

A Simple Proof of Suzumura's Extension Theorem for Finite Domains With Applications

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Abstract. In this paper we provide a simple proof of the extension theorem for partial orderings due to Suzumura [1983] when the domain of the partial order is finite. The extension theorem due to Szpilrajn [1930] follows from this theorem. Szpilrajn's extension theorem is used to show that an asymmetric binary relation is contained in the asymmetric part of a linear order if and only if it is acyclic. This theorem is then applied to prove three results. Finally we introduce the concept of a threshold choice function, and our third result says that such choice functions are the only ones to satisfy a property called functional acyclicity.

Keywords: Partial Orderings, Extension Theorem, Threshold Choice Function.

1. Introduction

In this paper we provide a simple proof of the extension theorem for partial orderings due to Suzumura [1983] when the domain of the partial order is finite. The extension theorem due to Szpilrajn [1930] follows from this theorem. Szpilrajn's extension theorem is used to show that an asymmetric binary relation is contained in the asymmetric part of a linear order if and only if it is acyclic. This theorem is then applied to prove three results. The first result implied by two theorems in Aizerman and Malishevsky [1981], (see Aizerman and Aleskerov [1995] as well) says that the asymmetric part of a quasi-transitive binary relation can be expressed as the intersection of the asymmetric parts of orders. The well known result due to Dushnik and Miller [1941], which states that any asymmetric and transitive binary relation is the intersection of linear orders follows as an immediate corollary of this result. The second result is a theorem in Lahiri [1999], which says that a choice function is a batch choice function if and only if it satisfies a property called the choice acyclicity property. We provide a new proof

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of this result. The concept of a batch choice function can be found in Aizerman and Aleskerov [1995] and in recent times it has been applied in the study of stable matching problems. Finally we introduce the concept of a threshold choice function, and our third result says that such choice functions are the only ones to satisfy a property called functional acyclicity. This last property can be traced to Aizerman and Aleskerov [1995] as well.

2. The Extension Theorems

Let X be a finite, non-empty set. Given a binary relation R , let $P(R) = \{(x, y) \in R / (y, x) \notin R\}$ and $I(R) = \{(x, y) \in R / (y, x) \in R\}$. $P(R)$ is called the asymmetric part of R and $I(R)$ is called the symmetric part of R . A binary relation R on X is said to be (a) reflexive if $\forall x \in X : (x, x) \in R$; (b) complete if $\forall x, y \in X$ with $x \neq y$, either $(x, y) \in R$ or $(y, x) \in R$; (c) transitive if $\forall x, y, z \in X$, $[(x, y) \in R \ \& \ (y, z) \in R \text{ implies } (x, z) \in R]$; (d) asymmetric if $\forall x, y \in X : (x, y) \in R \text{ implies } (y, x) \notin R$; (e) quasi-transitive if $\forall x, y, z \in X$, $(x, y) \in P(R)$ and $(y, z) \in P(R)$ implies $(x, z) \in P(R)$. Given a binary relation R on X a binary relation Q on X is said to extend (be an extension of) R if $R \subset Q$ and $P(R) \subset P(Q)$.

A binary relation R on X is said to be a partial order if it is reflexive and transitive. It is said to be an order if it is a complete partial order. A binary relation R on X is said to be a linear order if it is an order and further $I(R) = \Delta_X \equiv \{(x, x) / x \in X\}$.

Given a binary relation R on X and given any non-empty subset S of X , let $M(S, R)$ denote $\{x \in S / (y, x) \in P(R) \text{ implies } y \notin S\}$.

Given a binary relation R on X define binary relations $T(R)$ ($: T^\circ(R)$) on X as follows: $(x, y) \in T(R)$ ($: T^\circ(R)$) if and only if there exists a positive integer K and x_1, \dots, x_K in X with (i) $x_1 = x, x_K = y$; (ii) $(x_i, x_{i+1}) \in R \forall i \in \{1, \dots, K-1\}$ (and $(x_i, x_{i+1}) \in P(R)$ for $i \in \{1, \dots, K-1\}$). $T(R)$ is called the transitive hull of R . Clearly $T(R)$ is always transitive. Further $T(I(R)) \subset I(T(R))$. Note that $T(R) \setminus T(I(R)) \subset T^\circ(R)$.

A binary relation R on X is said to be acyclic if $T(P(R))$ is asymmetric. It is said to be consistent if there does not exist any x in X such that $(x, x) \in T^\circ(R)$.

THEOREM 1 (Suzumura's Extension Theorem): *If R is a reflexive binary relation on X then it has an extension Q which is an order if and only if R is consistent.*

Proof: Since $T(R)$ is transitive, it is clearly acyclic. Thus whenever S is a non-empty subset of X , $M(S, T(R))$ is non-empty. Let $A_1 = M(X, T(R))$

and having defined A_n , let $A_{n+1} = M(X \setminus \bigcup_{i=1}^n A_i, T(R))$. Since X is finite, there exists a positive integer r such that $A_r \neq \phi$ and $X = \bigcup_{i=1}^r A_i$. Further if $i \neq j$, then $A_i \cap A_j = \phi$. Define $f : X \rightarrow \mathfrak{R}$ (the set of real numbers) as follows : $f(x) = r - i + 1$ if $x \in A_i$. Suppose $(x, y) \in P(T(R))$. Then $x \in A_i, y \in A_j$ implies by our method of construction that $i < j$. Thus $f(x) > f(y)$. Now suppose $(x, y) \in T(R)$ and towards a contradiction suppose that $f(y) > f(x)$. Hence if $y \in A_j$ and $x \in A_i$, clearly $j < i$. Thus, $A_j = M(X \setminus \bigcup_{k=1}^{j-1} A_k, T(R)), X \setminus \bigcup_{k=1}^{j-1} A_k$ is finite and $T(R)$ is transitive implies that there exists $z \in A_j$ such that $(z, x) \in P(T(R))$ since $x \in (X \setminus \bigcup_{k=1}^{j-1} A_k) \setminus A_j$. By transitivity of $T(R)$, $(z, y) \in P(T(R))$, contradicting $y \in A_j$. Thus, $f(x) \geq f(y)$. Let $(x, y) \in P(R)$. Thus $(x, y) \in T(R)$. If $(y, x) \in T(R)$, then along with $(x, y) \in P(R)$ it follows that $(y, y) \in T^\circ(R)$ contradicting that R is consistent. Thus $(x, y) \in P(T(R))$. Thus $f(x) > f(y)$. Now suppose that $(x, y) \in R$ and towards a contradiction suppose that $f(y) > f(x)$. Then as before there exists $z \in X$ such that $f(z) = f(y), (z, x) \in P(T(R))$. Thus $(z, y) \in T^\circ(R)$. If $(y, z) \in T(R)$ then $(z, z) \in T(R)$ contradicting the requirement that R is consistent. Thus, $(z, y) \in P(T(R))$. Thus, $f(z) > f(y)$ which contradicts $f(z) = f(y)$. Thus, $(x, y) \in R$ implies $f(x) \geq f(y)$. Let $Q = \{(x, y) \in X \times X / f(x) \geq f(y)\}$. Thus, Q is an order which extends R . \square

Corollary 1 (Szpilrajn's Extension Theorem): *If R is a partial order on X then it has an extension Q which is an order.*

Proof: Follows easily from Suzumura's Extension Theorem by noting that a partial order is always consistent. \square

The following lemma proves useful in establishing subsequent results.

LEMMA 1 *Let $f : X \rightarrow \mathfrak{R}$ (the set of real numbers) be given. Then, there exists a positive integer n and one to one functions $f_i : X \rightarrow N$ (:the set of natural numbers), $i \in \{1, \dots, n\}$ such that $\{(x, y) \in X \times X / f(x) \geq f(y)\} = \{(x, y) \in X \times X / f_i(x) \geq f_i(y) \text{ for some } i \in \{1, \dots, n\}\}$.*

Proof: Let $\{f(x) / x \in X\} = \{s_1, \dots, s_q\}$ where q is a positive integer and $s_j < s_{j+1} \forall j \in \{1, \dots, q-1\}$. Let $n_j = \{x \in X / f(x) = s_j\}$ and let $n = (n_1)! \times \dots \times (n_q)!$

Let $g : X \rightarrow N$ be defined as follows: $g(x) = n_1$, if $f(x) = s_1$

$g(x) = n_1 + \dots + n_j$, if $f(x) = s_j$.

Clearly, $\forall x, y \in X : [f(x) \geq f(y) \text{ if and only if } [g(x) \geq g(y)]$.

A function $\pi : \{1, \dots, n_1 + \dots + n_q\} \rightarrow X$ is called a restricted permutation if $\forall k \in \{1, \dots, n_1 + \dots + n_q\}$: (1) $[\pi(k) \in \{x \in X / f(x) = s_1\}$ if and only $(1 \leq k \leq n_1)]$ & (2) $[\pi(k) \in \{x \in X / f(x) = s_i\}$ if and only $(n_{i-1} \leq k \leq n_i$ and $1 < i \leq q)]$. Let Π denote the set of all restricted permutations. Since X is finite so is Π . For $\pi \in \Pi$, define $f_\pi: X \rightarrow \{1, \dots, n_1 + \dots + n_q\}$ as follows: $\forall x \in X, f_\pi(x) = k$ if and only if $\pi(k) = x$. It is now easy to verify that, $\{(x, y) \in X \times X / f(x) \geq f(y)\} = \{(x, y) \in X \times X / g(x) \geq g(y)\} = \{(x, y) \in X \times X / f_\pi(x) \geq f_\pi(y) \text{ for some } \pi \in \Pi\}$. This proves the lemma. \square

The following theorem is rather interesting and to an extent original:

THEOREM 2 *Let P be any asymmetric binary relation on X . Then there exists a linear order Q on X such that $P \subset P(Q)$ if and only if P is acyclic.*

Proof: Suppose P is an asymmetric binary relation on X and suppose there exists a linear order Q on X such that $P \subset P(Q)$. Towards a contradiction suppose P is not acyclic. Then there exists $x \in X$ such that $(x, x) \in T(P)$. Since $P \subset P(Q)$, $(x, x) \in T(P(Q))$. Since $P(Q)$ is transitive, $(x, x) \in P(Q)$, contradicting the asymmetry of $P(Q)$. Hence P must be acyclic.

Now suppose P is an asymmetric and acyclic binary relation on X . Let $R = T(P \cup \Delta)$. Clearly, R is reflexive and transitive. Hence by Szpilrajn's Extension Theorem there exists a reflexive, complete and transitive binary relation L on X such that $R \subset L$ and $P(R) \subset P(L)$. Since P is asymmetric and acyclic $P \subset P(R)$. Hence $P \subset P(L)$.

Since L is transitive, it is clearly acyclic. Thus whenever S is a non-empty subset of X , $M(S, L)$ is non-empty. Let $A_1 = M(X, L)$ and having defined A_n , let $A_{n+1} = M(X \setminus \bigcup_{i=1}^n A_i, L)$. Since X is finite, there exists a positive

integer r such that $A_r \neq \phi$ and $X = \bigcup_{i=1}^r A_i$. Further if $i \neq j$, then $A_i \cap$

$A_j = \phi$. Define $f : X \rightarrow \mathfrak{R}$ (the set of real numbers) as follows : $f(x) = r - i + 1$ if $x \in A_i$. Clearly, $L = \{(x, y) \in X \times X / f(x) \geq f(y)\}$. By Lemma 1, there exists a positive integer n and one to one functions $f_i: X \rightarrow \mathbb{N}, i \in \{1, \dots, n\}$ such that $\{(x, y) \in X \times X / f(x) \geq f(y)\} = \{(x, y) \in X \times X / f_i(x) \geq f_i(y) \text{ for some } i \in \{1, \dots, n\}\}$. For $i \in \{1, \dots, n\}$, let $Q_i = \{(x, y) \in X \times X / f_i(x) \geq f_i(y)\}$. Now $(x, y) \in P(L)$ implies and is implied by $f(x) > f(y)$ which is equivalent to $f_i(x) > f_i(y)$ for all $i \in \{1, \dots, n\}$. Thus $P(L) = \bigcap \{P(Q_i) / i \in \{1, \dots, n\}\}$. Thus $P \subset P(Q_1)$ where Q_1 is a linear order on X . \square

The following theorem, is really a consequence of two theorems in Aizerman and Malishevsky [1981] and these two theorems have been reproduced

in Aizerman and Aleskerov [1995]. It is important enough to merit an independent proof.

THEOREM 3 *If R is a quasi-transitive binary relation then $P(R) = \bigcap \{P(Q)/Q \in A\}$ where $\phi \neq A \subset \{Q \subset X \times X/Q \text{ is a linear order}\}$.*

Proof: Let $P = P(R)$. P is asymmetric and transitive. Hence by Theorem 2, there exists a linear order R^1 on X such that $P \subset P(R^1)$. Let $A = \{Q/Q \text{ is a linear order on } X \text{ with } P \subset P(Q)\}$. Thus, $P \subset \bigcap \{P(Q) / Q \in A\}$.

Now suppose $(x, y) \in \bigcap \{P(Q)/Q \in A\}$. Towards a contradiction suppose $(x, y) \notin P$. Since $(y, x) \in P \subset \bigcap \{P(Q)/Q \in A\}$ contradicts $[(x, y) \in P(Q) \text{ whenever } Q \in A]$, clearly $(y, x) \notin P$. Further, $(x, y) \in \bigcap \{P(Q)/Q \in A\}$ implies $[(y, x) \notin P(Q) \text{ whenever } Q \in A]$.

Let $\bar{P} = P \cup \{(y, x)\}$. Clearly, \bar{P} is asymmetric. Suppose towards a contradiction that $(z, z) \in T(\bar{P})$ for some $z \in X$. Thus there exists a positive integer m and elements z_1, \dots, z_m in X with $z = z_1 = z_m$ and $(z_i, z_{i+1}) \in P \cup \{(y, x)\} \forall i \in \{1, \dots, m-1\}$. If $(z_i, z_{i+1}) \in P \forall i \in \{1, \dots, m-1\}$, then we get by transitivity of P , that $(z_1, z_m) \in P(R)$ i.e. $(z, z) \in P$, contradicting asymmetry of P . Hence $(z_i, z_{i+1}) = (y, x)$ for some $i \in \{1, \dots, m-1\}$.

Observe that 'm' is greater than three, for if $m \leq 3$, then (z_1, z_2) and (z_2, z_1) belong to $P \cup \{(y, x)\}$ which is not possible since by hypothesis $x \neq y$ and (x, y) does not belong to $P(R)$.

Case 1: Cardinality of $\{i \in \{1, \dots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$ is one.

If $(z_1, z_2) = (y, x)$, then $z_m = y$ implies by transitivity of P that $(x, y) \in P$ which is a contradiction.

If $i > 1$, then $(z_1, y) \in P$ and $(x, z_1) \in P$ by transitivity of P , so that $(x, y) \in P$ by transitivity of P which is a contradiction.

Case 2: Cardinality of $\{i \in \{1, \dots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$ is greater than one.

Let $j = \min \{i \in \{1, \dots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$ and $k = \min \{i \in \{j+1, \dots, m-1\}/(z_i, z_{i+1}) = (y, x)\}$. Thus $z_{j+1} = x, z_k = y$ and by transitivity of P , $(x, y) \in P$ which is a contradiction.

Thus $(z, z) \notin T(\bar{P})$ whenever $z \in X$. Thus, \bar{P} is acyclic. By Theorem 2, there exists a linear order R° such that $\bar{P} \subset P(R^\circ)$. Thus $P \subset \bar{P} \subset P(R^\circ)$ and hence $R^\circ \in A$. However, $(y, x) \in \bar{P}$ implies $(y, x) \in P(R^\circ)$. This contradicts $(x, y) \in \bigcap \{P(Q)/Q \in A\}$. Thus $(x, y) \in P$. Hence the proof is complete. \square

The following well known theorem due to Dushnik and Miller [1941] follows as an immediate corollary of Theorem 3:

THEOREM 4 *Let P be any asymmetric and transitive binary relation on X . Then $P = \bigcap \{P(Q)/Q \in B\}$, where, $\phi \neq B \subset \{Q \in X \times X/Q \text{ is a linear order}\}$.*

3. Batch Choice Functions

Given any non-empty subset S of X , let $[S]$ denote the set of all non-empty subsets of S . Hence in particular, $[X]$ denotes the set of all non-empty subsets of X . A choice function C on X is a function $C : [X] \rightarrow [X]$ such that $C(S) \subset S \forall S \in [X]$.

A choice function C on X is said to satisfy the Choice Acyclicity Property (CAP) if there does not exist a positive integer K and sets $S_1, \dots, S_K \in [X]$ such that : (i) $\forall i \in \{1, \dots, K-1\} : C(S_i) \in [S_{i+1}] \setminus \{C(S_{i+1})\}$; and (ii) $C(S_K) \in [S_1] \setminus \{C(S_1)\}$.

A choice function C on X is said to be a batch choice function if there exists a linear order Q on $[X]$ such that $\forall S \in [X], C(S) = \{A \in [S] / \forall B \in [S] : (A, B) \in Q\}$.

THEOREM 5 (Lahiri [1999]) *C is a batch choice function if and only if C satisfies CAP.*

Proof: If C is a batch choice function it clearly satisfies CAP. Hence suppose C satisfies CAP. If X has just one element then C is obviously a batch choice function. Hence suppose that X has atleast two elements. Let $P = \{ (C(S), A) / A \in [S] \setminus \{C(S)\}, S \in [X] \text{ and } S \text{ has atleast two elements} \}$. Clearly P is asymmetric. Further, since C satisfies CAP, P is acyclic. By Theorem 2, there exists a linear order Q on $[X]$ such that $P \subset P(Q)$. Given $S \in [X]$, since $(C(S), A) \in P \forall A \in [S] \setminus \{C(S)\}$, $C(S) = \{A \in [S] / \forall B \in [S] : (A, B) \in Q\}$. Thus, C is a batch choice function. \square

Remark 1 : It is worth observing that there exists a choice function C on X which does not satisfy the CAP and yet there does not exist sets $S_1, S_2 \in [X]$ such that : (i) $C(S_1) \in [S_2] \setminus \{C(S_2)\}$ and (ii) $C(S_2) \in [S_1] \setminus \{C(S_1)\}$.

Example: Let $X = \{x, y, z\}$. Let $C(\{x, y\}) = \{y\}$, $C(\{y, z\}) = \{z\}$, $C(\{x, z\}) = \{z\}$, $C(A) = A$, otherwise. Clearly, there does not exist sets $S_1, S_2 \in [X]$ such that : (i) $C(S_1) \in [S_2] \setminus \{C(S_2)\}$ and (ii) $C(S_2) \subset [S_1] \setminus \{C(S_1)\}$.

However C does not satisfy CAP: $C(\{x, y\}) \in [\{y, z\}] \setminus \{C(\{y, z\})\}$, $C(\{y, z\}) \in [\{x, z\}] \setminus \{C(\{x, z\})\}$ and $C(\{x, z\}) \in [\{x, y\}] \setminus \{C(\{x, y\})\}$. Towards a contradiction suppose there exists an order Q on $[X]$ such that $\forall S \in [X], C(S) = \{A \in [S] / \forall B \in [S] : (A, B) \in Q\}$. Then, $(\{y\}, \{x\}) \in P(Q)$, $(\{x\}, \{z\}) \in P(Q)$ and $(\{z\}, \{y\}) \in P(Q)$ contradicting the assumption that Q is an order on $[X]$. Thus C is not a batch choice function.

4. Functional Acyclicity

The following property in Aizerman and Aleskerov [1995] known as functional acyclicity implies CAP:

A choice function C on X is said to satisfy Functional Acyclicity (FA) if there does not exist a positive integer K and sets $S_1, \dots, S_K \in [X]$ such that : (i) $\forall i \in \{1, \dots, K-1\} : C(S_i) \cap (S_{i+1} \setminus C(S_{i+1})) \neq \phi$; and (ii) $C(S_K) \cap (S_1 \setminus C(S_1)) \neq \phi$. However the following example reveals that the converse need not be true:

Example: Let $X = \{x, y, z\}$. Let $C(X) = \{x, y\}$, $C(\{x, z\}) = \{z\}$ and $C(A) = A$ otherwise. Clearly, C satisfies CAP. However, $(X \setminus C(X)) \cap \{x, z\} \neq \phi$ and $(\{x, z\} \setminus C(\{x, z\})) \cap X \neq \phi$ contradicting FA.

A choice function C is said to be a threshold choice function if there exists a function $V : [X] \rightarrow X$ and a linear order Q such that : (i) $\forall S \in [X] : V(S) \in S$; (ii) $C(S) = \{x \in S / (x, V(S)) \in Q\}$.

The following theorem is equivalent to Theorem 3.15 in Aizerman and Aleskerov [1995] but unlike others we prove it here by appealing to Theorem 2.

THEOREM 6 *A choice correspondence C is a threshold choice function if and only if it satisfies FA.*

Proof: Let C be a threshold choice function. Thus, there exists a function $V : [X] \rightarrow X$ and a linear order Q such that : (i) $\forall S \in [X] : V(S) \in S$; (ii) $C(S) = \{x \in S / (x, V(S)) \in Q\}$. Towards a contradiction suppose that there exists a positive integer K and sets $S_1, \dots, S_K \in [X]$ such that : (i) $\forall i \in \{1, \dots, K-1\} : C(S_i) \cap (S_{i+1} \setminus C(S_{i+1})) \neq \phi$; and (ii) $C(S_K) \cap (S_1 \setminus C(S_1)) \neq \phi$. Let $x_t \in C(S_t) \cap (S_{t+1} \setminus C(S_{t+1}))$, for $t = 1, \dots, K-1$ and let $x_K \in C(S_K) \cap (S_1 \setminus C(S_1))$. Thus $(x_t, V(S_t)) \in Q$, for $t = 1, \dots, K$, $(V(S_{t+1}), x_t) \in P(Q)$ for $t = 1, \dots, K-1$, and $(V(S_1), x_K) \in P(Q)$. Since Q is transitive we get $(x_K, x_K) \in P(Q)$, contradicting the asymmetry of $P(Q)$. This contradiction implies that C must satisfy FA.

Now suppose that C satisfies FA. Let $P = \{C(S)x(S \setminus C(S)) / S \in [X]\}$. P is asymmetric and by Functional Acyclicity P is acyclic. By Theorem 2, there exists a linear order Q on X such that $P \subset P(Q)$.

Given $S \in [X]$, let $\{V(S)\} = \{x \in C(S) / \forall y \in C(S) : (y, x) \in Q\}$.

Clearly, if $x \in C(S)$ then $(x, V(S)) \in Q$. Now, suppose $x \in S$ and $(x, V(S)) \in Q$ and towards a contradiction suppose $x \notin C(S)$. Thus, $(V(S), x) \in P$. Thus by the above $(V(S), x) \in P(Q)$ which contradicts $(x, V(S)) \in Q$. Thus $x \in S$, $(x, V(S)) \in Q$ implies $x \in C(S)$. \square

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