

*Research Article*

## **Computational Exploration of the Biological Basis of Black-Scholes Expected Utility Function**

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It has often been argued that there exists an underlying biological basis of utility functions. Taking this line of argument a step further in this paper, we have aimed to computationally demonstrate the biological basis of the Black-Scholes functional form as applied to classical option pricing and hedging theory. The evolutionary optimality of the classical Black-Scholes function has been computationally established by means of a haploid genetic algorithm model. The objective was to minimize the dynamic hedging error for a portfolio of assets that is built to replicate the payoff from a European multi-asset option. The functional form that is seen to evolve over successive generations which best attains this optimization objective is the classical Black-Scholes function extended to a multiasset scenario.

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### **1. Option basics**

An *option* provides the holder with the right to buy or sell a specified quantity of an underlying asset at a fixed price (called a strike price or an exercise price) at or before the expiration date of the option. Since it is a right and not an obligation, the holder can choose not to exercise the right and allow the option to expire. There are two basic types of options—*call* options (right to buy) and *put* options (right to sell). A call option gives the buyer of the option the right to buy the underlying asset at a fixed price (strike price or  $K$ ) at any time prior to the expiration date of the option. The buyer pays a price for this right. At expiration, if the value of the underlying asset is greater than the strike price, the buyer makes the difference as his or her profit. If the value of the underlying asset is less than the strike price, then the buyer simply does not exercise the option. More

Table 1.1. The effect of determining variables on the price of call and put options.

<i>Factor</i>	<i>Call value</i>	<i>Put value</i>
Increase in asset price	Increases	Decreases
Increase in strike price	Decreases	Increases
Increase in variance of asset price	Increases	Increases
Increase in time to expiration	Increases	Increases
Increase in interest rates	Increases	Decreases
Increase in dividends paid	Decreases	Increases

generally, the value of a call option increases as the value of the underlying asset increases and decreases as the value of the underlying asset decreases. The reverse is true for a put option.

An *American* option can be exercised at any time prior to its expiration, while a *European* option can be exercised only at expiration. The possibility of early exercise makes American options more valuable than otherwise similar European options. However, in most cases, the time premium associated with the remaining life of an option makes early exercise suboptimal.

A number of economic variables affect option prices. These are the value of underlying asset; variance in that value—as the variance increases price of both calls and puts increase—since all options have limited downside and depend upon price volatility for upside; expected dividends on the asset—which are likely to reduce the price appreciation component of the asset thereby reducing the value of calls and increasing the value of puts; strike price of option—the right to buy (sell) at a fixed price becomes more (less) valuable at a lower strike price; life of the option—both calls and puts benefit from a longer life; and the level of interest rates—as interest rates increase, the right to buy (sell) at a fixed price in the future becomes more (less) valuable (see Table 1.1).

To cover the risk (i.e., hedge) of writing (selling) an option, an option writer often takes recourse to a *replicating portfolio*. The objective is to use a combination of risk-free borrowing/lending and the underlying asset to create the same cash flows as the option being valued. Replicating a call requires borrowing at the available lowest borrowing rate and using the proceeds to buy  $D$  units of the underlying asset. Replicating a put requires selling  $D$  units of the underlying asset and lending out the proceeds. The magnitude of  $D$  is determined by means of the *option delta*; which is the ratio of change in option value to change in the price of the underlying asset. That is, in mathematical terms, the option delta is the first-order partial derivative of the option value with respect to the price of the underlying asset. Applying the principle of zero-arbitrage, the value of the option has to be equal to the value of the replicating portfolio. Any difference between the two is known in financial parlance as the *hedging error*. The replicating portfolio will then have to be periodically *rebalanced* (i.e., the delta recalculated with corresponding changes made to  $D$ ).

## 2. Problem specification

Sinn [4] has supplied a formal proof that expected utility maximization with logarithmic utility acts as the dominant preference in biological selection process in the sense that a population following any other preference rule for decision-making under risk will, with a probability that approaches certainty, become extinct over time relative to the population following the preference rule based on logarithmic utility. Chen [5] also argues on similar lines advocating an essentially biologically driven derivation of economic rationale underlying the utilitarian aspects governing production and competition in organized societies.

For an option writer who is looking to hedge his or her position, the expected utility maximization criterion will require the hedging error to be at a minimum at each point of rebalancing, where the hedging error is the difference between the expected payoff on the best-of option and the replicating portfolio value at that point. Given the fundamental premise of log-normality of the *Black-Scholes option pricing function* (Black and Scholes, [6]), we contend that the Black-Scholes functional form can be shown to have an evolutionary supremacy over any other functional form in maximizing the expected utility of the hedger by minimizing the hedging error. We do not seek to argue the applicational efficiency of a genetic algorithm-based hedging strategy but our main motivation here is rather to computationally explore and extricate the subtle features of evolutionary optimality naturally embedded in the Black-Scholes functional form.

If the Black-Scholes functional form indeed possesses evolutionary optimality, then, given a necessarily biological basis of the evolution of utility forms (Robson [2, 3]; Becker [1]), the dominant surviving population of hedging solutions in a haploid genetic algorithm model (which can be shown to be statistically equivalent to multiple *multiarmed bandit processes*) should show satisfactory convergence with any hedging scheme based on such function. That is, a hedging solution derived from the Black-Scholes expected payoff function should gradually evolve out of the future generations as the largely predominant genotype. Of course, all implicit assumptions about underlying asset price behavior have to be same as the Black-Scholes model.

## 3. The Black-Scholes function as used in classical options pricing and hedging

As the time interval between successive points of rebalancing the replicating portfolio is progressively shortened, the *limiting distribution* can take one of two forms: either it corresponds to a distribution allowing a continuous price process or one that allows for price “jumps.” The Black-Scholes pricing function is based on the assumption that the price process is continuous and the limiting distribution is the log-normal distribution.

The version of the model presented by Black and Scholes was designed to value European options, which were dividend-protected. The value of a call option in the Black-Scholes model can be written as a function of the following variables:

$S$  = current price of the underlying asset;

$X$  = strike price of the option;

$t$  = current time;

$T$  = time at maturity;

$i$  = risk-less interest rate corresponding to the life of the option;  
 $\sigma^2$  = variance in the  $\log_n$  (price) of the underlying asset.

The value of a European call option using the Black-Scholes pricing function is given as

$$C = SN(d_1) - Xe^{-i(T-t)}N(d_2), \quad \text{where } d_1 = \frac{[\log_n(S/K) + (i + \sigma^2/2)(T-t)]}{\{\sigma^2\sqrt{(T-t)}\}}, \quad (3.1)$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)}.$$

As is obvious from (3.1), the Black-Scholes function is nothing but an expected payoff function, yielding the present value of the expected payoff from holding the option. The call option delta is obtained from (3.1) as  $(\partial C/\partial S)$ , that is,  $N(d_1)$ . The replicating portfolio is actually embedded in the Black-Scholes model: to replicate the option one would need to invest  $N(d_1)$ -fraction of total portfolio value in the underlying asset and borrow an amount equal to  $Xe^{-it}N(d_2)$ .

In terms of the hedging problem, expected utility maximization is linked to option-pricing through the *indifference pricing technique* in the form of a continuous-time stochastic optimization problem. The hedger's expected utility from a hedging strategy is maximized when the cumulative hedging error is minimized. Out of a starting population of number of hedging solutions, that predicated by the Black-Scholes function should emerge as the dominant surviving population in order to validate our position concerning its evolutionary supremacy.

#### 4. Hedging problem involving a multiasset option

An option whose payoff is dependent on the value of more than one underlying asset is called a multiasset option. A hedging problem involving a multiasset option using a similar paradigm incorporates an extension of the Black-Scholes model to a multi-dimensional form (Braddock [7]; Fabozzi [8]). So far, closed-form pricing functions have been derived for options involving up to a maximum of two underlying assets but a closed-form pricing solution for the  $n$ -assets option remains an open problem (Stulz [9]; Johnson [10]; Martin [11]).

Let the value of a European multiasset option be contingent on  $J$  different assets such that the investor has the right to claim the return on the best-performing asset after a stipulated lock-in period. Given that one of the  $J$  assets is a risk-free asset, the investor gets assured of a minimum return equal to the risk-free rate  $i$  on his or her invested capital at the termination of the stipulated lock-in period. This effectively means that the nominal value of his or her initial investment can become endogenously insured, because the terminal wealth, even in the worst-case scenario, cannot be lower in value to the initial wealth plus the return earned on the risk-free asset minus a finite cost of *portfolio insurance* which is basically the cost of the multiasset option (Leland and Rubinstein [12]).

Then the *expected present value* of the terminal option payoff  $\hat{E}(r)_{t=T}$  is obtained as follows:

$$\hat{E}(r)_{t=T} = \text{Max}[w, \text{Max}_j \{e^{it}E(r_j)_{t=T}\}], \quad j = 1, 2, \dots, J-1. \quad (4.1)$$

In the above equation,  $r$  is the rate of return on the risk-free asset and  $T$  is the length of the investment horizon in continuous time and  $w$  is the initial wealth invested, that is, ignoring insurance cost, if the risk-free asset outperforms all other assets, then we get

$$\hat{E}(r)_{t=T} = \frac{we^{iT}}{e^{iT}} = w. \quad (4.2)$$

Thus to replicate the payoff from such an option, one requires a replicating portfolio consisting of all the underlying assets invested in accordance with their respective option deltas. Like the single-asset counterpart, the replicating portfolio corresponding to the multiasset option also needs to be periodically rebalanced as the option deltas are not static but rather change over time.

## 5. A genetic algorithm model for the multiasset option replication problem

**5.1. The setup.** At each point of rebalancing, the hedging error has to be minimized if the difference between the expected option payoff and the replicating portfolio value is to be minimized. The more significant this difference, the more will be the cost of rebalancing associated with correcting the hedging error; and as these costs cumulate, the less will be the ultimate utility of the hedge at the end of the lock-in period. Then the cumulative hedging error over the lock-in period is given as

$$\sum_t |\varepsilon_t| = \sum_t |E(r)_t - v_t|. \quad (5.1)$$

Here  $E(r)_t$  is the expected best-of option payoff at time-point  $t$  and  $v_t$  is the replicating portfolio value at that point of time. Then the replicating portfolio value at time  $t$  is obtained as the following linear form:

$$v_t = (p_0)_t e^{it} + \sum_j \{(p_j)_t (S_j)_t\}, \quad j = 1, 2, \dots, J-1. \quad (5.2)$$

Here  $(S_j)_t$  is the realized return on asset  $j$  at time-point  $t$  and  $p_1, p_2, \dots, p_{J-1}$  are the respective allocation proportions (weights) of investment funds among the  $J-1$  risky assets at time-point  $t$  and  $(p_0)_t$  is the allocation for the risk-free asset at time-point  $t$ . Of course the following relation holds at any point of time  $t$ :

$$(p_0)_t = 1 - \sum_j (p_j)_t, \quad j = 1, 2, \dots, J-1. \quad (5.3)$$

It is the *portfolio weights*, that is, the  $p_0$  and  $p_j$  values that are of critical importance in determining the size of the hedging error. The correct selection of these portfolio weights will ensure that the replicating portfolio accurately tracks the option. Biologically then, over a few successive generations, the predominant genotype will evolve as the one that best meets the fitness criterion based on the magnitude of the aforesaid hedging error.

The computational haploid genetic algorithm model we have devised has been programmed in Borland C, Release 5.02 and performs the three basic genetic functions of

*reproduction*, *crossover*, and *mutation* with the premise that in each subsequent generation  $x$  number of *chromosomes* from the previous generation will be reproduced based on the principal of *natural selection*. Following the reproduction function,  $2(x - 1)$  number of additional chromosomes will be produced through the crossover function, whereby every  $g$ th chromosome included in the *mating pool* will be crossed with the  $(g + 1)$ th chromosome at a preassigned *crossover locus*. There is also a provision in our program to introduce a certain number of mutations in each current chromosome population in order to enable rapid adaptation.

**5.2. The haploid genetic algorithm as a generalized bandit process.** Let a simple two-armed bandit process be considered whereby it is known that one of the arms pays a reward  $m_1$  with variance  $s_1^2$  and the other arm pays a reward  $m_2$  with variance  $s_2^2$  such that  $m_2 \leq m_1$ . Then this gives rise to the classical bandit problem dilemma regarding which of the two arms to play so as to optimize the tradeoff between *information usage* and *information acquisition* (Berry and Fristedt [13]). A common solution method would be to allocate an equal number of trials between the two arms, say  $f$ . Suppose that we have a total number of  $F$  possible trials. Say we select  $f$  such that  $2f < F$ . This will consist of the “training” phase. Thereafter, in the subsequent “testing” phase, the remaining  $F - 2f$  trials are allocated to the arm with the best observed payoff in the “training” phase. Then the expected loss is calculated as follows:

$$\lambda(F, f) = |m_1 - m_2| [(F - f)q(f) + f\{1 - q(f)\}]. \quad (5.4)$$

In the above equation,  $q(f)$  is the probability of selecting the wrong arm as the best arm during the “training” phase (De Jong [14]). The value of  $q(f)$  is fairly approximated by the tail of the normal distribution as follows:

$$q(f) \approx \frac{1}{\sqrt{2\pi}} \left\{ \exp \left( \frac{-z^2/2}{z} \right) \right\}, \quad \text{where } z = (\sqrt{f}) \left\{ \frac{(m_1 - m_2)}{\sqrt{(s_1^2 + s_2^2)}} \right\}. \quad (5.5)$$

The optimal “training” sample size  $f^*$  for minimizing the loss function  $\lambda(F, f)$  may be obtained by setting  $\lambda_f = 0$ ; as per the usual, *first-order condition* for minimization.

However, though procedure sounds simple enough, it is not necessarily the most optimal as was shown by Holland. According to his calculations, expected losses could be further minimized by allocating exponentially increasing number of trials to the observed better arm (Holland [15]). Though this approach is untenable because it entails perfect future knowledge, it at least sets an *upper bound* to the best solution technique, whereby any good technique should asymptotically approach this bound. The three-function genetic algorithm performs best in terms of asymptotically approaching the upper bound as it allocates an exponentially increasing number of trials to the best-observed chromosomes in terms of the fitness criterion (Goldberg [16]). Accordingly, we have based our computational model on the premise that  $2(x - 1)$  number of additional chromosomes

will be produced through the crossover function, for  $x$  number of chromosomes included in the mating pool through the process of natural selection. Every  $g$ th chromosome in the mating pool is crossed with the  $(g + 1)$ th chromosome at a predetermined *crossover locus*. Then, given that  $x$  number of chromosomes are initially included in the mating pool following the reproduction function, we will get a total number of  $x + 2(x - 1)$  chromosomes in the first generation's population. Here,  $x$  number are chromosomes retained from the previous generation based on the observed best fitness criterion. One or more of these  $x$  chromosomes can be, optionally, allowed to mutate (i.e., swap of bit positions between 1 and 0 from a particular locus onwards in the encoded bit-string). The remaining chromosomes in the first generation are the ones that come out as a direct result of our crossover function. Therefore, we get the following linear difference equation as the governing equation determining chromosome population size in the  $n$ th generation (derivation provided in the appendices):

$$G_n = G_{n-1} + 2(G_{n-1} - 1) = 3G_{n-1} - 2. \quad (5.6)$$

**5.3. Constructing a hypothetical *best of 3 assets* option.** The terminal payoff from such an option would be that on the asset ending up as the best performer among the three assets within the envelope, that is,

$$r_{t=T} = \text{Max}(S_0, S_1, S_2)_{t=T}. \quad (5.7)$$

Two assets with realized returns  $S_1$  and  $S_2$  can be considered *risky*, that is,  $\sigma_{S_1}^2, \sigma_{S_2}^2 > 0$ , while  $S_0$  may be considered *risk-free*, for example, the return on a government treasury bill, that is,  $\sigma_{S_0}^2 \approx 0$ . Then, a dynamic hedging scheme for this option would be to invest in a replicating portfolio consisting of the three assets, with funds allocated in particular proportions in accordance with the objective of maximizing expected utility, that is, minimizing the cumulative hedging error. Then the replicating portfolio at  $t$  for our 3-asset option is

$$v_t = (p_0)_t e^{it} + (p_1)_t (S_1)_t + (p_2)_t (S_2)_t. \quad (5.8)$$

Then the hedging error at time-point  $t$  is given as the difference between the payoff on the option at time-point  $t$  and the value of the replicating portfolio at that time:

$$|\varepsilon_t| = | \text{Max}(S_0, S_1, S_2)_t - v_t | = | \text{Max}(S_0, S_1, S_2)_t - \{ (p_0)_t e^{it} + (p_1)_t (S_1)_t + (p_2)_t (S_2)_t \} |. \quad (5.9)$$

If  $|\varepsilon_t| \approx 0$ , then the option writer is perfectly hedged at time-point  $t$ . If  $|\varepsilon_t| \approx 0$  can be maintained for the entire lock-in period, then one can say that the dynamic hedging scheme has worked perfectly, resulting in utility maximization for the option writer. We have used the following hypothetical data set for conducting our computational study with the haploid genetic algorithm scheme to minimize the hedging error  $|\varepsilon_t|$ .

We assume that all three assets start off with a realized return of unity at  $t = 0$ . The correlation between the two risky assets is assumed constant at a moderate 50%. The first

risky asset is taken to have a slightly higher mean return (20%) but a rather high volatility (30%), compared to the mean return (18%) and volatility (20%) of the second one. The risk-free rate is 5% p.a. and the lock-in period is taken as one year. The replicating portfolio is rebalanced at the end of each month over the period.

Thus, if the best performer is the first risky asset at time-point  $t$ ,  $|\varepsilon_t|$  is minimized when maximum allocation is made to the first risky asset at  $t$  while if the best performer is the second risky asset at time-point  $t$ ,  $|\varepsilon_t|$  is minimized if the maximum allocation is made to that asset at  $t$ . If neither of the two risky assets can outperform the guaranteed return on the risk-free asset at  $t$ , then  $|\varepsilon_t|$  is minimized if maximum allocation is made to the risk-free asset. Short selling is not permitted in our model.

To minimize programming involvement, our haploid genetic algorithm model is designed to handle only univariate optimization models. However, the problem we are studying is one of multivariate optimization with three parameters corresponding to the portfolio weights of the three assets underlying the option. Therefore, we have taken the allocation for the risk-free asset as given. This then essentially reduces our problem to one of univariate optimization, whereby we have to minimize the cumulative hedging error given by the following objective function:

$$\Sigma_t |\varepsilon_t| = \Sigma_t | \text{Max}(S_0, S_1, S_2)_t - \{c e^{it} + p_t (S_1)_t + (1 - c - p_t) (S_2)_t\} |. \quad (5.10)$$

Here,  $p_t$  is the allocation to be made to the first risky asset at every point of rebalancing and  $c$  is the given allocation always made to the risk-free asset thereby allowing us to substitute  $(p_2)_t$  with  $(1 - c - p_t)$ ; as portfolio weights sum up to unity. Then the *expected utility maximizing behavior* of the option writer will compel him or her to find out the optimal functional values of  $p_t$  at every  $t$  so as to minimize the total error (Goldberger [17]).

It is quite logical to assume that the  $p_t$  values will have to be related in some way to the *sensitivity* of the change in potential option payoff to the change in performance of the observed best asset within the envelope. With continuous rebalancing, one can theoretically achieve  $|\varepsilon_t| \approx 0$  if portfolio weights are selected in accordance with the partial derivatives of the option value with respect to the underlying asset returns, as per usual dynamic hedging technique in a Black-Scholes environment. Thus the utility maximization goal would be to suppress  $|\varepsilon_t|$  to a value as close as possible to zero at every  $t$  so that overall  $\Sigma_t |\varepsilon_t|$  is consequently minimized.

## 6. Computational observations

The hypothetical data presented in Table 6.1 is used to run a *Monte Carlo* simulation to generate the *potential* payoffs for the option on best of three assets at the end of each month for  $t = 0, 2, \dots, 11$ . The word potential is crucial in the sense that our multiasset option is essentially European and *path-independent*, that is, basically to say only the terminal payoff counts. However, the replicating portfolio has to track the option all through its life in order to ensure an optimal hedge, and therefore we have evaluated potential



Table 6.1. Hypothetical data for computational study.

$S_{1(t=0)}$	1.00
$S_{2(t=0)}$	1.00
$S_{0(t=0)}$	1.00
Correlation ( $S_1, S_2$ )	0.50
$\mu_{S_1}$	20%
$\mu_{S_2}$	18%
$\sigma_{S_1}$	30%
$\sigma_{S_2}$	20%
$I$	5%
$T$	12 months
Rebalancing frequency	Monthly

Table 6.2. Results of the Monte Carlo model.

$T$	$S_1$	$S_2$	$S_0$	Best asset	Potential payoff
0	1	1	1	—	—
1	1.0169	1.0219	1.0003	Asset 2	0.0216
2	1.0311	1.0336	1.0010	Asset 2	0.0326
3	1.0801	1.0820	1.0021	Asset 2	0.0799
4	1.1076	1.0960	1.0035	Asset 1	0.1041
5	1.1273	1.1280	1.0052	Asset 2	0.1228
6	1.1694	1.1694	1.0073	Asset 2	0.1621
7	1.2008	1.1923	1.0098	Asset 1	0.1910
8	1.2309	1.2160	1.0126	Asset 1	0.2183
9	1.2836	1.2489	1.0157	Asset 1	0.2679
10	1.3362	1.3030	1.0193	Asset 1	0.3169
11	1.3617	1.3196	1.0232	Asset 1	0.3385

payoffs at each  $t$ . The potential payoffs are computed as  $\text{Max}[(S_1)_t - (S_0)_t, (S_2)_t - (S_0)_t, 0]$  and the results are presented in Table 6.2.  $S_1$  and  $S_2$  are assumed to evolve over time following the stochastic diffusion process of a geometric Brownian motion (Black and Jones [18]).

The risk-free return  $S_0$  is continuously compounded approximately at a rate of 0.41% per month giving a 5% annual yield. We have run our Monte Carlo simulation model with the hypothetical data in Table 1.1 over the one-year lock-in period and calculated the potential option payoffs. All formal definitions pertaining to our computational model are given in the appendices.

The results shown in Table 6.2 and Figure 6.1 show that the second risky asset is the best performer towards the beginning and in the middle of lock-in period but the first one catches up and in fact outperforms the second one towards the end of the period.

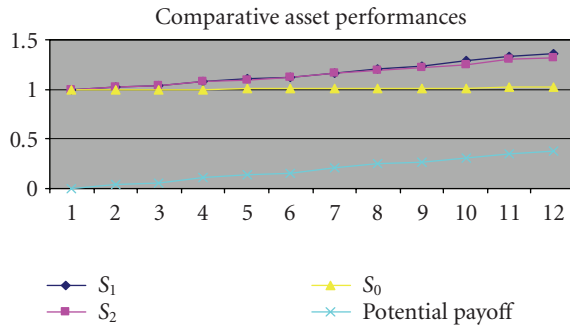


Figure 6.1

Table 6.3. Replicating portfolio value at each monthly rebalancing point.

$T$	$p_1 S_1$	$p_2 S_2$	$p_0 S_0$	$v_t$
0	0.4500	0.4500	0.1000	1.0000
1	0.4576	0.4598	0.1000	1.0175
2	0.0000	0.9303	0.1001	1.0304
3	0.0000	0.9738	0.1002	1.0740
4	0.0000	0.9864	0.1003	1.0867
5	1.0145	0.0000	0.1005	1.1151
6	0.0000	1.0524	0.1007	1.1532
7	0.0000	1.0731	0.1008	1.1741
8	1.1078	0.0000	0.1013	1.2091
9	1.1552	0.0000	0.1016	1.2568
10	1.2025	0.0000	0.1019	1.3045
11	1.2255	0.0000	0.1023	1.3279

For an initial input of \$1, apportioned at  $t = 0$  as 45% between  $S_1$  and  $S_2$  and 10% for  $S_0$ , we have constructed five replicating portfolios according to a simple *rule-based logic*:  $k\%$  of funds are allocated to the observed best performing risky asset and the balance proportion, that is,  $(90 - k)\%$  to the other risky asset (keeping the portfolio self-financing after the initial investment) at every monthly rebalancing point.

We have reduced  $k$  by 10% for each portfolio starting from 90% and going down to 50%. As is shown in the appendices, this simple hedging scheme performs quite well over the lock-in period when  $k = 90\%$ . But the performance falls away steadily as  $k$  is reduced every time.

### 7. Inferential remarks based on the computation results

The fact that a dominance pattern can be noticed for  $80\% < k^* \leq 90\%$  is quite evident from the output data in Tables 6.3 and 7.1 and the graphical plot of that data accompanying tabulated figures in the appendices. It may be noted to be of significance that

Table 7.1

Choice of $k$	Cumulative hedging error
90%	0.1064
80%	0.1172
70%	0.1281
60%	0.1390
50%	0.1498

$k^* \approx 90\%$  indeed comes closest to the percentage allocation for the best performing asset made according to the sensitivity of change in the potential payoff on the option to a change in performance of the observed best performer. This indeed satisfies the dynamic hedging principle in a Black-Scholes environment to the maximal extent possible given monthly rebalancing, a fixed allocation to  $S_0$  and no shorting, as were our imposed conditions. This therefore lends credence to our position that the Black-Scholes expected payoff function indeed has embedded evolutionary optimality when it comes to maximizing the utility of a hedging strategy via minimization of the cumulative hedging error.

## Appendices

### A. Formal definitions

A *geometric Brownian motion* is assumed to be the ubiquitous stochastic diffusion process driving asset prices and performances in organized financial markets. The discrete time version of this model, as is relevant in most computational applications, is given as follows:

$$\frac{\Delta S}{S} = \mu \Delta t + \varepsilon \sqrt{(\sigma^2 \Delta t)}. \quad (\text{A.1})$$

The variable  $\Delta S$  is the change in the asset price  $S$  and  $\varepsilon$  is a random sample from the standard normal distribution  $N(0, 1)$ . The expected return per unit of time on the asset is denoted as  $\mu$  and the variance of the asset price is denoted as  $\sigma^2$ .  $\Delta S/S$  is actually the proportional return provided by the asset in a very short interval of time  $\Delta t$ . The term  $\mu \Delta t$  then stands for the expected value of this return, which is deterministic and  $\varepsilon \sqrt{(\sigma^2 \Delta t)}$  is then the stochastic component of this return.

Therefore, the variance of the return on the asset is given by  $\sigma^2 \Delta t$ . That is to say  $\Delta S/S$  follows a normal distribution with mean  $\mu \Delta t$  and variance  $\sigma^2 \Delta t$ . Thus a path for an asset price or performance can be simulated by repeated sampling from  $N(\mu \Delta t, \sigma^2 \Delta t)$ . This is exactly how the Monte Carlo simulation model we have used in this study was designed. Since we have used monthly rebalancing, we have  $\Delta t = 1/12$ , that is,  $\Delta t \approx 0.0833$ . The asset performances  $S_1$  and  $S_2$  have been generated by repeatedly sampling from  $N(0.0167, 0.0075)$  and  $N(0.0150, 0.0033)$ , respectively, and averaging. As the performances

of the two risky assets are correlated, the random samples were drawn according to the following formula using Microsoft Excel's inbuilt RAND ( $\cdot$ ) function:

$$\begin{aligned} r_1 &= \text{RAND}(\cdot), \\ r_2 &= r_1 + \text{RAND}(\cdot) * (1 - 0.50)^2 = r_1 + \text{RAND}(\cdot) * 0.50^2. \end{aligned} \quad (\text{A.2})$$

### B. Mathematical derivation of the size of $G_n$

According to our haploid genetic algorithm reproduction and crossover functions, the size of the  $n$ th generation, that is, the number of chromosomes in the population at the end of the  $n$ th generation is given by the following first order, linear difference equation:

$$G_n = G_{n-1} + 2(G_{n-1} - 1) = 3G_{n-1} - 2. \quad (\text{A.1})$$

If  $x$  initial number of chromosomes are introduced at  $n = 0$ , we have  $G_0 = x$ . Then, obviously,  $G_1 = x + 2(x - 1) = 3x - 2 = 3^1(x - 1) + 1$ . Extending the recursive logic to  $G_2$  and  $G_3$  we get  $G_2 = 9x - 8 = 3^2(x - 1) + 1$  and  $G_3 = 27x - 26 = 3^3(x - 1) + 1$ . Therefore, extending to  $G_t$  we can write the following relation:

$$G_t = 3^t(x - 1) + 1. \quad (\text{A.2})$$

Therefore,  $G_{t+1} = 3^{t+1}(x - 1) + 1$  but  $G_{t+1} = 3G_t - 2$ . Substituting for  $G_t$ , we thereby get  $G_{t+1} = 3\{3^t(x - 1) + 1\} - 2 = 3^{t+1}(x - 1) + 3 - 2 = 3^{t+1}(x - 1) + 1$ . Therefore, the case is proved for  $G_{t+1}$ . But we have already proved it for  $G_1$ ,  $G_2$ , and  $G_3$ . Therefore, by the *principle of mathematical induction*, the general formula is derived as follows:

$$G_n = 3^n(x - 1) + 1. \quad (\text{A.3})$$

We can verify that our genetic algorithm model has indeed reproduced this  $G_n$  number of chromosomes in each generation for an initial input of  $G_0 = x = 5$ . Accordingly, our algorithm reproduced 13, 37, and 109 chromosomes for  $n = 1, 2, 3$ .

### C. Computational output of the GA performance

For an initial input of \$1, apportioned at  $t = 0$  as 45% between  $S_1$  and  $S_2$  and 10% for  $S_0$ , we have constructed five replicating portfolios according to a simple *rule-based logic*:  $k\%$  of funds are allocated to the observed best performing risky asset and the balance  $(90 - k)\%$  to the other risky asset (keeping the portfolio self-financing after the initial investment) at every monthly rebalancing point. We have reduced  $k$  by 10% for each portfolio starting from 90% and going down to 50%. As is shown in Figures A.1–A.5, this simple hedging scheme performs quite well over the lock-in period when  $k = 90\%$ . But the performance falls away steadily as  $k$  is reduced every time.

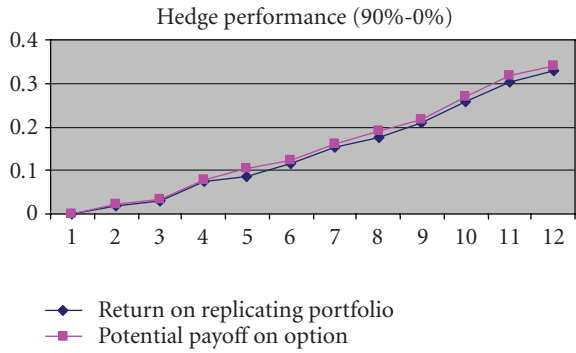


Figure A.1

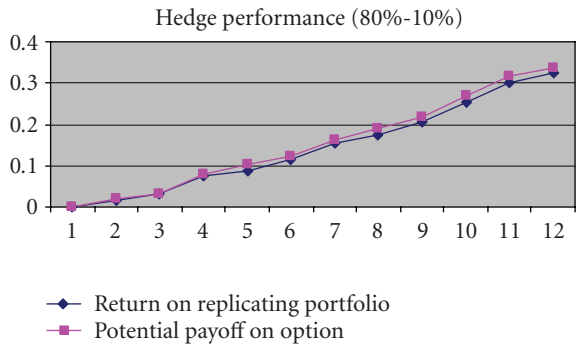


Figure A.2

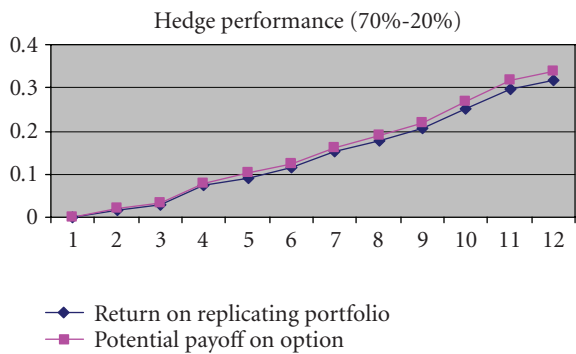


Figure A.3

Evaluation of the fitness criterion over successive generations corresponding to the optimal choice of  $k$  ( $80% < k = 90%$ ) is shown in Table A.1.

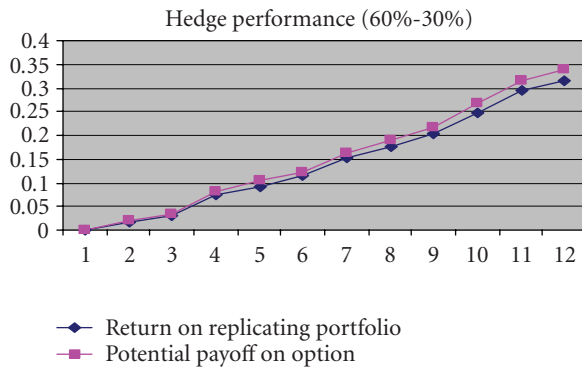


Figure A.4

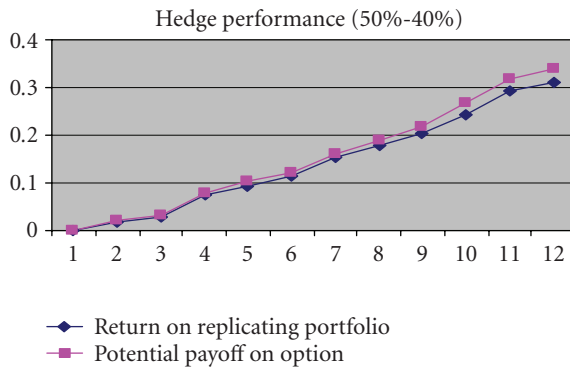


Figure A.5

Table A.1

Generation	Number of chromosomes	$80\% < k^* \leq 90\%$	% in range (fitness)
0	5	1	20%
1	13	4	31%
2	37	19	51%
3	109	61	56%

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