

*Research Article*

**Invariant Regions and Global Existence of Solutions for Reaction-Diffusion Systems with a Tridiagonal Matrix of Diffusion Coefficients and Nonhomogeneous Boundary Conditions**

Abdelmalek Salem

Received 4 June 2007; Accepted 28 September 2007

Recommended by Bernard Geurts

The purpose of this paper is the construction of invariant regions in which we establish the global existence of solutions for reaction-diffusion systems (three equations) with a tridiagonal matrix of diffusion coefficients and with nonhomogeneous boundary conditions after the work of Kouachi (2004) on the system of reaction diffusion with a full 2-square matrix. Our techniques are based on invariant regions and Lyapunov functional methods. The nonlinear reaction term has been supposed to be of polynomial growth.

Copyright © 2007 Abdelmalek Salem. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**1. Introduction**

We consider the reaction-diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$

$$\frac{\partial v}{\partial t} - c\Delta u - a\Delta v - b\Delta w = g(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$

$$\frac{\partial w}{\partial t} - c\Delta v - a\Delta w = h(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.3}$$

with the boundary conditions

$$\begin{aligned} \lambda u + (1 - \lambda)\partial_\eta u &= \beta_1 & \text{in } \mathbb{R}^+ \times \partial\Omega, \\ \lambda v + (1 - \lambda)\partial_\eta v &= \beta_2 & \text{in } \mathbb{R}^+ \times \partial\Omega, \\ \lambda w + (1 - \lambda)\partial_\eta w &= \beta_3 & \text{in } \mathbb{R}^+ \times \partial\Omega, \end{aligned} \tag{1.4}$$

and the initial data

$$u(0, x) = u_0(x), \quad v(0, x) = v_0(x), \quad w(0, x) = w_0(x) \quad \text{in } \Omega. \tag{1.5}$$

(i) For nonhomogeneous Robin boundary conditions, we use

$$0 < \lambda < 1, \quad \beta_i \in \mathbb{R} \quad i = 1, 2, 3. \tag{1.6}$$

(ii) For homogeneous Neumann boundary conditions, we use

$$\lambda = \beta_i = 0, \quad i = 1, 2, 3. \tag{1.7}$$

(iii) For homogeneous Dirichlet boundary conditions, we use

$$1 - \lambda = \beta_i = 0, \quad i = 1, 2, 3. \tag{1.8}$$

Here,  $\Omega$  is an open bounded domain of class  $C^1$  in  $\mathbb{R}^N$ , with boundary  $\partial\Omega$  and  $\partial/\partial\eta$  denotes the outward normal derivative on  $\partial\Omega$ . The  $a, b$ , and  $c$  are positive constants satisfying the condition  $\sqrt{2a} \geq (b + c)$  which reflects the parabolicity of the system and implies at the same time that the matrix of diffusion

$$A = \begin{pmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{pmatrix} \tag{1.9}$$

is positive definite; that is, the eigenvalues  $\lambda_1, \lambda_2$ , and  $\lambda_3$  ( $\lambda_1 < \lambda_2 < \lambda_3$ ) of its transposed are positive.

The initial data are assumed to be in the following region:

$$\Sigma = \left\{ \begin{array}{l} (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \left\{ \begin{array}{l} \sqrt{2\mu}|v_0| \leq u_0 + \mu w_0, \\ u_0 \leq \mu w_0 \end{array} \right. \text{ if } \left\{ \begin{array}{l} \sqrt{2\mu}|\beta_2| \leq \beta_1 + \mu\beta_3, \\ \beta_1 \leq \mu\beta_3 \end{array} \right. \\ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \left\{ \begin{array}{l} |u_0 + \mu w_0| \leq \sqrt{2\mu}v_0, \\ u_0 \leq \mu w_0 \end{array} \right. \text{ if } \left\{ \begin{array}{l} |\beta_1 + \mu\beta_3| \leq \sqrt{2\mu}\beta_2, \\ \beta_1 \leq \mu\beta_3 \end{array} \right. \\ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \left\{ \begin{array}{l} \sqrt{2\mu}|v_0| \leq u_0 + \mu w_0, \\ \mu w_0 \leq u_0 \end{array} \right. \text{ if } \left\{ \begin{array}{l} \sqrt{2\mu}|\beta_2| \leq \beta_1 + \mu\beta_3, \\ \mu\beta_3 \leq \beta_1 \end{array} \right. \\ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \left\{ \begin{array}{l} |u_0 + \mu w_0| \leq \sqrt{2\mu}v_0, \\ \mu w_0 \leq u_0 \end{array} \right. \text{ if } \left\{ \begin{array}{l} |\beta_1 + \mu\beta_3| \leq \sqrt{2\mu}\beta_2, \\ \mu\beta_3 \leq \beta_1 \end{array} \right. \end{array} \right\}, \tag{1.10}$$

where

$$\mu = \frac{b}{c}. \tag{1.11}$$

One will treat the first case, the others will be discussed in the last section.

We suppose that the reaction terms  $f$ ,  $g$ , and  $h$  are continuously differentiable, polynomially bounded on  $\Sigma$  satisfying

$$f\left(\sqrt{2\mu}v - \mu w, v, w\right) - \sqrt{2\mu}g\left(\sqrt{2\mu}v - \mu w, v, w\right) + \mu h\left(\sqrt{2\mu}v - \mu w, v, w\right) \geq 0, \quad (1.12)$$

$$-f(\mu w, v, w) + \mu h(\mu w, v, w) \geq 0, \quad (1.13)$$

$$f\left(-\sqrt{2\mu}v - \mu w, v, w\right) + \sqrt{2\mu}g\left(-\sqrt{2\mu}v - \mu w, v, w\right) + \mu h\left(-\sqrt{2\mu}v - \mu w, v, w\right) \geq 0 \quad (1.14)$$

for all  $v, w \geq 0$ . And for positive constants  $C'_1 \leq 1/\mu$ ,  $C''_1 \leq \sqrt{2/\mu}$ ,  $\alpha_1 \geq 1/\mu$  and  $\beta_1 \geq \sqrt{2/\mu}$ ,

$$C'_1 f(u, v, w) + C''_1 g(u, v, w) + h(u, v, w) \leq C_1 (\alpha_1 u + \beta_1 v + w + 1); \quad (1.15)$$

for all  $u, v, w$  in  $\Sigma$ , where  $C_1$  is a positive constant.

In the trivial case where  $b = c = 0$ , nonnegative solutions exist globally in time.

This class of systems motivated us to construct this type of functionals considered in this paper in the aim to prove global existence of solutions.

## 2. Existence

In this section, we prove that if  $(f, g, h)$  points into  $\Sigma$  on  $\partial\Sigma$ , then  $\Sigma$  is an invariant region for problem (1.1)–(1.5), that is, the solution remains in  $\Sigma$  for any initial data in  $\Sigma$ . Once the invariant regions are constructed, both problems of the local and global existence become easier to be established.

**2.1. Local existence.** The usual norms in spaces  $L^p(\Omega)$ ,  $L^\infty(\Omega)$ , and  $C(\bar{\Omega})$  are denoted, respectively, by

$$\begin{aligned} \|u\|_p^p &= \frac{1}{|\Omega|} \int_{\Omega} |u(x)|^p dx, \\ \|u\|_\infty &= \max_{x \in \Omega} |u(x)|. \end{aligned} \quad (2.1)$$

For any initial data in  $C(\bar{\Omega})$  or  $L^p(\Omega)$ ,  $p \in (1, +\infty)$  local existence and uniqueness of solutions to the initial value problem (1.1)–(1.5) follow from the basic existence theory for abstract semilinear differential equations (see Friedman [1], Henry [2], and Pazy [3]). The solutions are classical on  $[0; T_{\max} [$ ; where  $T_{\max}$  denotes the eventual blowing-up time in  $L^\infty(\Omega)$ .

**2.2. Invariant regions.** The main result of this subsection is the following.

**PROPOSITION 2.1.** *Suppose that the functions  $f$ ,  $g$ , and  $h$  point into the region  $\Sigma$  on  $\partial\Sigma$ , then for any  $(u_0; v_0, w_0)$  in  $\Sigma$ , the solution  $(u(t; \cdot); v(t; \cdot), w(t; \cdot))$  of the problem (1.1)–(1.5) remains in  $\Sigma$  for any time  $[0; T^* [$ .*

*Proof.* The proof follows the same way to that of Kouachi (see [4]). Then we multiply (1.1) by  $c$ , (1.2) by  $\sqrt{2bc}$ , and (1.3) by  $b$ . Adding the first result to the third one and subtracting the second result, we get (2.2). Subtracting the first result from the third one, we get (2.3). Adding the first, the second, and the third results to each other, we get (2.4):

$$\frac{\partial U}{\partial t} - \lambda_1 \Delta U = F(U, V, W) \quad \text{in } ]0, T^*[\times\Omega, \quad (2.2)$$

$$\frac{\partial V}{\partial t} - \lambda_2 \Delta V = G(U, V, W) \quad \text{in } ]0, T^*[\times\Omega, \quad (2.3)$$

$$\frac{\partial W}{\partial t} - \lambda_3 \Delta W = H(U, V, W) \quad \text{in } ]0, T^*[\times\Omega \quad (2.4)$$

with the boundary conditions

$$\begin{aligned} \lambda U + (1 - \lambda) \partial_\eta U &= \rho_1 & \text{in } ]0, T^*[\times\partial\Omega, \\ \lambda V + (1 - \lambda) \partial_\eta V &= \rho_2 & \text{in } ]0, T^*[\times\partial\Omega, \\ \lambda W + (1 - \lambda) \partial_\eta W &= \rho_3 & \text{in } ]0, T^*[\times\partial\Omega, \end{aligned} \quad (2.5)$$

and the initial data

$$U(0, x) = U_0(x), \quad V(0, x) = V_0(x), \quad W(0, x) = W_0(x) \quad \text{in } \Omega, \quad (2.6)$$

where

$$\begin{aligned} U(t, x) &= cu(t, x) - \sqrt{2bc}v(t, x) + bw(t, x), \\ V(t, x) &= -cu(t, x) + bw(t, x), \\ W(t, x) &= cu(t, x) + \sqrt{2bc}v(t, x) + bw(t, x) \end{aligned} \quad (2.7)$$

for all  $(t, x)$  in  $]0, T^*[\times\Omega$ ,

$$\begin{aligned} F(U, V, W) &= (cf - \sqrt{2bc}g + bh)(u, v, w) \\ G(U, V, W) &= (-cf + bh)(u, v, w) \\ H(U, V, W) &= (cf + \sqrt{2bc}g + bh)(u, v, w) \end{aligned} \quad (2.8)$$

for all  $(u, v, w)$  in  $\Sigma$ ,

$$\begin{aligned} \lambda_1 &= a - \sqrt{2bc}, \\ \lambda_2 &= a, \\ \lambda_3 &= a + \sqrt{2bc}, \\ \rho_1 &= c\beta_1 - \sqrt{2bc}\beta_2 + b\beta_3, \\ \rho_2 &= -c\beta_1 + b\beta_3, \\ \rho_3 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3. \end{aligned} \quad (2.9)$$

First, let us notice that the condition of the parabolicity of the system (1.1)–(1.3) implies the one of the (2.2)–(2.4) system; since  $\sqrt{2a} \geq (b + c) \Rightarrow a > \sqrt{2bc}$ .

Now, it suffices to prove that the region

$$\{(U_0, V_0, W_0) \text{ such that } U_0 \geq 0, V_0 \geq 0, W_0 \geq 0\} = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \quad (2.10)$$

is invariant for system (2.2)–(2.4).

Since, from (1.12)–(1.14), we have that  $F(0, V, W) \geq 0$  suffices to be  $f(\sqrt{2\mu\nu} - \mu w, v, w) - \sqrt{2\mu}g(\sqrt{2\mu\nu} - \mu w, v, w) + \mu h(\sqrt{2\mu\nu} - \mu w, v, w) \geq 0$ , for all  $V, W \geq 0$  and all  $v, w \geq 0$ , then  $U(t, x) \geq 0$ , for all  $(t, x)$  in  $]0, T^*[ \times \Omega$ , thanks to the invariant region method (see Smoller [5]), and Because  $G(U, 0, W) \geq 0$  suffices to be  $-f(\mu w, v, w) + \mu h(\mu w, v, w) \geq 0$  for all  $U, W \geq 0$  and all  $v, w \geq 0$ , then  $V(t, x) \geq 0$  for all  $(t, x)$  in  $]0, T^*[ \times \Omega$ , and  $H(U, V, 0) \geq 0$  suffices to be  $f(-\sqrt{2\mu\nu} - \mu w, v, w) + \sqrt{2\mu}g(-\sqrt{2\mu\nu} - \mu w, v, w) + \mu h(-\sqrt{2\mu\nu} - \mu w, v, w) \geq 0$  for all  $U, V \geq 0$  and all  $v, w \geq 0$ , then  $W(t, x) \geq 0$  for all  $(t, x)$  in  $]0, T^*[ \times \Omega$ , then  $\Sigma$  is an invariant region for the system (1.1)–(1.3).  $\square$

Then system (1.1)–(1.3), with boundary conditions (1.4) and initial data in  $\Sigma$ , is equivalent to system (2.2)–(2.4) with boundary conditions (2.5) and positive initial data (2.6). As it has been mentioned at the beginning of this section and since  $\rho_1, \rho_2$ , and  $\rho_3$  are positive, for any initial data in  $C(\bar{\Omega})$  and  $L^p(\Omega)$ ,  $p \in (1, +\infty)$ . (The local existence and uniqueness of the solutions to the initial value problem (2.2)–(2.6) gives us directly those of (1.1)–(1.5).)

Once invariant regions are constructed, one can apply Lyapunov technique and establish global existence of unique solutions for (1.1)–(1.5).

**2.3. Global existence.** As the determinant of the linear algebraic system (2.7) with regard to variables  $u, v$ , and  $w$ , is different from zero, then to prove global existence of solutions of problem (1.1)–(1.5), one needs to prove it for problem (2.2)–(2.6). To this subject; it is well-known that (see Henry [2]) it suffices to derive an uniform estimate of  $\|F(U, V, W)\|_p, \|G(U, V, W)\|_p$ , and  $\|H(U, V, W)\|_p$  on  $[0, T^*[$  for some  $p > N/2$ .

Let us put  $A_{12} = (a - \sqrt{bc}/2)/\sqrt{a(a - \sqrt{2bc})}$ ,  $A_{13} = a/2\sqrt{a^2 - 2bc}$ ,  $A_{23} = (a + \sqrt{bc}/2)/\sqrt{a(a + \sqrt{2bc})}$ ,  $\theta$ , and  $\sigma$  are as two positive constants sufficiently large such that

$$\theta > A_{12}, \quad (2.11)$$

$$(\theta^2 - A_{12}^2)(\sigma^2 - A_{23}^2) > (A_{13} - A_{12}A_{23})^2. \quad (2.12)$$

Let us define, for any positive integer  $n$ , the two finite sequences

$$\begin{aligned} \theta_q &= \theta^{(p-q)^2}, & q &= 0, \dots, p, \\ \sigma_p &= \sigma^{(n-p)^2}, & p &= 0, \dots, n. \end{aligned} \quad (2.13)$$

The main result of the paper is as follows.

**THEOREM 2.2.** Let  $(U(t, \cdot), V(t, \cdot), W(t, \cdot))$  be any positive solution of (2.2)–(2.6) and assume the functional

$$L(t) = \int_{\Omega} H_n(U(t, x), V(t, x), W(t, x)) dx, \tag{2.14}$$

where

$$H_n(U, V, W) = \sum_{p=0}^n \sum_{q=0}^p C_n^p C_p^q \theta_q \sigma_p U^q V^{p-q} W^{n-p}. \tag{2.15}$$

Then, the functional  $L$  is uniformly bounded on the interval  $[0, T^*]$ ,  $T^* < T_{\max}$ .

**COROLLARY 2.3.** Suppose that the functions  $f, g,$  and  $h$  are continuously differentiable on  $\Sigma$ , point into  $\partial\Sigma$  and satisfy condition (1.15). Then all solutions of (1.1)–(1.5) with initial data in  $\Sigma$  and uniformly bounded on  $\Omega$  are in  $L^\infty(0, T^*; L^p(\Omega))$  for all  $p \geq 1$ .

**PROPOSITION 2.4.** Under the hypothesis of Corollary 2.3, if the reactions  $f, g,$  and  $h$  are polynomially bounded, then all solutions of (1.1)–(1.5) with the initial data in  $\Sigma$  and uniformly bounded on  $\Omega$  are global time.

*Proofs.* For the proof of Theorem 2.2, we need some preparatory lemmas.

**LEMMA 2.5.** Let  $H_n$  be the homogeneous polynomial defined by (2.15). Then

$$\begin{aligned} \partial_U H_n &= n \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q \theta_{(q+1)} \sigma_{(p+1)} U^q V^{p-q} W^{(n-1)-p}, \\ \partial_V H_n &= n \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q \theta_q \sigma_{(p+1)} U^q V^{p-q} W^{(n-1)-p}, \\ \partial_W H_n &= n \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q \theta_q \sigma_p U^q V^{p-q} W^{(n-1)-p}. \end{aligned} \tag{2.16}$$

*Proof of Lemma 2.5.* See Kouachi [6]. □

**LEMMA 2.6.** The second partial derivatives of  $H_n$  are given by

$$\begin{aligned} \partial_{U^2} H_n &= n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^p C_{n-2}^p C_p^q \theta_{(q+2)} \sigma_{(p+2)} U^q V^{p-q} W^{(n-2)-p}, \\ \partial_{UV} H_n &= n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^p C_{n-2}^p C_p^q \theta_{(q+1)} \sigma_{(p+2)} U^q V^{p-q} W^{(n-2)-p}, \\ \partial_{UW} H_n &= n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^p C_{n-2}^p C_p^q \theta_{(q+1)} \sigma_{(p+1)} U^q V^{p-q} W^{(n-2)-p}, \\ \partial_{V^2} H_n &= n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^p C_{n-2}^p C_p^q \theta_q \sigma_{(p+2)} U^q V^{p-q} W^{(n-2)-p}, \end{aligned}$$

$$\begin{aligned}
 \partial_V W H_n &= n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^p C_{n-2}^p C_p^q \theta_q \sigma_{(p+1)} U^q V^{p-q} W^{(n-2)-p}, \\
 \partial_W^2 H_n &= n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^p C_{n-2}^p C_p^q \theta_q \sigma_q U^q V^{p-q} W^{(n-2)-p}.
 \end{aligned} \tag{2.17}$$

*Proof of Lemma 2.6.* See Kouachi [6].  $\square$

*Proof of Theorem 2.2.* Differentiating  $L$  with respect to  $t$  yields

$$\begin{aligned}
 L'(t) &= \int_{\Omega} \left( \partial_U H_n \frac{\partial U}{\partial t} + \partial_V H_n \frac{\partial V}{\partial t} + \partial_W H_n \frac{\partial W}{\partial t} \right) dx \\
 &= \int_{\Omega} (\lambda_1 \partial_U H_n \Delta U + \lambda_2 \partial_V H_n \Delta V + \lambda_3 \partial_W H_n \Delta W) dx \\
 &\quad + \int_{\Omega} (F \partial_U H_n + G \partial_V H_n + H \partial_W H_n) dx = I + J.
 \end{aligned} \tag{2.18}$$

Using Green's formula and applying Lemma 2.5, we get  $I = I_1 + I_2$ , where

$$\begin{aligned}
 I_1 &= \int_{\partial\Omega} (\lambda_1 \partial_U H_n \partial_{\eta} u + \lambda_2 \partial_V H_n \partial_{\eta} v + \lambda_3 \partial_W H_n \partial_{\eta} w) dx, \\
 I_2 &= -n(n-1) \int_{\Omega} \sum_{p=0}^{n-2} \sum_{q=0}^p C_{n-2}^p C_p^q (B_{qp} T) \cdot T dx,
 \end{aligned} \tag{2.19}$$

where

$$B_{qp} = \begin{bmatrix} \lambda_1 \sigma_{(p+2)} \theta_{(q+2)} & \frac{\lambda_1 + \lambda_2}{2} \sigma_{(p+2)} \theta_{(q+1)} & \frac{\lambda_1 + \lambda_3}{2} \sigma_{(p+1)} \theta_{(q+1)} \\ \frac{\lambda_1 + \lambda_2}{2} \sigma_{(p+2)} \theta_{(q+1)} & \lambda_2 \sigma_{(p+2)} \theta_q & \frac{\lambda_2 + \lambda_3}{2} \sigma_{(p+1)} \theta_q \\ \frac{\lambda_1 + \lambda_3}{2} \sigma_{(p+1)} \theta_{(q+1)} & \frac{\lambda_2 + \lambda_3}{2} \sigma_{(p+1)} \theta_q & \lambda_3 \sigma_p \theta_q \end{bmatrix} \tag{2.20}$$

for  $q = \overline{0, p}, p = \overline{0, n-2}$  and  $T = (\nabla U, \nabla V, \nabla W)^t$ .

We prove that there exists a positive constant  $C_2$  independent of  $t \in [0, T_{\max} [$  such that

$$I_1 \leq C_2 \quad \forall t \in [0, T_{\max} [ \tag{2.21}$$

and that

$$I_2 \leq 0 \tag{2.22}$$

for several boundary conditions.

(i) If  $0 < \lambda < 1$ , using the boundary conditions (1.4), we get

$$I_1 = \int_{\partial\Omega} (\lambda_1 \partial_u H_n(\gamma_1 - \alpha U) + \lambda_2 \partial_v H_n(\gamma_2 - \alpha V) + \lambda_3 \partial_w H_n(\gamma_3 - \alpha Z)) dx, \tag{2.23}$$

where  $\alpha = \lambda/(1 - \lambda)$  and  $\gamma_i = \rho_i/(1 - \lambda)$ ,  $i = 1, 2, 3$ .

Since  $K(U, V, W) = \lambda_1 \partial_u H_n(\gamma_1 - \alpha U) + \lambda_2 \partial_v H_n(\gamma_2 - \alpha V) + \lambda_3 \partial_w H_n(\gamma_3 - \alpha W)$ , where  $P_{n-1}$  and  $Q_n$  are polynomials with positive coefficients and respective degrees  $(n - 1)$  and  $n$ . Since the solution is positive, then

$$\limsup_{(|U|+|V|+|W|) \rightarrow +\infty} K(U, V, W) = -\infty \tag{2.24}$$

which proves that  $K$  is uniformly bounded on  $\mathbb{R}_+^3$  and consequently (2.21).

(ii) If  $\lambda = 0$ , then  $I_1 = 0$  on  $[0, T_{\max} [$ .

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on  $[0, T_{\max} [ \times \Omega$  implies that  $\partial_\eta U \leq 0, \partial_\eta V \leq 0$  and  $\partial_\eta W \leq 0$  on  $[0, T_{\max} [ \times \partial\Omega$ . Consequently, one gets again (2.21) with  $C_3 = 0$ .

Now, we prove (2.22). The quadratic forms (with respect to  $\nabla U, \nabla V$ , and  $\nabla W$ ) associated with the matrixes  $B_{qp}$ ,  $q = 0, \dots, p$ , and  $p = 0, \dots, n - 2$  are positive since their main determinants  $\Delta_1, \Delta_2$ , and  $\Delta_3$  are positive, too. To see this, we have

- (1)  $\Delta_1 = \lambda_1 \sigma_{(p+2)} \theta_{(q+2)} > 0$  for  $q = 0, \dots, p$  and  $p = 0, \dots, n - 2$ ;
- (2)

$$\begin{aligned} \Delta_2 &= \begin{vmatrix} \lambda_1 \sigma_{(p+2)} \theta_{(q+2)} & \frac{\lambda_1 + \lambda_2}{2} \sigma_{(p+2)} \theta_{(q+1)} \\ \frac{\lambda_1 + \lambda_2}{2} \sigma_{(p+2)} \theta_{(q+1)} & \lambda_2 \sigma_{(p+2)} \theta_q \end{vmatrix} \\ &= \lambda_1 \lambda_2 \sigma_{(p+2)}^2 \theta_{(q+1)}^2 (\theta^2 - A_{12}^2), \end{aligned} \tag{2.25}$$

for  $q = 0, \dots, p$ , and  $p = 0, \dots, n - 2$ . Using (2.11), we get  $\Delta_2 > 0$ .

(3)  $\Delta_3 = |B_{qp}| = \lambda_1 \lambda_2 \lambda_3 \sigma_{(p+2)}^2 \sigma_{(p+1)}^2 \theta_{(q+1)}^2 \theta_q [(\theta^2 - A_{12}^2)(\sigma^2 - A_{23}^2) - (A_{13} - A_{12} A_{23})^2]$  for  $q = 0, \dots, p$  and  $p = 0, \dots, n - 2$ . Using (2.12), we get  $\Delta_3 > 0$  (see Kouachi [6]).

Substituting the expressions of the partial derivatives given by Lemma 2.5 in the second integral yields

$$\begin{aligned} J &= \int_{\Omega} n \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q [\sigma_{(p+1)} \theta_{(q+1)} F + \sigma_{(p+1)} \theta_q G + \sigma_p \theta_q H] U^q V^{p-q} W^{(n-1)-p} dx \\ &= \int_{\Omega} n \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q U^q V^{p-q} W^{(n-1)-p} \left[ F + \frac{\theta_q}{\theta_{(q+1)}} G + \frac{\sigma_p}{\sigma_{(p+1)}} \frac{\theta_q}{\theta_{(q+1)}} H \right] \sigma_{(p+1)} \theta_{(q+1)} dx. \end{aligned} \tag{2.26}$$



Using the expressions (2.8), we obtain

$$\begin{aligned}
 J &= \int_{\Omega} n \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q U^q V^{p-q} W^{(n-1)-p} \\
 &\times \left( \left( 1 - \frac{\theta_q}{\theta_{(q+1)}} + \frac{\sigma_p}{\sigma_{(p+1)}} \frac{\theta_q}{\theta_{(q+1)}} \right) cf + \left( -1 + \frac{\sigma_p}{\sigma_{(p+1)}} \frac{\theta_q}{\theta_{(q+1)}} \right) \sqrt{2bcg} \right. \\
 &\quad \left. + \left( 1 + \frac{\theta_q}{\theta_{(q+1)}} + \frac{\sigma_p}{\sigma_{(p+1)}} \frac{\theta_q}{\theta_{(q+1)}} \right) bh \right) \sigma_{(p+1)} \theta_{(q+1)} dx. \tag{2.27}
 \end{aligned}$$

And so, we get the following inequality:

$$\begin{aligned}
 J &\leq \int_{\Omega} n \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q U^q V^{p-q} W^{(n-1)-p} \\
 &\times \left( \frac{1 - \theta_q/\theta_{(q+1)} + (\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})}{1 + \theta_q/\theta_{(q+1)} + (\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})} cf \right. \\
 &\quad \left. + \frac{-1 + (\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})}{1 + \theta_q/\theta_{(q+1)} + (\sigma_p/\sigma_{(p+1)})(\theta_q/\theta_{(q+1)})} \sqrt{2bcg} + bh \right) \sigma_{(p+1)} \theta_{(q+1)} dx, \tag{2.28}
 \end{aligned}$$

using condition (1.15) and relation (2.7) successively, we get

$$J \leq C_4 \int_{\Omega} \sum_{p=0}^{n-1} \sum_{q=0}^p C_{n-1}^p C_p^q U^q V^{p-q} W^{(n-1)-p} (U + V + W + 1) dx. \tag{2.29}$$

Following the same reasoning as in Kouachi [6], a straightforward calculation shows that

$$L'(t) \leq C_5 L(t) + C_6 L^{(n-1)/n}(t) \quad \text{on } [0, T^*], \tag{2.30}$$

which for  $Z = L^{1/n}$  can be written as

$$nZ' \leq C_5 Z + C_6. \tag{2.31}$$

A simple integration gives the uniform bound of the functional  $L$  on the interval  $[0, T^*]$ ; this ends the proof of the Theorem 2.2.  $\square$

*Proof of corollary.* The proof of this corollary is an immediate consequence of Theorem 2.2 and the inequality

$$\int_{\Omega} (U + V + W)^p dx \leq C_8 L(t) \quad \text{on } [0, T^* [ \tag{2.32}$$

for some  $p \geq 1$ , taking into consideration expressions (2.7).  $\square$

*Proof of proposition.* As it has been mentioned above; it suffices to derive a uniform estimate of  $\|F(U, V, W)\|_p$ ,  $\|G(U, V, W)\|_p$ , and  $\|H(U, V, W)\|_p$  on  $[0, T^*[$  for some  $p > N/2$ . Since the functions  $f$ ,  $g$ , and  $h$  are polynomially bounded on  $\Sigma$ , then using relations (2.5), (2.7), and (2.8), we get that  $F$ ,  $G$ , and  $H$  are polynomially bounded, too, and the proof becomes an immediate consequence of Corollary 2.3.  $\square$

### 3. Final remarks

The second, the third, and the fourth cases are to be studied in the same way as we have done with the first case.

The second case: If  $\{|\beta_1 + \mu\beta_3| \leq \sqrt{2\mu}\beta_2$  and  $\beta_1 \leq \mu\beta_3\}$ , then system (1.1)–(1.3) can be rewritten as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} - a\Delta u - b\Delta v &= f(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial w}{\partial t} - a\Delta w - c\Delta v &= h(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} - c\Delta u - b\Delta w - a\Delta v &= g(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega \end{aligned} \tag{3.1}$$

with the same boundary conditions (1.4) and initial data (1.5).

In this case, the diffusion matrix of the system becomes

$$A = \begin{pmatrix} a & 0 & b \\ 0 & a & c \\ c & b & a \end{pmatrix}. \tag{3.2}$$

Then all the previous results remain valid in the region

$$\Sigma = \left\{ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that: } |u_0 + \mu w_0| \leq \sqrt{2\mu}v_0 \text{ and } u_0 \leq \mu w_0 \right\}, \tag{3.3}$$

where

$$\mu = \frac{b}{c}. \tag{3.4}$$

And system (2.2)–(2.4) becomes

$$\begin{aligned} \frac{\partial U}{\partial t} - a\Delta U &= F_1(U, V; W), \\ \frac{\partial V}{\partial t} - (a + \sqrt{2}\sqrt{bc})\Delta V &= G_1(U, V; W), \\ \frac{\partial W}{\partial t} - (a - \sqrt{2}\sqrt{bc})\Delta W &= H_1(U, V; W), \end{aligned} \tag{3.5}$$

where

$$\begin{aligned}
 U(t,x) &= -cu(t,x) + bw(t,x), \\
 V(t,x) &= cu(t,x) + \sqrt{2bc}v(t,x) + bw(t,x), \\
 W(t,x) &= -cu(t,x) + \sqrt{2bc}v(t,x) - bw(t,x)
 \end{aligned} \tag{3.6}$$

for all  $(t,x)$  in  $]0, T^*[\times\Omega$ ,

$$\begin{aligned}
 F_1(U, V; W) &= (-cf + bh)(u, v, w), \\
 G_1(U, V; W) &= (cf + \sqrt{2}\sqrt{bc}g + bh)(u, v, w), \\
 H_1(U, V; W) &= (-cf + \sqrt{2}\sqrt{bc}g - bh)(u, v, w)
 \end{aligned} \tag{3.7}$$

for all  $(u, v, w)$  in  $\Sigma$ ; with the boundary conditions

$$\begin{aligned}
 \lambda U + (1 - \lambda)\partial_\eta U &= \rho_1 \quad \text{in } ]0, T^*[\times\partial\Omega, \\
 \lambda V + (1 - \lambda)\partial_\eta V &= \rho_2 \quad \text{in } ]0, T^*[\times\partial\Omega, \\
 \lambda W + (1 - \lambda)\partial_\eta W &= \rho_3 \quad \text{in } ]0, T^*[\times\partial\Omega,
 \end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
 \rho_1 &= -c\beta_1 + b\beta_3, \\
 \rho_2 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3, \\
 \rho_3 &= -c\beta_1 + \sqrt{2bc}\beta_2 - b\beta_3,
 \end{aligned} \tag{3.9}$$

and initial data (1.5).

The conditions (1.12)–(1.15) become, respectively,

$$\begin{aligned}
 -f(\mu w, v, w) + \mu h(\mu w, v, w) &\geq 0, \\
 f(-\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(-\sqrt{2\mu}v - \mu w, v, w) + \mu h(-\sqrt{2\mu}v - \mu w, v, w) &\geq 0, \\
 -f(\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(\sqrt{2\mu}v - \mu w, v, w) - \mu h(\sqrt{2\mu}v - \mu w, v, w) &\geq 0
 \end{aligned} \tag{3.10}$$

for all  $v, w \geq 0$ , and for positive constants  $C'_2 \leq \sqrt{1/2\mu}$ ,  $C'_1 \leq \sqrt{\mu/2}$ ,  $\alpha_2 \geq \sqrt{1/2\mu}$ , and  $\beta_2 \geq \sqrt{\mu/2}$

$$C'_2 f(u, v, w) + g(u, v, w) + C'_1 h(u, v, w) \leq C_1 (\alpha_2 u + v + \beta_2 w + 1); \tag{3.11}$$

for all  $(u, v, w)$  in  $\Sigma$ , where  $C_1$  is a positive constant.

The third case: If  $\{\sqrt{2\mu}|\beta_2| \leq \beta_1 + \mu\beta_3$  and  $\mu\beta_3 \leq \beta_1\}$ , then system (1.1)–(1.3) can be rewritten as follows:

$$\begin{aligned} \frac{\partial v}{\partial t} - a\Delta v - b\Delta w - c\Delta u &= g(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial w}{\partial t} - c\Delta v - a\Delta w &= h(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial u}{\partial t} - b\Delta v - a\Delta u &= f(u, v, w) \quad \text{in } \mathbb{R}^+ \times \Omega, \end{aligned} \tag{3.12}$$

with the same boundary conditions (1.4) and initial data (1.5).

In this case, the diffusion matrix of the system becomes

$$A = \begin{pmatrix} a & b & c \\ c & a & 0 \\ b & 0 & a \end{pmatrix}. \tag{3.13}$$

Then all the previous results remain valid in the region

$$\Sigma = \left\{ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that: } \sqrt{2\mu} |v_0| \leq u_0 + \mu w_0 \text{ and } \mu w_0 \leq u_0 \right\}, \tag{3.14}$$

where

$$\mu = \frac{b}{c}. \tag{3.15}$$

And systems (2.2)–(2.4) become

$$\begin{aligned} \frac{\partial U}{\partial t} - a\Delta U &= F_2(U, V; W), \\ \frac{\partial V}{\partial t} - (a + \sqrt{2}\sqrt{bc})\Delta V &= G_2(U, V; W), \\ \frac{\partial W}{\partial t} - (a - \sqrt{2}\sqrt{bc})\Delta W &= H_2(U, V; W), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} U(t, x) &= cu(t, x) - bw(t, x), \\ V(t, x) &= cu(t, x) + \sqrt{2bc}v(t, x) + bw(t, x), \\ W(t, x) &= cu(t, x) - \sqrt{2bc}v(t, x) + bw(t, x) \end{aligned} \tag{3.17}$$

for all  $(t, x)$  in  $]0, T^*[\times\Omega$ ,

$$\begin{aligned} F_2(U, V; W) &= (cf - bh)(u, v, w), \\ G_2(U, V; W) &= (cf + \sqrt{2}\sqrt{bc}g + bh)(u, v, w), \\ H_2(U, V; W) &= (cf - \sqrt{2}\sqrt{bc}g + bh)(u, v, w) \end{aligned} \tag{3.18}$$

for all  $(u, v, w)$  in  $\Sigma$ ; with the boundary conditions

$$\begin{aligned}\lambda U + (1 - \lambda)\partial_\eta U &= \rho_1 & \text{in } ]0, T^*[\times\partial\Omega, \\ \lambda V + (1 - \lambda)\partial_\eta V &= \rho_2 & \text{in } ]0, T^*[\times\partial\Omega, \\ \lambda W + (1 - \lambda)\partial_\eta W &= \rho_3 & \text{in } ]0, T^*[\times\partial\Omega,\end{aligned}\tag{3.19}$$

where

$$\begin{aligned}\rho_1 &= c\beta_1 - b\beta_3, \\ \rho_2 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3, \\ \rho_3 &= c\beta_1 - \sqrt{2bc}\beta_2 + b\beta_3\end{aligned}\tag{3.20}$$

and the initial data (1.5).

The conditions (1.12)–(1.15) become, respectively,

$$\begin{aligned}f(\mu w, v, w) - \mu h(\mu w, v, w) &\geq 0, \\ f(-\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(-\sqrt{2\mu}v - \mu w, v, w) + \mu h(-\sqrt{2\mu}v - \mu w, v, w) &\geq 0, \\ f(\sqrt{2\mu}v - \mu w, v, w) - \sqrt{2\mu}g(\sqrt{2\mu}v - \mu w, v, w) + \mu h(\sqrt{2\mu}v - \mu w, v, w) &\geq 0\end{aligned}\tag{3.21}$$

for all  $v, w \geq 0$ , and for positive constants  $C'_3 \leq \sqrt{2\mu}$ ,  $C''_1 \leq \mu$ ,  $\alpha_3 \geq \sqrt{2\mu}$ , and  $\beta_3 \geq \mu$ ,

$$f(u, v, w) + C'_3 g(u, v, w) + C''_1 h(u, v, w) \leq C_1(u + \alpha_3 v + \beta_3 w + 1);\tag{3.22}$$

for all  $(u, v, w)$  in  $\Sigma$ , where  $C_1$  is a positive constant.

The fourth case. If  $\{|\beta_1 + \mu\beta_3| \leq \sqrt{2\mu}\beta_2$  and  $\mu\beta_3 \leq \beta_1\}$ , then system (1.1)–(1.3) can be rewritten as follows:

$$\begin{aligned}\frac{\partial w}{\partial t} - a\Delta w - c\Delta v &= h(u, v, w) & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial u}{\partial t} - a\Delta u - b\Delta v &= f(u, v, w) & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} - b\Delta w - c\Delta u - a\Delta v &= g(u, v, w) & \text{in } \mathbb{R}^+ \times \Omega\end{aligned}\tag{3.23}$$

with the same boundary conditions (1.4) and initial data (1.5).

In this case, the diffusion matrix of the system becomes

$$A = \begin{pmatrix} a & 0 & c \\ 0 & a & b \\ b & c & a \end{pmatrix}.\tag{3.24}$$

Then all the previous results remain valid in the region

$$\Sigma = \left\{ (u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that: } |u_0 + \mu w_0| \leq \sqrt{2\mu}v_0 \text{ and } \mu w_0 \leq u_0 \right\}, \quad (3.25)$$

where

$$\mu = \frac{b}{c}. \quad (3.26)$$

And system (2.2)–(2.4) becomes

$$\begin{aligned} \frac{\partial U}{\partial t} - a\Delta U &= F_3(U, V; W), \\ \frac{\partial V}{\partial t} - (a + \sqrt{2}\sqrt{bc})\Delta V &= G_3(U, V; W), \\ \frac{\partial W}{\partial t} - (a - \sqrt{2}\sqrt{bc})\Delta W &= H_3(U, V; W), \end{aligned} \quad (3.27)$$

where

$$\begin{aligned} U(t, x) &= cu(t, x) - bw(t, x), \\ V(t, x) &= cu(t, x) + \sqrt{2bc}v(t, x) + bw(t, x), \\ W(t, x) &= -cu(t, x) + \sqrt{2bc}v(t, x) - bw(t, x) \end{aligned} \quad (3.28)$$

for all  $(t, x)$  in  $]0, T^*[\times\Omega$ ,

$$\begin{aligned} F_3(U, V; W) &= (cf - bh)(u, v, w), \\ G_3(U, V; W) &= (cf + \sqrt{2}\sqrt{bc}g + bh)(u, v, w), \\ H_3(U, V; W) &= (-cf + \sqrt{2}\sqrt{bc}g - bh)(u, v, w) \end{aligned} \quad (3.29)$$

for all  $(u, v, w)$  in  $\Sigma$ ; with the boundary conditions

$$\begin{aligned} \lambda U + (1 - \lambda)\partial_\eta U &= \rho_1 \quad \text{in } ]0, T^*[\times\partial\Omega, \\ \lambda V + (1 - \lambda)\partial_\eta V &= \rho_2 \quad \text{in } ]0, T^*[\times\partial\Omega, \\ \lambda W + (1 - \lambda)\partial_\eta W &= \rho_3 \quad \text{in } ]0, T^*[\times\partial\Omega, \end{aligned} \quad (3.30)$$

where

$$\begin{aligned} \rho_1 &= c\beta_1 - b\beta_3, \\ \rho_2 &= c\beta_1 + \sqrt{2bc}\beta_2 + b\beta_3, \\ \rho_3 &= -c\beta_1 + \sqrt{2bc}\beta_2 - b\beta_3. \end{aligned} \quad (3.31)$$

The conditions (1.12)–(1.15) become, respectively:

$$\begin{aligned}
 & f(\mu w, v, w) - \mu h(\mu w, v, w) \geq 0, \\
 & f(-\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(-\sqrt{2\mu}v - \mu w, v, w) + \mu h(-\sqrt{2\mu}v - \mu w, v, w) \geq 0, \\
 & -f(\sqrt{2\mu}v - \mu w, v, w) + \sqrt{2\mu}g(\sqrt{2\mu}v - \mu w, v, w) - \mu h(\sqrt{2\mu}v - \mu w, v, w) \geq 0
 \end{aligned} \tag{3.32}$$

for all  $v, w \geq 0$ , and for positive constants  $C'_4 \leq \sqrt{1/2\mu}$ ,  $C''_4 \leq \sqrt{\mu/2}$ ,  $\alpha_4 \geq \sqrt{1/2\mu}$ , and  $\beta_4 \geq \sqrt{\mu/2}$ ,

$$C'_4 f(u, v, w) + g(u, v, w) + C''_4 h(u, v, w) \leq C_1(\alpha_4 u + v + \beta_4 w + 1); \tag{3.33}$$

for all  $(u, v, w)$  in  $\Sigma$ , where  $C_1$  is a positive constant.

## References

- [1] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1964.
- [2] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, vol. 840 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1984.
- [3] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.
- [4] S. Kouachi, “Global existence of solutions for reaction-diffusion systems with a full matrix of diffusion coefficients and nonhomogeneous boundary conditions,” *Electronic Journal of Qualitative Theory of Differential Equations*, no. 2, pp. 1–10, 2002.
- [5] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, vol. 258 of *Fundamental Principles of Mathematical Science*, Springer, New York, NY, USA, 1983.
- [6] S. Kouachi, “Existence of global solutions to reaction-diffusion systems with nonhomogeneous boundary conditions via a Lyapunov functional,” *Electronic Journal of Differential Equations*, no. 88, pp. 1–13, 2002.

Abdelmalek Salem: Department of Mathematics and Informatiques, University Center of Tebessa, Tebessa 12002, Algeria

Email address: a.salem@gawab.com

## Special Issue on Decision Support for Intermodal Transport

### Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

Before submission authors should carefully read over the journal's Author Guidelines, which are located at <http://www.hindawi.com/journals/jamds/guidelines.html>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/>, according to the following timetable:

Manuscript Due	June 1, 2009
First Round of Reviews	September 1, 2009
Publication Date	December 1, 2009

### Lead Guest Editor

**Gerrit K. Janssens**, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; [Gerrit.Janssens@uhasselt.be](mailto:Gerrit.Janssens@uhasselt.be)

### Guest Editor

**Cathy Macharis**, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; [Cathy.Macharis@vub.ac.be](mailto:Cathy.Macharis@vub.ac.be)