UNSTEADY HEAT TRANSFER OF A MONATOMIC GAS BETWEEN TWO COAXIAL CIRCULAR CYLINDERS

A. M. ABOURABIA, M. A. MAHMOUD, AND W. S. ABDEL KAREEM

Received 16 August 2001

We consider a kinetic-theory treatment of the cylindrical unsteady heat transfer. A model kinetic equation of the BGK (Bhatnager-Gross-Krook) type is solved using the method of moments with a two-sided distribution function. We study the relations between the different macroscopic properties of the gas as the temperature, density, and heat flux with both the radial distance r and the time t. Also we study the problem from the viewpoint of irreversible thermodynamics and estimate the entropy, entropy production, entropy flux, thermodynamic forces, kinetic coefficients, the change in internal energy, and verify Onsager's relation for nonequilibrium thermodynamic properties of the system.

1. Introduction

The Couette problem with heat transfer is one of the important situations in gas dynamics, which involve the nature of a rarefied gas near a solid surface. From the kinetic viewpoint, the rarefied cylindrical Couette flow has been analyzed by many authors. One of the main methods of constructing the transfer theory at arbitrary Knudsen number consists of the use of moments obtained from Boltzmann equation. The idea behind the method of moments consists of transforming the boundary value problems from the microscopic form to the form of equations of the continuum in which the principle variables that define the state of the system are certain moments of the distribution function. The motion of a rarefied gas between two coaxial cylinders: one is fixed and the other rotates with constant angular velocity, was studied in [2], using

Journal of Applied Mathematics 2:3 (2002) 141–161
2000 Mathematics Subject Classification: 74A25, 80Axx, 74A15, 62P30
URL: http://dx.doi.org/10.1155/S1110757X02108023

the moments method for obtaining a suitable solution for any Knudsen number. The flow of a gas between two coaxial cylinders, the inside cylinder being at rest with temperature T_i, while the outside cylinder rotates at a constant angular velocity with temperature T^*, was studied in [12]. A numerical solution to the problem of a cylinder rotating in a rarefied gas and a comparison with the approximate analytical solution are given in [9]. The problem of flow over a right circular cylinder within the framework of the kinetic theory of gases is studied in [1]. The heat transfer of a cylindrical Couette flow of a rarefied gas with porous surfaces was investigated in [5] in the framework of the kinetic theory of gases. In [3], the cylindrical Couette flow problem of rarefied gases was numerically analyzed. This estimate is based upon the characteristic equations, which are equivalent to the BGK (Bhatnager-Gross-Krook) model of Boltzmann equation. Over a wide range of Knudsen numbers, it was found that the BGK solutions show good agreement with the other numerical solutions and with the existing experimental data of density profiles and drag coefficients for light gases such as argon and air. The free cylindrical Couette flow of a rarefied gas with heat transfer, porous surfaces, and arbitrary reflection coefficient was discussed in [7], solving the moment equations with convenient boundary conditions concerning heat transfer, porosity, and reflection at the surfaces using the small parameters method. The behavior of the velocity, density, and temperature was predicted by Mahmoud [8], he studied steady motion of a rarefied gas between two coaxial cylinders: one is fixed and the other is rotating with angular velocity Ω. The free unsteady expansion of an ideal gas into a vacuum was discussed by Kraiko [6], starting with one-dimensional isentropic unsteady gas flow, he derived an asymptotic expansion for the density, and considering only the first term. It was concluded that the density decreases as a negative power of the time. In [11], a problem of a steady radial gas flow between two infinitely long coaxial cylinders, with boundary conditions of evaporation (emission) and condensation (absorption) which is formulated for a nonlinear kinetic equation with a model operator of collisions was studied. This problem is solved by the finite difference method. Considerable attention is paid to the flow from the inner evaporation cylinder to the absolutely absorbent outer one. Sone et al. [13, 14] studied the steady behaviour of the gas between two rotating cylinders in the basis of the kinetic theory from the continuum to the Knudsen limit. In this paper, we study the unsteady heat transfer of a gas using the moments and perturbation methods, and we study the problem from the standpoint of irreversible thermodynamics to estimate the macroparameters and verify Onsager's relations applied to the system.

2. The physical problem and mathematical formulation

Consider an axially symmetric problem of an unsteady heat transfer of a rarefied gas between two coaxial cylinders of infinite length and circular cross-section with the radii $r = r_1$ and $r = r_2$, where $r_1 < r_2$, under the conditions of evaporation from the surface of the inner cylinder and absorption on the surface of the outer cylinder. The gas is evaporated from the cylindrical surface $r = r_1$ with the parameters of saturated vapor. The axially symmetric state of a rarefied gas at the distance r from the symmetry axis OZ of the Cartesian coordinate system, is determined by a distribution function $f(r, c_z, c_r, c_\theta, t)$ of molecules over velocities, where c_z, c_r, and c_θ are the components of the molecular velocity in the axial, radial, and azimuthal directions, respectively. In the space of molecular velocities, we also use the cylindrical coordinates (c_z, c_n, ψ) related to the orthogonal coordinates (c_z, c_r, c_θ) by the formulas

$$\begin{aligned} c_z &= c_z, \\ c_r &= c_n \sin\psi, \\ c_\theta &= c_n \cos\psi. \end{aligned} \tag{2.1}$$

Here, c_n is the component of molecular velocity that lies in the plane perpendicular to the symmetry axis so that $c_n^2 = c_r^2 + c_\theta^2$ and ψ is the angle between the vectors c_n and r, where r is the radius vector of a point of the physical space in the cylindrical coordinate system (z, r, θ). Assuming that the distribution function satisfies the kinetic equation with the BGK approximate collision operator, we will solve the problem in a simplified statement. The kinetic equation can be written as follows:

$$Df = \frac{1}{\tau}(f_0 - f), \quad D = \frac{\partial}{\partial t} + \vec{c} \cdot \frac{\partial}{\partial \vec{r}}, \tag{2.2}$$

where f_0 is the local Maxwellian distribution function and τ is the relaxation time. Hence, we obtain the transfer equation in cylindrical coordinates in the form

$$\begin{aligned} r\frac{\partial}{\partial t}\int Q_i f\, d\vec{c} + \frac{\partial}{\partial r}\left(r\int Q_i c_r f\, d\vec{c}\right) - \int c_\theta^2 f \frac{\partial Q_i}{\partial c_r}\, d\vec{c} + \int c_r c_\theta f \frac{\partial Q_i}{\partial c_\theta}\, d\vec{c} \\ = \frac{r}{\tau}\int (f_0 - f) Q_i\, d\vec{c}, \end{aligned} \tag{2.3}$$

where Q_i is a function of the velocity. The moment $\overline{Q_i}$ of $Q_i(\vec{c})$ is given by

$$\overline{Q_i} = \int Q_i(\vec{c}) f\, d\vec{c} = \int_{\alpha}^{\pi-\alpha}\int_0^{\infty}\int_{-\infty}^{\infty} Q_i f_1 c_n\, dc_z\, dc_n\, d\psi + \int_{\pi-\alpha}^{2\pi+\alpha}\int_0^{\infty}\int_{-\infty}^{\infty} Q_i f_2 c_n\, dc_z\, dc_n\, d\psi, \tag{2.4}$$

where $\cos\alpha = r_1/r$. Since the particles between the two cylinders are colliding with each other via binary collisions, the cone of influence will be generated [12]. Let n_1, T_1 be the density and temperature in region one of the cone and n_2, T_2 be the density and temperature in region two. It is assumed from the beginning that the gas flows in a coordinate system moving with the mean gas velocity, furthermore, the distribution function is divided as follows:

$$f(\vec{c}, r, t) = \begin{cases} f_1 = \dfrac{n_1}{(2\pi RT_1)^{3/2}} \exp\left(-\dfrac{c^2}{2RT_1}\right) & \text{if } \alpha \le \psi \le \pi - \alpha, \\ f_2 = \dfrac{n_2}{(2\pi RT_2)^{3/2}} \exp\left(-\dfrac{c^2}{2RT2}\right) & \text{if } \pi - \alpha \le \psi \le 2\pi + \alpha, \end{cases} \tag{2.5}$$

and the local Maxwillian distribution function f_0 is

$$f_0 = \frac{n}{(2\pi RT)^{3/2}} \exp\left(-\frac{c^2}{2RT}\right). \tag{2.6}$$

All necessary macroparameters of the gas, such as the number density, temperature, pressure, and radial heat flux are expressed in terms of the distribution function in the usual way:

$$\begin{aligned} n &= \int f\, dc = \frac{(\pi - 2\alpha)n_1 + (\pi + 2\alpha)n_2}{2\pi}, \\ T &= \frac{1}{3Rn}\int c^2 f\, dc = \frac{(\pi - 2\alpha)n_1 T_1 + (\pi + 2\alpha)n_2 T_2}{(\pi - 2\alpha)n_1 + (\pi + 2\alpha)n_2}, \\ \overline{p_{rr}} &= \frac{p_{rr}}{R} = \int c_r^2 f\, d\vec{c} \\ &= \frac{1}{2\pi}\big((\pi - 2\alpha + \sin 2\alpha)n_1 T_1 + (\pi + 2\alpha - \sin 2\alpha)n_2 T_2\big), \\ \overline{q_r} &= \frac{q_r}{mR^{3/2}} = \frac{m}{2}\int c_r c^2 f\, d\vec{c} = 2\sqrt{2}\big(n_1 T_1^{3/2} - n_2 T_2^{3/2}\big)\cos\alpha. \end{aligned} \tag{2.7}$$

On the surface of each cylinder, we specify the flow of particles from the cylinder or, analogously, the distribution function for molecular velocities directed into the domain of integration. On the outer cylinder

$r = r_2$, we assume that this function is the Maxwellian distribution with the known macroparameters n_s and T_s. If we then take $Q = c_r, c^2, c_r c^2, 1$ and substitute in (2.3), using the normalized quantities

$$n_i = n_i' n_s, \quad T_i = T_i' T_s, \quad q = \frac{r_1}{r_2}, t = t'\tau, \quad r = r' r_2, \quad i = 1,2, \tag{2.8}$$

we get the following equations in nondimensional form

$$\begin{aligned}
&\frac{\partial}{\partial t'}\left(n_1' T_1'^{1/2} - n_2' T_2'^{1/2}\right) \\
&\quad + \frac{\gamma}{2\pi q^2}\frac{\partial}{\partial r'}\left(r'\{(\pi - 2\alpha + \sin 2\alpha) n_1' T_1' + (\pi + 2\alpha - \sin 2\alpha) n_2' T_2'\}\right) \\
&\quad - \frac{\gamma}{2\pi q^2}\left(\{(\pi - 2\alpha + \sin 2\alpha) n_1' T_1' + (\pi + 2\alpha - \sin 2\alpha) n_2' T_2'\}\right) = 0,
\end{aligned} \tag{2.9}$$

$$\frac{\partial}{\partial t'}\left((\pi - 2\alpha) n_1' T_1' + (\pi + 2\alpha) n_2' T_2'\right) + \frac{4\gamma}{r'}\frac{\partial}{\partial r'}\left(n_1' T_1'^{3/2} - n_2' T_2'^{3/2}\right) = 0, \tag{2.10}$$

$$\begin{aligned}
&\frac{\partial}{\partial t'}\left(n_1' T_1'^{3/2} - n_2' T_2'^{3/2}\right) \\
&\quad + \frac{5\gamma}{8\pi q^2}\frac{\partial}{\partial r'}\left(r'\{(\pi - 2\alpha + \sin 2\alpha) n_1' T_1'^2 + (\pi + 2\alpha - \sin 2\alpha) n_2' T_2'^2\}\right) \\
&\quad - \frac{5\gamma}{8\pi q^2}\left(\{(\pi - 2\alpha + \sin 2\alpha) n_1' T_1'^2 + (\pi + 2\alpha - \sin 2\alpha) n_2' T_2'^2\}\right) \\
&\quad = -\left(n_1' T_1'^{3/2} - n_2' T_2'^{3/2}\right),
\end{aligned} \tag{2.11}$$

$$\frac{\partial}{\partial t'}\left((\pi - 2\alpha) n_1' + (\pi + 2\alpha) n_2'\right) + \frac{4\gamma}{r'}\frac{\partial}{\partial r'}\left(n_1' T_1'^{1/2} - n_2' T_2'^{1/2}\right) = 0. \tag{2.12}$$

The boundary conditions can be taken as follows:

$$n_2(r_2, t) = n_s, \qquad T_2'(r_2, t) = T_s, \qquad T_1'(r_1, t) = (1 + \gamma) T_s, \tag{2.13}$$

where

$$\gamma = q K_n \ll 1, \quad K_n = \frac{\lambda_s}{r_2}, \quad \lambda_s = \tau\sqrt{2\pi R T_s}, \tag{2.14}$$

here K_n is the Knudsen number and λ_s is the mean free path at the outer cylinder.

The initial and boundary conditions can be taken as

$$n_2'(1,0)=1, \qquad T_2'(1,0)=1, \qquad T_1'(q,0)=1+\gamma. \tag{2.15}$$

Equations (2.9), (2.10), (2.11), and (2.12) are nonlinear. Since γ is small, we consider the two perturbing quantities (after dropping the primes)

$$n_i=1+\gamma n_i^{(1)}, \qquad T_i=1+\gamma T_i^{(1)}. \tag{2.16}$$

Substituting from expression (2.16) into (2.9), (2.10), (2.11), and (2.12), we get the following equations taking into consideration terms of equal powers of γ and then integrating.

For free terms of γ

$$n_1^{(1)}+\frac{1}{2}T_1^{(1)}-n_2^{(1)}-\frac{1}{2}T_2^{(1)}=0, \tag{2.17}$$

$$(\pi-2\alpha)\left(n_1^{(1)}+T_1^{(1)}\right)+(\pi+2\alpha)\left(n_2^{(1)}+T_2^{(1)}\right)=G(r), \tag{2.18}$$

$$n_1^{(1)}+\frac{3}{2}T_1^{(1)}-n_2^{(1)}-\frac{3}{2}T_2^{(1)}=F(r)\exp(-t). \tag{2.19}$$

For the first power of γ

$$\begin{aligned}\frac{\partial}{\partial t}&\left((\pi-2\alpha)n_1^{(1)}T_1^{(1)}+(\pi+2\alpha)n_2^{(1)}T_2^{(1)}\right)\\&+\frac{4}{r}\frac{\partial}{\partial r}\left(n_1^{(1)}+\frac{3}{2}T_1^{(1)}-n_2^{(1)}-\frac{3}{2}T_2^{(1)}\right)=0,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(n_1^{(1)}T_1^{(1)}-n_2^{(1)}T_2^{(1)}\right)&+\frac{r}{\pi q^2}\frac{\partial}{\partial r}\left\{(\pi-2\alpha+\sin 2\alpha)\left(n_1^{(1)}+T_1^{(1)}\right)\right.\\&\left.+(\pi+2\alpha-\sin 2\alpha)\left(n_2^{(1)}+T_2^{(1)}\right)\right\}=0.\end{aligned} \tag{2.21}$$

For the second power of γ

$$\begin{aligned}\frac{\partial}{\partial r}&\left((\pi-2\alpha+\sin 2\alpha)n_1^{(1)}T_1^{(1)}+(\pi+2\alpha-\sin 2\alpha)n_2^{(1)}T_2^{(1)}\right)\\&+\frac{2\sin 2\alpha}{r}\left(n_1^{(1)}T_1^{(1)}-n_2^{(1)}T_2^{(1)}\right)=0,\end{aligned} \tag{2.22}$$

$$n_1^{(1)}T_1^{(1)}-n_2^{(1)}T_2^{(1)}=H(t). \tag{2.23}$$

Substituting from (2.23) into (2.22), we obtain

$$\pi\frac{\partial}{\partial r}\left(n_1^{(1)}T_1^{(1)}+n_2^{(1)}T_2^{(1)}\right)-2H(t)\frac{\partial\alpha}{\partial r}+H(t)\frac{\partial(\sin 2\alpha)}{\partial r}+\frac{2\sin 2\alpha}{r}H(t)=0, \tag{2.24}$$

using (2.23) again we find that

$$n_1^{(1)}T_1^{(1)}+n_2^{(1)}T_2^{(1)}=H(t)+2n_2^{(1)}T_2^{(1)}. \tag{2.25}$$

Hence, introducing (2.25) into (2.24), we get

$$2\pi\frac{\partial y}{\partial r}-\frac{4r_1H(t)\sqrt{r^2-r_1^2}}{r^3}(1-r)=0, \tag{2.26}$$

where $y=n_2^{(1)}T_2^{(1)}$. Now integrating (2.26), we get

$$y=\frac{2r_1H(t)}{\pi}\left(\frac{1}{2r_1}\left(\alpha-\frac{\sin 2\alpha}{2}\right)-\ln(\sec\alpha+\tan\alpha)+\sin\alpha\right)+D(t). \tag{2.27}$$

We let $D(t)=0$ for simplicity. By using (2.20) and (2.27), we obtain

$$\exp(t)\frac{dH(t)}{dt}+\frac{4}{L(r)}\frac{dF(r)}{dr}=0, \tag{2.28}$$

where

$$L(r)=r\left((\pi-2\alpha)+2\left(\alpha-\frac{\sin 2\alpha}{2}\right)-4r_1\ln(\sec\alpha+\tan\alpha)+4r_1\sin\alpha\right). \tag{2.29}$$

Solving (2.28) by separation of variables, we get

$$H(t)=a\int\exp(-t)\,dt=-a\exp(-t)+c_1, \tag{2.30}$$

$$\begin{aligned}F(r)&=\frac{a}{4}\int L(r)\,dr\\&=-\frac{a\pi}{8}r^2+\frac{ar_1^2}{2}(\tan\alpha-\alpha)+\frac{ar_1^2}{2}\left(\tan^2\alpha\right)\ln(\sec\alpha+\tan\alpha)\\&\quad-\frac{5ar_1^3}{4}\left(\ln(\sec\alpha+\tan\alpha)+\frac{3}{5}\sec\alpha\tan\alpha\right)+c_2,\end{aligned} \tag{2.31}$$

where a is the separation constant and c_1, c_2 are the constants of integration. From (2.18), (2.21), and (2.23) we obtain

$$\frac{dH(t)}{dt}+\frac{r}{\pi q^2}\frac{dG}{dr}+\frac{2r_1 r}{\pi q^2}U\left(\frac{2r_1^2-r^2}{r^3\sqrt{r^2-r_1^2}}\right)+\frac{2r_1\sqrt{r^2-r_1^2}}{\pi q^2 r}\frac{\partial y}{\partial r}=0, \quad (2.32)$$

where

$$U=n_1^{(1)}+T_1^{(1)}-n_2^{(1)}-T_2^{(1)}. \quad (2.33)$$

From (2.17) and (2.19), we obtain

$$U=\frac{1}{2}F(r)\exp(-t). \quad (2.34)$$

Solving (2.32) using (2.34), we obtain

$$\begin{aligned}
G(r)=&-\frac{ab\pi r_1^2}{4}\left(\alpha+\frac{1}{2}\tan\alpha\right)+\frac{abr_1^3}{2}\sin^2\alpha\\
&+\frac{5abr_1^3}{4}\sec\alpha+\frac{abr_1}{4}\beta(r)\ln(\sec\alpha+\tan\alpha)\\
&+\pi abq^2\ln r+\frac{br_1}{6}\eta(r)\cos\alpha\\
&-\frac{abr_1^2}{2}\left((1+r_1)\cos\alpha-\ln\cos\alpha+\alpha\sin\alpha\right)-(bc_2)\alpha\\
&+\frac{\pi abr_1}{4}\sin\alpha+\frac{abr_1^3}{4}(6r_1^2-7)\Omega+c_3,
\end{aligned} \quad (2.35)$$

where

$$\begin{aligned}
\Omega&=\int_0^{\alpha}\alpha\sec\alpha\,d\alpha,\\
\eta(r)&=\left((12c_2-3a)+ar_1\cos\alpha(r_1\cos\alpha-3)\right),\\
\beta(r)&=7r_1^2\alpha-2r_1^2\left(\sin 2\alpha+\tan\alpha-3r_1^2\alpha\right)\\
&\quad-2r_1\left(\ln(\sec\alpha+\tan\alpha)+2\sin\alpha\right)-\pi.
\end{aligned} \quad (2.36)$$

Hence we get the following four algebraic equations:

$$
\begin{gathered}
n_1^{(1)} + \frac{3}{2}T_1^{(1)} - n_2^{(1)} - \frac{3}{2}T_2^{(1)} = F(r)\exp(-t), \\
n_1^{(1)} + \frac{1}{2}T_1^{(1)} - n_2^{(1)} - \frac{1}{2}T_2^{(1)} = 0, \\
n_1^{(1)}T_1^{(1)} - n_2^{(1)}T_2^{(1)} = H(t), \\
(\pi - 2\alpha)\left(n_1^{(1)} + T_1^{(1)}\right) - (\pi + 2\alpha)\left(n_2^{(1)} + T_2^{(1)}\right) = G(r).
\end{gathered}
\tag{2.37}
$$

Solving these equations simultaneously, we obtain

$$
\begin{aligned}
n_1^{(1)} &= \frac{2H(t)}{3F(r)}\exp(t) + \frac{1}{6\pi}G(r) + \frac{2\alpha - 3\pi}{12\pi}F(r)\exp(-t), \\
n_2^{(1)} &= \frac{2H(t)}{3F(r)}\exp(t) + \frac{1}{6\pi}G(r) + \frac{2\alpha + 3\pi}{12\pi}F(r)\exp(-t), \\
T_1^{(1)} &= -\frac{2H(t)}{3F(r)}\exp(t) + \frac{1}{3\pi}G(r) + \frac{2\alpha + 3\pi}{6\pi}F(r)\exp(-t), \\
T_2^{(1)} &= -\frac{2H(t)}{3F(r)}\exp(t) + \frac{1}{3\pi}G(r) + \frac{2\alpha - 3\pi}{6\pi}F(r)\exp(-t).
\end{aligned}
\tag{2.38}
$$

Under the initial and boundary conditions

$$
\begin{aligned}
n_1^{(1)}(q,0) &= 1, \qquad n_2^{(1)}(1,0) = 0, \\
T_1^{(1)}(q,0) &= 1, \qquad T_2^{(1)}(1,0) = 0,
\end{aligned}
\tag{2.39}
$$

we obtain the values of the constants a and b as follows:

$$
a = -\frac{1}{\pi F^*(q)}, \qquad b = -\frac{2(\arccos r_1) + 3\pi}{2G^*(1)}F^*(1),
\tag{2.40}
$$

where

$$
\begin{aligned}
F^*(1) = {}& -\frac{\pi}{8} - \frac{r_1\sqrt{1-r_1^2}}{4} - \frac{r_1^2 \arccos r_1}{2} \\
& + \frac{1}{2}\left(\ln\left(\frac{1+\sqrt{1-r_1^2}}{r_1}\right)\right)\left(1 - r_1^2 - \frac{5r_1^2}{2} + \frac{c_2}{a}\right),
\end{aligned}
\tag{2.41}
$$

$$F^*(q) = -\frac{\pi q^2}{8} - \frac{r_1 q\sqrt{1-r_2^2}(1-3q/2)}{2} - \frac{r_1^2 \arccos r_2}{2} + \frac{r_1}{2}\left(\ln\left(\frac{1+\sqrt{1-r_2^2}}{r_1}\right)\right)\left(q^2 - \frac{7r_1^2}{2} + \frac{c_2}{a}\right), \tag{2.42}$$

$$\begin{aligned} G^*(1) = {}& -\frac{\pi r_1^2}{4}\left(\arccos r_1 + \frac{1}{2}\tan\left(\arccos r_1\right)\right) + \frac{r_1^3}{2}\sin^2\left(\arccos r_1\right) \\ &+ \frac{r_1^2}{2}\ln r_1 - \frac{r_1^2}{2}\arccos r_1 \sin\left(\arccos r_1\right) - \frac{r_1^3}{2} - \frac{r_1^4}{2} - (bc_2)\alpha \\ &+ \zeta\ln\left(\sec\arccos r_1 + \tan\arccos r_1\right) - \frac{7r_1^3}{8}\left(\arccos r_1\right)^2 - \frac{r_1^2}{2} \\ &- \frac{7r_1^3}{32}\left(\arccos r_1\right)^4 - \frac{r_1^3}{2}\left(\tan\arccos r_1\right) + \frac{5r_1^3}{4}\sec\left(\arccos r_1\right) \\ &+ \frac{r_1^6}{6} + \frac{3r_1^5}{4}\arccos^2 r_1 + \frac{3r_1^5}{16}\arccos^4 r_1 - \left(2r_1^2 c_2\right) - \frac{r_1^4}{2} + c_3 \\ &+ \frac{\pi r_1}{4}\sin\left(\arccos r_1\right) - r_1^2\left(\sin\left(\arccos r_1\right)\right), \end{aligned} \tag{2.43}$$

where

$$\begin{aligned} \zeta = {}& \frac{7}{4}r_1^3 \arccos r_1 - \frac{1}{2}r_1^3 \sin\left(2\arccos r_1\right) \\ &- \frac{1}{2}r_1^3 \tan\left(\arccos r_1\right) - \frac{3}{2}r_1^5 \arccos r_1 - \frac{\pi}{4}r_1 \\ &+ \frac{1}{2}r_1^2 \ln\left(\sec\arccos r_1 + \tan\arccos r_1\right) - r_1^2 \sin\left(\arccos r_1\right). \end{aligned} \tag{2.44}$$

Also we evaluate the constants c_1, c_2, and c_3

$$\begin{aligned} &c_1 = a, \qquad c_2 = -\frac{\pi a r_1^2}{8}, \\ &c_3 = -abr_1^2 + \frac{11abr_1^3}{12} + \frac{\pi abr_1^3}{4} - abr_1. \end{aligned} \tag{2.45}$$

3. The nonequilibrium thermodynamic predictions of the problem

In order to study the irreversible thermodynamic properties of the system, we begin with the evaluation of the entropy per unit mass $\bar{s}$. It is written in nondimensional form as

$$\bar{s} = -\int f \ln f \, d\vec{c} = -\left(\int f_1 \ln f_1 \, d\vec{c} + \int f_2 \ln f_2 \, d\vec{c} \right)$$
$$= \frac{3}{4\pi} n - \frac{1}{2\pi} \left(\begin{array}{c} (\pi - 2\alpha) n_1 \ln\left(\dfrac{n_1}{(2\pi R T_1)^{3/2}} \right) \\ +(\pi + 2\alpha) n_2 \ln\left(\dfrac{n_2}{(2\pi R T_2)^{3/2}} \right) \end{array} \right), \tag{3.1}$$

also we get the entropy flux in the radial direction

$$\overline{J_r} = -\int c_r f \ln f \, d\vec{c} = -\left(\int c_r f_1 \ln f_1 \, d\vec{c} + \int c_r f_2 \ln f_2 \, d\vec{c} \right)$$
$$= \frac{r_1}{r} \sqrt{\frac{1}{2\pi}} \left(n_2 T_2^{1/2} \ln\left(\frac{n_2}{(2\pi R T_2)^{3/2}} \right) - n_1 T_1^{1/2} \ln\left(\frac{n_1}{(2\pi R T_1)^{3/2}} \right) \right). \tag{3.2}$$

The law of increase of entropy is written in the local form [4] as

$$\frac{\partial \bar{s}}{\partial t} + \frac{\partial \overline{J_r}}{\partial r} = \sigma, \tag{3.3}$$

where σ is the entropy production, hence

$$\sigma = \frac{\gamma}{4\pi} \left(\frac{4\pi}{F(r)} c_1 \exp(t) - \frac{\alpha}{\pi} F(r) \exp(-t) \right)$$
$$- \frac{1}{2\pi} \left(\frac{2\gamma c_1}{3F(r)} \exp(t)(2\pi + A) - \frac{\gamma}{12\pi} F(r) \exp(-t)(4\alpha + B) \right)$$
$$+ r_1 \sqrt{\frac{1}{2\pi}} \left(\frac{1}{r} \left(\frac{T_2^{1/2}}{n_s T_s^{1/2}} \frac{\partial n_2}{\partial r} \ln\left(\frac{n_2 T_s^{3/2}}{n_s (2\pi R T_2)^{3/2}} \right) + C \right) - \frac{1}{r^2} D \right), \tag{3.4}$$

where

$$A = \frac{3}{2} \left(\frac{(\pi - 2\alpha) n_1 T_2 + (\pi + 2\alpha) n_2 T_1}{T_1 T_2} \right) \frac{T_s}{n_s}$$
$$+ (\pi - 2\alpha) \ln\left(\frac{n_1 T_s^{3/2}}{(2\pi R T_1)^{3/2}} \right) + (\pi + 2\alpha) \ln\left(\frac{n_2 T_s^{3/2}}{(2\pi R T_2)^{3/2} n_s} \right), \tag{3.5}$$

$$B = 3\left(\frac{(2\alpha+3\pi)n_1T_2+(2\alpha-3\pi)n_2T_1}{T_1T_2}\right)\frac{T_s}{n_s}$$
$$+(2\alpha-3\pi)\ln\left(\frac{n_1T_s^{3/2}}{(2\pi RT_1)^{3/2}n_s}\right)+(2\alpha+3\pi)\ln\left(\frac{n_2T_s^{3/2}}{(2\pi RT_2)^{3/2}n_s}\right), \tag{3.6}$$

$$C = \frac{n_2}{2n_sT_2}\frac{\partial T_2}{\partial r}\ln\left(\frac{n_2T_s^{3/2}}{(2\pi RT_2)^{3/2}n_s}\right)$$
$$-\left(\frac{T_1^{1/2}}{n_sT_s^{1/2}}\frac{\partial n_1}{\partial r}+\frac{n_1T_s}{2T_1n_s}\right)\ln\left(\frac{n_1T_s^{3/2}}{(2\pi RT_1)^{3/2}n_s}\right), \tag{3.7}$$

$$D = \frac{n_2T_2^{1/2}}{n_sT_s^{1/2}}\ln\left(\frac{n_2T_s^{3/2}}{(2\pi RT_2)^{3/2}n_s}\right)-\frac{n_1T_1^{1/2}}{n_sT_s^{1/2}}\ln\left(\frac{n_1T_s^{3/2}}{(2\pi RT_1)^{3/2}n_s}\right). \tag{3.8}$$

Following the general theory of irreversible thermodynamics [10], we could estimate the thermodynamic forces corresponding to the parameters n_s and T_s at the boundary:

$$X_1 = \frac{\partial\bar{s}}{\partial n_s} = -\frac{n}{4\pi n_s^2}+\frac{1}{2\pi n_s^2}\left(\begin{array}{c}(\pi-2\alpha)n_1\ln\left(\frac{n_1T_s^{3/2}}{n_s(2\pi RT_1)^{3/2}}\right)\\ +(\pi+2\alpha)n_2\ln\left(\frac{n_2T_s^{3/2}}{n_s(2\pi RT_2)^{3/2}}\right)\end{array}\right), \tag{3.9}$$

$$X_2 = \frac{\partial\bar{s}}{\partial T_s} = \frac{3n\sqrt{T_s}}{4\pi n_s}. \tag{3.10}$$

According to Onsager theorem there are kinetic coefficients that relate the entropy production to the thermodynamic forces via the relationship

$$\sigma = \sum_{i,j=1}^{2}L_{ij}X_iX_j = L_{11}X_1^2+L_{12}X_1X_2+L_{21}X_2X_1+L_{22}X_2^2, \tag{3.11}$$

hence, we get

$$L_{11} = \frac{1}{2}\left(\frac{\partial^2\sigma}{\partial X_1^2}\right)_{X_2} = \left(\frac{4\pi n_s^4}{n(1+n_s-\ln n_s)}\right)^2\frac{\partial^2\sigma}{\partial n_s^2}+2E, \tag{3.12}$$

where

$$E = \left(\frac{4\pi n_s^4}{n(1+n_s-\ln n_s)}\right)\left(\frac{\partial\sigma}{\partial n_s}\right)\left(\frac{4\pi n_s^3(5+3n_s-4\ln n_s)}{n(1+n_s-\ln n_s)^2}\right). \tag{3.13}$$

Similarly, we obtain

$$L_{22} = \frac{1}{2}\left(\frac{\partial^2 \sigma}{\partial X_2^2}\right)_{X_1} = \frac{\partial^2 \sigma}{\partial T_s^2}\left(\frac{8\pi T_s^{1/2}}{3n}\right)^2 + \frac{64\pi^2}{9n^2}\frac{\partial \sigma}{\partial T_s}. \tag{3.14}$$

Also we get the nondiagonal coefficients from the relations

$$L_{12} = \frac{\partial^2 \sigma}{\partial X_1 \partial X_2}, \qquad L_{21} = \frac{\partial^2 \sigma}{\partial X_2 \partial X_1}. \tag{3.15}$$

These kinetic coefficients must satisfy Onsager relation such that the diagonal coefficients must be positive and the following inequality must hold true:

$$L_{ii}L_{jj} \geq \frac{1}{4}\left(L_{ij} + L_{ji}\right)^2. \tag{3.16}$$

The temperature gradient between the two cylinders causes a work done on the gas, which gains energy from the surroundings. According to the first and second laws of thermodynamics

$$dU = dQ + dW = T\,ds - p\,dV, \tag{3.17}$$

where

$$\begin{aligned} ds &= \left(\frac{\partial s}{\partial r}\right)\delta r + \left(\frac{\partial s}{\partial t}\right)\delta t, \quad dV = -\frac{dn}{n^2}, \\ dn &= \left(\frac{\partial n}{\partial r}\right)\delta r + \left(\frac{\partial n}{\partial t}\right)\delta t, \quad \delta r = 1,\ \delta t = 2. \end{aligned} \tag{3.18}$$

4. Discussion

This paper deals theoretically with a problem of actual interest in the field of evaporation and condensation processes. In all calculations and figures we take the ratio $q = 0.25$ and the parameter γ=0.15. Due to the monotone increase with time and monotone decrease with radial distance of the temperature (Figure 4.1), the gas is evaporated from the inner cylinder and in the course of time and radial distance tends to condensate at the outer cylinder, this behaviour agrees with the numerical results in [14, Figure 9-a] and [13, Figure 11-a], also at a constant Knudsen number ~0.6 the temperature behaves in the same manner with radial distance [13, Figure 12]. The reverse process appears clearly in Figure 4.2 for the number density, which in the course of time reaches

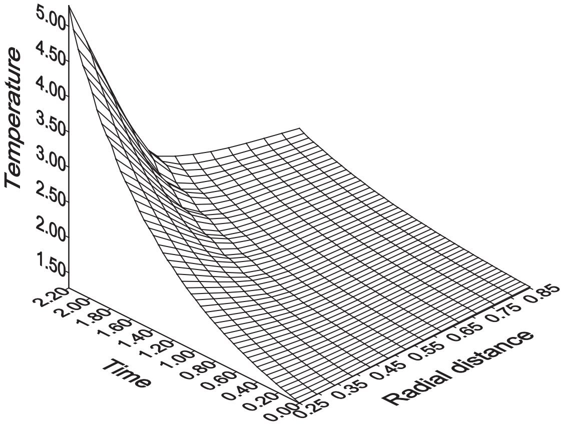

FIGURE 4.1. Variation of the temperature with radial distance and time.

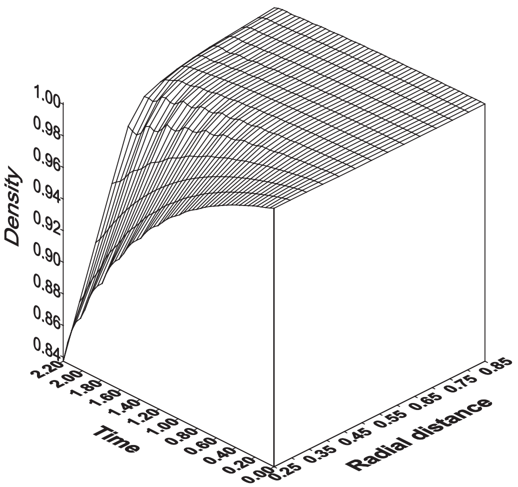

FIGURE 4.2. Variation of the density with radial distance and time.

its maximum and minimum values at the outer and inner walls, respectively, this agrees with the numerical results made by [11] in the same range of Knudsen number. As the temperature decreases from the inner to the outer cylinder, the radial heat flux vector $\bar{q}_r$ behaves similarly. In spite of the fact that $\bar{q}_r$ increases nonlinearly with time in a nonmonetary

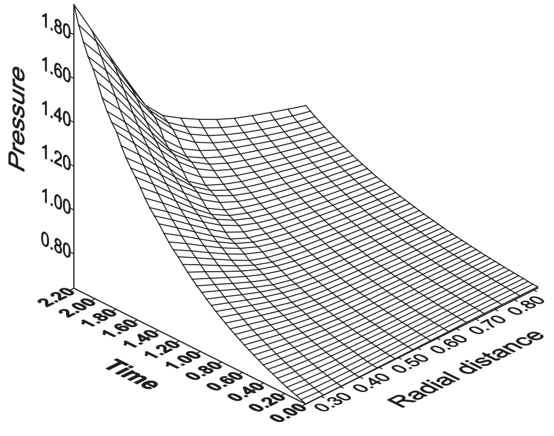

FIGURE 4.3. Variation of the heat flux with radial distance and time.

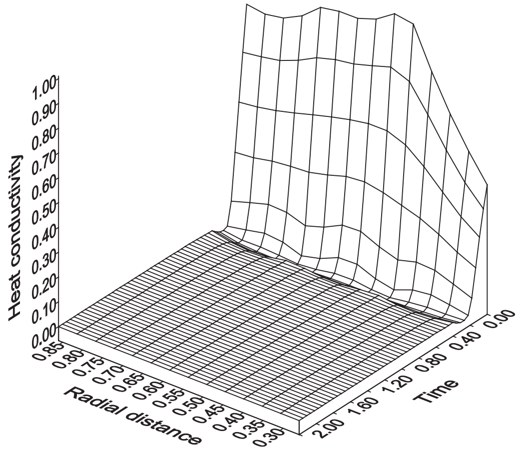

FIGURE 4.4. Variation of the heat conductivity with radial distance and time.

manner (see Figure 4.3), the heat conductivity $\varkappa$ which is derived from (Fourier law) $\bar{q}_r = -\varkappa(\partial T/\partial r)$ is always a positive quantity. In the beginning of the process it takes maximum values along the radial distance and suddenly decreases with time, then it takes nearly constant values between the two cylinders (see Figure 4.4). The pressure behaves similarly like the temperature which is in agreement with the numerical study of [13, Figure 11-a] (see Figure 4.5). We studied the state of

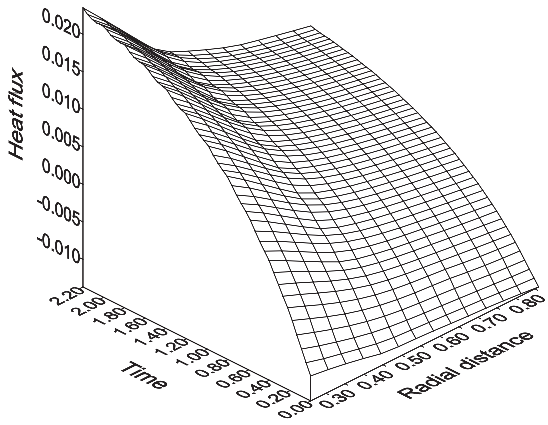

FIGURE 4.5. Variation of the pressure with radial distance and time.

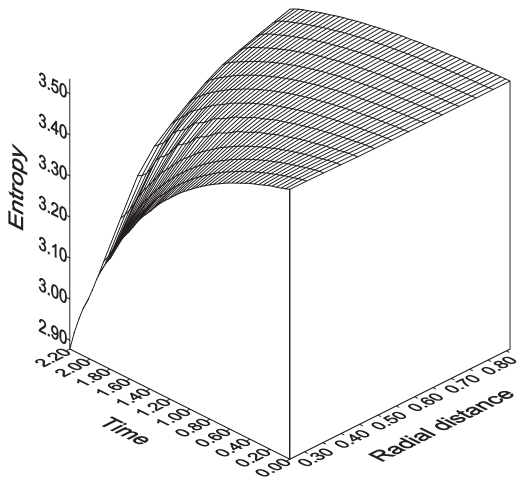

FIGURE 4.6. Variation of the entropy with radial distance and time.

the system from the viewpoint of thermodynamics for irreversible processes. As the system is adiabatic, the temporal rate of the entropy will be positive (see Figure 4.6), consequently there is a source of entropy or entropy production σ which is always a positive value with respect to the radial distance r and time t, but it is an increasing function of time and a decreasing function of radial distance (see Figure 4.7). By Onsager

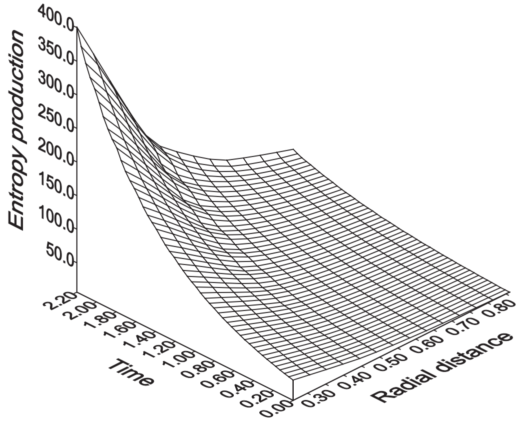

FIGURE 4.7. Variation of the entropy production with radial distance and time.

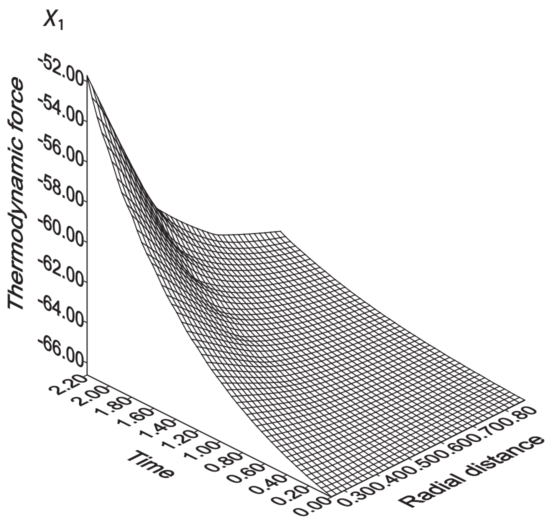

FIGURE 4.8. Variation of the thermodynamic force X_1 with radial distance and time.

relations we determined the thermodynamic forces X_1 and X_2 as functions of r and t, they are opposite to each other, the first one behaves similar to the temperature (see Figure 4.8), and the second behaves similar to the number density (see Figure 4.9). The diagonal coefficients are shown in Figures 4.10 and 4.11, they are positive quantities with respect

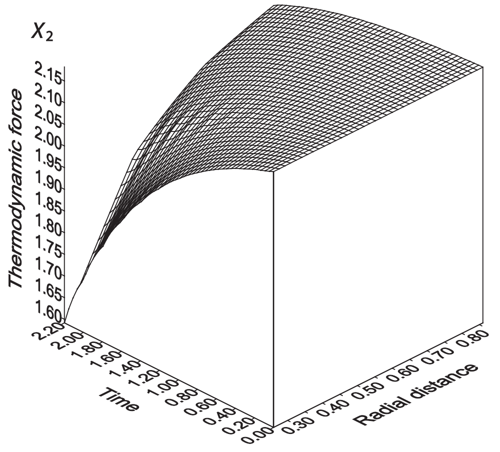

FIGURE 4.9. Variation of the thermodynamic force X_2 with radial distance and time.

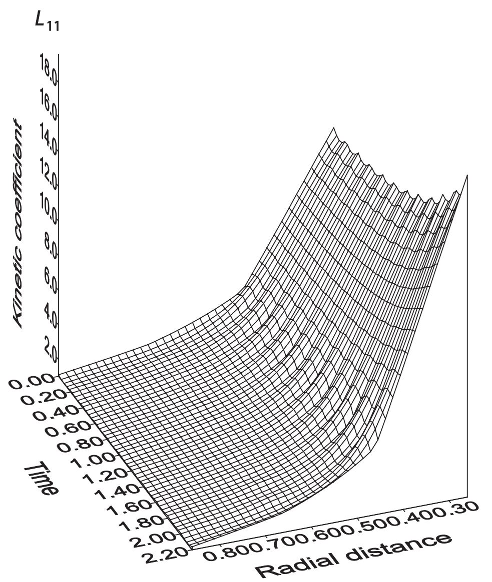

FIGURE 4.10. Variation of the kinetic coefficient L_{11} with radial distance and time.

to r and t. Figure 4.12 shows the validity of inequality (3.16) which is in good agreement with the general rules of irreversible thermodynamics.

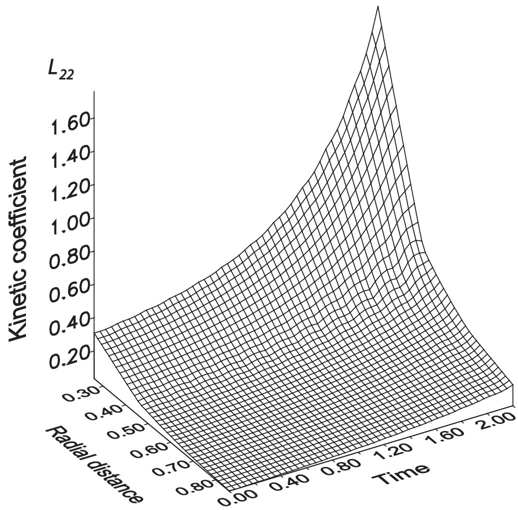

FIGURE 4.11. Variation of the kinetic coefficient L_{22} with radial distance and time.

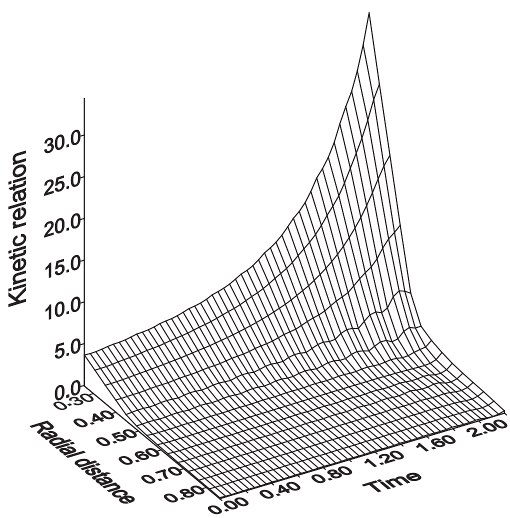

FIGURE 4.12. Variation of the kinetic relation $L_{11}L_{22} - (1/4)(L_{12} + L_{21})^2$ with radial distance and time.

For the monatomic gas, the total energy is conserved. At the inner cylinder, where the temperature is maximum, the atoms gain their

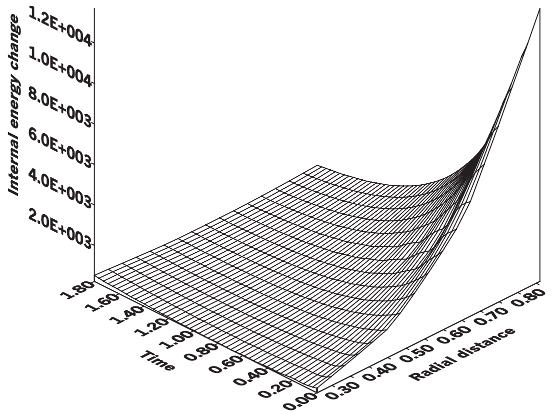

FIGURE 4.13. Variation of the internal energy dU with radial distance and time.

maximum kinetic energy and minimum internal energy. Their potential energy increases to a maximum till they reach the outer cylinder, where the temperature is minimum, see Figure 4.13.

References

[1] M. A. Abdel-Gaid, M. A. Khidr, and F. M. Hady, *Kinetic theory description of flow over a cylinder at low speeds under constant force*, Rev. Roumaine Sci. Tech. Sér. Méc. Appl. **24** (1979), no. 5, 699–706.

[2] V. Galkin, *The cylindrical Couette flow in a rarefied gas*, Inzhenerno-Fizicheskii Zhurnal **5** (1965), no. 3, 12–21 (Russian).

[3] M. Hiroshi and I. Tsunaichi, *Numerical analysis of rarefied cylindrical Couette flows. Rarefied gas dynamics*, Proceedings of the 14th International Symposium on Rarefied Gas-Dynamics (Tsukuba Science City, Japan), vol. 1, 1984, pp. 159–166.

[4] D. Jou, J. Casas-Vázquez, and G. Lebon, *Extended Irreversible Thermodynamics*, Springer-Verlag, Berlin, 1993.

[5] M. A. Khidr and M. A. Abdel-Gaid, *Cylindrical Couette flow with heat transfer of rarefied gas, and porous surface*, Rev. Roumaine Sci. Tech. Sér. Méc. Appl. **25** (1980), no. 4, 549–557.

[6] A. Kraiko, *On the free unsteady expansion of an ideal gas*, Fluid Dynamics **28** (1994), no. 4, 553–559.

[7] M. A. Mahmoud, *Study of some problems in rarefied gases*, Ph.D. thesis, Menoufiya University, Egypt, 1985.

[8] ———, *Steady motion of a rarefied gas between two coaxial circular cylinders*, Canad. J. Phys. **69** (1991), 1429–1436.

[9] B Piero, B. Caloric, and Poolos, *The problem of rotating cylinders in a rarefied gas. Rarefied gas dynamics*, Adv. in Appl. Mech. suppl. 4 **1** (1976), 505–516.

[10] Yu. Rumer and M. S. Ryvkin, *Thermodynamics, Statistical Physics and Kinetics*, MIR Publishers, Moscow, 1980.

[11] E. M. Shakhov, *Numerical solution of the kinetic equation for the evaporation-condensation problem*, Zh. Vychisl. Mat. Mat. Fiz. **38** (1998), no. 6, 1040–1053, translated in Comput. Math. Math. Phys. **38** (1998), no. 6, 994–1006.

[12] V. P. Shidlovskiy, *Introduction to the Dynamics of Rarefied Gases*, American Elsevier Publishing, New York, 1967.

[13] Y. Sone, H. Sugimoto, and K. Aoki, *Cylindrical Couette flows of a rarefied gas with evaporation and condensation: reversal and bifurcation of flows*, Phys. Fluids **11** (1999), no. 2, 476–490.

[14] Y. Sone, S. Takata, and H. Sugimoto, *The behavior of a gas in the continuum limit in the light of kinetic theory: the case of cylindrical Couette flows with evaporation and condensation*, Phys. Fluids **8** (1996), no. 12, 3403–3413.

A. M. Abourabia: Department of Mathematics, Faculty of Science, Menoufiya University, Shebin El-koam 32511, Egypt

E-mail address: am_abourabia@yahoo.com

M. A. Mahmoud: Department of Mathematics, Faculty of Science, Zagazig University, Banha, Egypt

W. S. Abdel Kareem: Department of Mathematics, Faculty of Education, Suez Canal University, Suez, Egypt

Journal of Applied Mathematics and Decision Sciences

Special Issue on Decision Support for Intermodal Transport

Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal's Author Guidelines, which are located at http://www.hindawi.com/journals/jamds/guidelines.html. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/, according to the following timetable:

Manuscript Due	June 1, 2009
First Round of Reviews	September 1, 2009
Publication Date	December 1, 2009

Lead Guest Editor

Gerrit K. Janssens, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; Gerrit.Janssens@uhasselt.be

Guest Editor

Cathy Macharis, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; Cathy.Macharis@vub.ac.be