \mid \mathcal{Y}_n] \triangleq \sum_{x^1,\ldots,x^K} f(x^1, \ldots, x^K) q_n(x^1, \ldots, x^K). \tag{12}$$

However, recall that $\gamma_n^{-1} = \dfrac{R_k}{\phi(X_k^1, \ldots, X_n^K)} K^M \prod_{i=1}^{K} \left(\dfrac{X_k^i}{L_k}\right)^{Y_k^i}$ where

$$R_n = \phi\Big(\alpha^1 \circ X_{n-1}^1 + U_n - X_n^1, \alpha^2 \circ X_{n-1}^2 + (1-\alpha^1) \circ X_{n-1}^1$$
$$- X_n^2, \ldots, \alpha^K \circ X_{nk-1}^K + (1-\alpha^{K-1}) \circ X_{n-1}^{K-1} - X_n^K\Big)$$

and that $(X_n^1, \ldots, X_n^K)$ has distribution $\phi(.)$ under $\overline{P}$. Hence

$$\overline{E}[f(X_n^1, \ldots, X_n^K)\Gamma_n^{-1} \mid \mathcal{Y}_n] = \overline{E}[f(X_n^1, \ldots, X_n^K)\Gamma_{n-1}^{-1}\gamma_n^{-1} \mid \mathcal{Y}_n]$$
$$= \sum_{x^1,\ldots,x^K} f(x^1, \ldots, x^K) K^M \prod_{i=1}^{K} \left(\frac{x^i}{L_n}\right)^{Y_n^i} \frac{\phi(x^1, \ldots, x^K)}{\phi(x^1, \ldots, x^K)}$$
$$\times \overline{E}\Big[\phi\Big(\alpha^1 \circ X_{n-1}^1 + U_n - x^1, \ldots, \alpha^K \circ X_{n-1}^K$$
$$+ (1-\alpha^{K-1}) \circ X_{n-1}^{K-1} - x^K\Big)\Gamma_{n-1}^{-1}\Big|\mathcal{Y}_{n-1}\Big]$$

The expectation is simply

$$\sum_{z^1,\ldots,z^K} \phi(\alpha^1 \circ z^1 + U_n - x^1, \ldots, \alpha^K \circ z^K$$
$$+ (1-\alpha^{K-1}) \circ z^{K-1} - x^K) q_{n-1}(z^1, \ldots, z^K).$$

Therefore

$$\sum_{x^1,\dots,x^K} f(x^1,\dots,x^K) q_n(x^1,\dots,x^K)$$
$$= \sum_{x^1,\dots,x^K} f(x^1,\dots,x^K) K^M \prod_{i=1}^{K} \left(\frac{x^i}{L_n}\right)^{Y_n^i}$$
$$\sum_{z^1,\dots,z^K} \phi(\alpha^1 \circ z^1 + U_n - x^1,\dots,\alpha^K \circ z^K$$
$$+(1-\alpha^{K-1}) \circ z^{K-1} - x^K) q_{n-1}(z^1,\dots,z^K).$$

Since f is arbitrary the result follows.

4. Parameters Estimation

In this section, we shall use the so-called (EM) expectation maximization algorithm: see Dempster *et al.* [4] to re-estimate the parameters of our model.

We assume that the demand distribution $\phi(j_1,\dots,j_K)$ has finite support in each argument, that is, $1 \le j_i \le D$, $i = 1,\dots,K$.

Our model is determined by the set of parameters:

$$\theta \triangleq \{\phi(j_1,\dots,j_K); \quad \alpha^i, i = 1,\dots,K \quad j_i \le D\}, \tag{13}$$

which given the observed history $\mathcal{Y}_n$ we wish to update to a new set of parameters

$$\hat\theta \triangleq \{\hat\phi(j_1,\dots,j_K); \quad \hat\alpha^i, i = 1,\dots,K \quad j_i \le D\},$$

by maximizing the conditional pseudo-log-likelihood to be defined below. Write

$$\mathcal{H}_n = \sigma\{X_k^i, Y_k^i, D_k^i, i = 1,\dots,K; l = 1,\dots,L, k \le n\}, \tag{14}$$
$$\mathcal{Z}_n = \sigma\{Y_k^i, D_k^i, i = 1,\dots,K; l = 1,\dots,L, k \le n\} \tag{15}$$

and

$$\begin{aligned} M_n &= \prod_{k=1}^{n} \prod_{j_1=1}^{D} \cdots \prod_{j_K=1}^{D} \left(\frac{\hat{\phi}(j_1,\ldots,j_K)}{\phi(j_1,\ldots,j_K)} \right)^{I(D_k^1=j_1,\ldots,D_k^K=j_K)} \\ &\times \prod_{i=1}^{K} \left(\frac{\hat{\alpha}^i}{\alpha^i} \right)^{\alpha^i \circ X_{k-1}^i} \left(\frac{1-\hat{\alpha}^i}{1-\alpha^i} \right)^{X_{k-1}^i - \alpha^i \circ X_{k-1}^i} \\ &\triangleq \prod_{k=1}^{n} m_k. \end{aligned}$$

Note here that the exponents $\alpha^i \circ X_{k-1}^i$ in the expression for M_n simply give the number of items which survived from the previous period under probability α^i and it is not an explicit function of the parameter α^i. It is only a notation.

It can be shown that $\{M_n\}$ is a mean-one $\mathcal{H}_n$-martingale. Now one can set

$$E_\theta \left[\frac{dP_{\hat{\theta}}}{dP_\theta} \mid \mathcal{H}_n \right] = M_n. \tag{16}$$

The existence of $P_{\hat{\theta}}$ follows from Kolmogorov's extension theorem.
The log-likelihood is given by:

$$\begin{aligned} &\sum_{k=1}^{n} \sum_{j_1=1}^{D_k^1} \cdots \sum_{j_K=1}^{D_k^K} I(D_k^1 = j_1,\ldots,D_k^K = j_K) \log \hat{\phi}(j_1,\ldots,j_K) \\ &+ \sum_{k=1}^{n} \sum_{i=1}^{K} \left(\alpha^i \circ X_{k-1}^i \right) \log \hat{\alpha}^i \\ &+ \sum_{k=1}^{n} \sum_{i=2}^{K} \left(X_{k-1}^i - \alpha^i \circ X_{k-1}^i \right) \log(1-\hat{\alpha}^i) + R(\theta), \end{aligned}$$

where $R(\theta)$ does not contain terms in $\hat{\theta}$.
The conditional log-likelihood is:

$$\begin{aligned} &E_\theta \left[\log M_n \mid \mathcal{Z}_n \right] \\ &= \sum_{k=1}^{n} \sum_{j_1=1}^{D_k^1} \cdots \sum_{j_K=1}^{D^K} I(D_k^1 = j_1,\ldots,D_k^K = j_K) \log \hat{\phi}(j_1,\ldots,j_K) \end{aligned}$$

$$+ E_\theta\left[\sum_{k=1}^{n}\sum_{i=2}^{K}\alpha^i\circ X_{k-1}^i \mid \mathcal{Z}_n\right]\log\hat{\alpha}^i$$
$$+ E_\theta\left[\sum_{k=1}^{n}\sum_{i=2}^{K}\left(X_{k-1}^i-\alpha^i\circ X_{k-1}^i\right) \mid \mathcal{Z}_n\right]\log(1-\hat{\alpha}^i)+\hat{R}(\theta). \tag{17}$$

Write

$$\sum_{k=1}^{n} X_{k-1}^i \triangleq S_n^i, \tag{18}$$
$$\sum_{k=1}^{n} \alpha^i\circ X_{k-1}^i \triangleq S_n^i(\alpha^i). \tag{19}$$

Maximizing (17) subject to the constraint

$$\sum_{j_1,\dots,j_N}^{D}\hat{\phi}(j_1,\dots,j_N)=1,$$

yields the following result.

Theorem 4 *The new estimates $\hat{\phi}_n(.)$ and $\hat{\alpha}_n^i$ are given by the following relations:*

$$\hat{\phi}_n(j_1,\dots,j_N)=\frac{\sum_{k=1}^{n} I(D_k^1=j_1,\dots,D_k^K=j_K)}{\sum_{z^1,\dots,z^K}\sum_{k=1}^{n} I(D_k^1=z^1,\dots,D_k^K=z^K)}, \tag{20}$$

$$\hat{\alpha}_n^i=\frac{E_\theta[S_n^i(\alpha^i)\mid\mathcal{Z}_n]}{E_\theta[S_n^i\mid\mathcal{Z}_n]}=\frac{\overline{E}[S_n^i(\alpha^i)\Gamma_n^{-1}\mid\mathcal{Z}_n](\overline{E}[\Gamma_n^{-1}\mid\mathcal{Z}_n])^{-1}}{\overline{E}[S_n^i\Gamma_n^{-1}\mid\mathcal{Z}_n](\overline{E}[\Gamma_n^{-1}\mid\mathcal{Z}_n])^{-1}}$$
$$\triangleq\frac{\rho_n(S_n^i(\alpha^i))}{\rho_n(S_n^i)}. \tag{21}$$

Remark. In order to make the above estimators useful we need to derive recursions for $\rho_n(S_n^i(\alpha^i))$ and $\rho_n(S_n^i)$. However finite-dimensional recursions are possible for only expressions of the form

$$\overline{E}[S_n^i I(X_n^1=x^1,\dots,X_n^K=x^K)\Gamma_n^{-1}\mid\mathcal{Z}_n]\triangleq\rho_n(S_n^i,x^1,\dots,x^K), \tag{22}$$

etc. However

$$\sum_{x^1,\dots,x^K}\overline{E}[S_n^i I(X_n^1=x^1,\dots,X_n^K=x^K)\Gamma_n^{-1}\mid\mathcal{Z}_n]=\rho_n(S_n^i).$$

Theorem 5 *Finite-dimensional recursions for* $\rho_n(S_n^i, x^1, \ldots, x^K)$ *and* $\rho_n(S_n^i(\alpha^i), x^1, \ldots, x^K)$ *are as follows:*

$$\begin{aligned}
&\rho_n(S_n^i, x^1, \ldots, x^K) \\
&= K^M \prod_{j=1}^{K} \left(p_n^j\right)^{Y_n^j} \sum_{z^1,\ldots,z^K} R_n(U_n, z^1, \ldots, z^K, x^1, \ldots, x^K) \\
&\times \rho_{n-1}(S_{n-1}^i, z^1, \ldots, z^K) + q_n(x^1, \ldots, x^K) \sum_{z^1,\ldots,z^K} \phi(z^1, \ldots, z^K) z^i,
\end{aligned}$$

and

$$\begin{aligned}
&\rho_n(S_n^i(\alpha^i), x^1, \ldots, x^K) \\
&= K^M \prod_{j=1}^{K} \left(p_n^j\right)^{Y_n^j} \sum_{z^1,\ldots,z^K} R_n(U_n, z^1, \ldots, z^K, x^1, \ldots, x^K) \\
&\times \rho_{n-1}(S_{n-1}^i(\alpha^i), z^1, \ldots, z^K) + q_n(x^1, \ldots, x^K) \sum_{z^1,\ldots,z^K} \phi(z^1, \ldots, z^K) \alpha^i z^i.
\end{aligned}$$

Proof First note that $S_n^i = S_{n-1}^i + X_{n-1}^i$. Now recall that $\Gamma_n^{-1} = \Gamma_{n-1}^{-1}\gamma_n^{-1}$ where γ_n^{-1} can be obtained from (8). Therefore

$$\begin{aligned}
&\rho_n(S_n^i, x^1, \ldots, x^K) = \overline{E}\left[S_{n-1}^i \Gamma_{n-1}^{-1} \gamma_n^{-1} I(X_n^1 = x^1, \ldots, X_n^K = x^K) \mid \mathcal{Z}_n\right] \\
&+ \overline{E}\left[X_{n-1}^i \Gamma_n^{-1} I(X_n^1 = x^1, \ldots, X_n^K = x^K) \mid \mathcal{Z}_n\right] \qquad (23)
\end{aligned}$$

Again write $\dfrac{X_n^i}{L_n} \triangleq p_n^i$, and recall R_n from (9). In view of (8) and the distribution assumption under $\bar{P}$, the first expectation is simply

$$\begin{aligned}
&\overline{E}[S_{n-1}^i \Gamma_{n-1} I(X_n^1 = x^1, \ldots, X_n^K = x^K) \frac{R_n}{\phi(x^1, \ldots, x^K)} K^M \prod_{i=1}^{K} \left(p_n^i\right)^{Y_n^i} \mid \mathcal{Z}_n] \\
&= K^M \prod_{i=1}^{K} \left(p_n^i\right)^{Y_n^i} \overline{E}[R_n(U_n, X_{k-1}^1, \ldots, X_{k-1}^K, x^1, \ldots, x^K) S_{n-1}^i \Gamma_{n-1}^{-1} \mid \mathcal{Z}_{n-1}] \\
&= K^M \prod_{i=1}^{K} \left(p_n^i\right)^{Y_n^i} [\sum_{z^1,\ldots,z^K} R_n(U_n, z^1, \ldots, z^K, x^1, \ldots, x^K) \\
&\qquad \times \overline{E} I(X_{n-1}^1 = z^1, \ldots, X_{n-1}^K = z^K) S_{n-1}^i \Gamma_{n-1}^{-1} \mid \mathcal{Z}_{n-1}] \\
&= K^M \prod_{i=1}^{K} \left(p_n^i\right)^{Y_n^i} \sum_{z^1,\ldots,z^K} R_n(U_n, z^1, \ldots, z^K, x^1, \ldots, x^K) \\
&\times \rho_{n-1}(S_{n-1}^i, z^1, \ldots, z^K)
\end{aligned}$$

The second expectation in (23) is

$$q_n(x^1,\ldots,x^K)\sum_{z^1,\ldots,z^K}\phi(z^1,\ldots,z^K)z^i.$$

The rest of the proof is similar and is, therefore, skipped.

Note that in the model treated in this paper items were allowed to move down one class only. It seems to be adequate to have models that allow items items move down more than one class during the period. These models are appropriate for periods which are long enough to allow for this phenomena to occur.

In this paper a new discrete time discrete state inventory model for perishable items of a single product was introduced. Items in stock belonged to a one of a finite number of quality classes. At each discrete time items in the inventory may experience deterioration or get sold. Finite dimensional filters for the number of items in each class were proposed. Further, parameters estimation of the model were also discussed.

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