

# REGULARIZATION METHOD FOR PARABOLIC EQUATION WITH VARIABLE OPERATOR

VALENTINA BURMISTROVA

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Consider the initial boundary value problem for the equation  $u_t = -L(t)u$ ,  $u(1) = w$  on an interval  $[0, 1]$  for  $t > 0$ , where  $w(x)$  is a given function in  $L^2(\Omega)$  and  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ .  $L$  is the unbounded, nonnegative operator in  $L^2(\Omega)$  corresponding to a selfadjoint, elliptic boundary value problem in  $\Omega$  with zero Dirichlet data on  $\partial\Omega$ . The coefficients of  $L$  are assumed to be smooth and dependent of time. It is well known that this problem is ill-posed in the sense that the solution does not depend continuously on the data. We impose a bound on the solution at  $t = 0$  and at the same time allow for some imprecision in the data. Thus we are led to the constrained problem. There is built an approximation solution, found error estimate for the applied method, given preliminary error estimates for the approximate method.

## 1. Introduction

Consider the problem of solving a parabolic partial differential equation with variable operator backwards in time. For convenience we write the equation in the following abstract form

$$\begin{aligned}u_t &= -L(t)u, & 0 \leq t \leq 1, \\u(1) &= w.\end{aligned}\tag{1.1}$$

Here  $w(x)$  is a given function in  $L^2(\Omega)$ , and  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with a smooth boundary  $\partial\Omega$ .  $L$  is the unbounded, nonnegative operator in  $L^2(\Omega)$  corresponding to a selfadjoint, elliptic boundary value problem in  $\Omega$  with zero Dirichlet data on  $\partial\Omega$ . The coefficients of  $L$  are assumed to be smooth and dependent of time.

The system (1.1) is ill-posed because the solution does not depend continuously on the data. We impose a bound on the solution at  $t = 0$  and at the same time allow for some imprecision in the data. Now we are led to the constrained problem.

Get any solution of

$$\begin{aligned} u_t &= -L(t)u, \quad 0 \leq t \leq 1, \\ \|u(1) - w\| &\leq \delta, \\ \|u(0)\| &\leq M, \end{aligned} \tag{1.2}$$

where the norm is the  $L^2(\Omega)$ -norm, and  $\delta$  and  $M$  are given positive constants,  $\delta \ll M$ . Using logarithmic convexity (see [1], [7, page 11]), we have that any two solutions of (1.2),  $u_1$  and  $u_2$ , satisfy

$$\|u_1(t) - u_2(t)\| \leq 2\delta^t M^{1-t}. \tag{1.3}$$

Write down system

$$\begin{aligned} u_t &= -L(t)u, \quad 0 \leq t \leq 1, \\ w &= u(0)e^{-\int_0^1 L(\tau)d\tau} + \psi, \\ \|\psi\| &\leq \delta, \\ \|u(0)\| &\leq M \leq \delta. \end{aligned} \tag{1.4}$$

Thus for  $0 < t \leq 1$  we have continuous dependence on the data.

It is difficult to solve (1.2), because solutions are not unique. There are some methods for approximating solutions of (1.2), which are optimal in the sense that Hölder type error estimates (1.3) can be obtained for them.

We consider a method related to *the regularization method* of Tikhonov [5] and Phillips. This method for parabolic equation with operator  $L$  independent of time is learned in [4]. Now we consider more generalized case: parabolic equation with variable coefficients.

An approximate solution of (1.2) is given by

$$v(t) = \frac{e^{-\int_0^t L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} w, \quad \mu(t) = (\delta/M)(1-t)/t. \tag{1.5}$$

Let  $u$  be any solution of (1.2). Then, for  $0 \leq t \leq 1$ ,

$$\begin{aligned} \|u(t) - v(t)\| &\leq \delta^t M^{1-t}, \\ \|u - v\| &= \left\| u_0 e^{-\int_0^t L(\tau)d\tau} - \frac{e^{-\int_0^t L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} (u_0 e^{-\int_0^1 L(\tau)d\tau}) + \psi \right\| \\ &= \left\| e^{-\int_0^t L(\tau)d\tau} - \frac{e^{-\int_0^t L(\tau)d\tau} - e^{-\int_0^1 L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} \right\| \|u_0\| + \left\| \frac{e^{-\int_0^t L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} \right\| \|\psi\|. \end{aligned} \tag{1.6}$$

We now raise the following question. Can we discretize (1.5) in such a way that for the discrete approximation  $v_a$  we get an error estimate of type (1.6)

$$\|u(t) - v(t)\| \leq C\delta^t M^{1-t} \tag{1.7}$$

for some constant  $C$ ?

The answer to this question will have significance for the possibilities of solving numerically problems in two (or more) space dimensions, with nonrectangular geometry or nonconstant coefficients, since for such problems we must discretize in time and space.

In this paper, we give a partial answer to the above question. We consider approximating the exponential function in (1.5) in a way which corresponds to a time discretization. In Section 3, we show that if  $\exp(-\lambda)$  is approximated well enough for  $0 \leq \lambda \leq \log(M/\delta)$ , we can get error estimates of the form (1.7) with  $C = 2$ .

### 2. The regularization method for parabolic equation with variable coefficients

We show that the estimate (1.6) holds for the regularization method (1.5). The proof is quite simple and we use the same method in connection with discretization of (1.5). We also show that the same error estimate is valid if we use (1.5) in a step-by-step manner.

We assume that  $\delta$  and  $M$  have been chosen so that there exist solutions of (1.2).

**THEOREM 2.1.** *Let  $u(t)$  denote an arbitrary solution of (1.2), and for  $0 \leq t \leq 1$  let  $v(t)$  be defined by (1.5). Then*

$$\|u(t) - v(t)\| \leq \delta^t M^{1-t}. \tag{2.1}$$

*Proof.* The assumption about the existence of solutions of (1.2) is equivalent to there being functions  $u_0$  and  $\psi$  such that

$$\|u_0\| \leq M, \quad \|\psi\| \leq \delta, \quad w = \exp(-L)u_0 + \psi. \tag{2.2}$$

Putting  $u(t) = u_0 e^{-\int_0^t L(\tau) d\tau}$  we get

$$\begin{aligned} \|u - v\| &= \left\| u_0 e^{-\int_0^t L(\tau) d\tau} - \frac{e^{-\int_0^t L(\tau) d\tau}}{e^{-\int_0^1 L(\tau) d\tau} + \mu(t)} (u_0 e^{-\int_0^1 L(\tau) d\tau}) + \psi \right\| \\ &= \left\| e^{-\int_0^t L(\tau) d\tau} - \frac{e^{-\int_0^t L(\tau) d\tau} e^{-\int_0^1 L(\tau) d\tau}}{e^{-\int_0^1 L(\tau) d\tau} + \mu(t)} \right\| \|u_0\| + \left\| \frac{e^{-\int_0^t L(\tau) d\tau}}{e^{-\int_0^1 L(\tau) d\tau} + \mu(t)} \right\| \|\psi\|, \end{aligned} \tag{2.3}$$

where the operator norm is defined such way  $\|A\| = \sup \{\|Au\| : \|u\| = 1\}$ . We now use (2.2) and the fact that  $L$  is selfadjoint and nonnegative to get

$$\|u(t) - v(t)\| \leq \sup_{\lambda \geq 0} A(\lambda)M + \sup_{\lambda \geq 0} B(\lambda)\delta, \tag{2.4}$$

where

$$A(\lambda) = \left| e^{-\int_0^t L(\tau)d\tau} - \frac{e^{-\int_0^t L(\tau)d\tau - \int_0^1 L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} \right|, \tag{2.5}$$

$$B(\lambda) = \frac{e^{-\int_0^t L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)}.$$

We have  $A = \mu(t)e^{-\int_0^t L(\tau)d\tau} / (e^{-\int_0^1 L(\tau)d\tau} + \mu(t)) = \mu(t)B$ .

Let  $p = e^{-\int_0^1 L(\tau)d\tau} / (e^{-\int_0^1 L(\tau)d\tau} + \mu(t))$ ,  $1 - p = \mu(t) / (e^{-\int_0^1 L(\tau)d\tau} + \mu(t))$ . We use the fact from [4]. We have for  $0 \leq p, t \leq 1$  the inequality

$$p^t(1 - p)^{1-t} \leq t^t(1 - t)^{1-t} \tag{2.6}$$

is valid, we obtain

$$B(\lambda) = \frac{e^{-\int_0^t L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} \leq \frac{e^{-t\int_0^1 \chi L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)}$$

$$= \left( \frac{e^{-\int_0^t \chi L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} \right)^t \left( \frac{\mu(t)}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)} \right)^{1-t} \mu^{t-1}(t) \tag{2.7}$$

$$\leq t^t(1 - t)^{1-t} \left( \frac{\delta}{M} \right)^{t-1} \left( \frac{1 - t}{t} \right)^{t-1} = t \left( \frac{M}{\delta} \right)^{1-t},$$

$$A \leq \frac{\delta}{M} \frac{1 - t}{t} t \left( \frac{M}{\delta} \right)^{1-t} = \left( \frac{\delta}{M} \right)^t (1 - t)$$

(look at the definition (1.5) of  $\mu(t)$ ). Therefore we can estimate (2.4)

$$\|u(t) - v(t)\| \leq \left( \frac{\delta}{M} \right)^t (1 - t)M + t \left( \frac{M}{\delta} \right)^{1-t} \delta = \delta^t M^{1-t}. \tag{2.8}$$

□

The numerical of a *forward parabolic* problem is usually computed by a marching procedure, that is, a procedure which is recursive in time. We show that the method (1.5) for the backward problem can be generalized to a recursive formula in such a way that the procedure remains optimal in the above sense. Make a (possibly nonuniform) partitioning of the interval [0,1]

$$0 < t_1 < t_2 < \dots < t_s < 1, \tag{2.9}$$

and let the recursion be

$$v_s = v(t_s), \quad v_{i-1} = \frac{e^{-\int_0^{t_{i-1}} L(\tau)d\tau}}{e^{-\int_0^{t_i} L(\tau)d\tau} + \mu_i} v_i, \quad i = s, s - 1, \dots, 2, \tag{2.10}$$

where  $v(t_s)$  is given by (1.5) and

$$\mu_i = (\delta_i/M)(t_i - t_{i-1})/t_{i-1}, \quad \delta_i = \delta^i M^{1-t_i}. \quad (2.11)$$

**COROLLARY 2.2.** *Let  $u(t)$  denote an arbitrary solution of (1.2), and let  $(v_i)_{i=1}^s$  be defined by (2.10). Then*

$$\|u(t_i) - v_i\| \leq \delta^i M^{1-t_i}. \quad (2.12)$$

*Proof.* The result is obviously true for  $i = s$ . Then assume that it is true for  $i = k$ , and consider

$$u_t = -L(t)u \quad \text{for } 0 < t \leq t_k, \quad \|u(t_k) - v_k\| \leq \delta_k, \quad \|u(0)\| \leq M. \quad (2.13)$$

The recursion formula (2.10) is a straightforward generalization of (1.5) to the interval  $[0, t_k]$ , and, putting  $\tau_k = t_{k-1}/t_k$ , we obtain

$$\|u(t_{k-1}v_{k-1})\| \leq \delta^{\tau_k} M^{1-\tau_k} = \delta^{t_{k-1}} M^{1-t_{k-1}} = \delta_{k-1}. \quad (2.14)$$

□

### 3. Preliminary error estimation

We get now error estimates for

$$v = \frac{e^{-\int_0^t L(\tau)d\tau}}{e^{-\int_0^t L(\tau)d\tau} + \mu(t)} \leq \frac{e^{-t \int_0^1 L(\tau)d\tau}}{e^{-\int_0^1 L(\tau)d\tau} + \mu(t)}. \quad (3.1)$$

Let

$$v_a(t) = \frac{g^t}{g + \mu(t)} w, \quad g \approx e^{-\int_0^1 L(\tau)d\tau}, \quad (3.2)$$

which is (1.5) with the exponential function replaced by an approximation  $f$ , such that  $f(\lambda) \approx e^{-\lambda}$ . In the next section  $f(\lambda)$  will depend on  $N$ , where  $k = 1/N$  is a step length parameter, but here this dependence is suppressed. There we will be dealing explicitly with the class of approximations defined by  $\exp -\lambda \approx (Q(\lambda/N)/P(\lambda/N))^N$  (see [1, p. 54]), but in this section it will be sufficient to distinguish between two subclasses characterized by the following inequalities:

$$e^{-\int_0^t \lambda(\tau)d\tau} \leq g(\lambda) \leq 1, \quad \lambda \geq 0, \quad (3.3)$$

$$0 < g(\lambda) \leq e^{-\int_0^t \lambda(\tau)d\tau}, \quad 0 \leq e^{\int_0^t \lambda(\tau)d\tau} \leq \ln(M/\delta), \quad (3.4)$$

$$0 \leq g(\lambda) \leq 1, \quad e^{\int_0^t \lambda(\tau)d\tau} \geq \ln(M/\delta).$$

First we give an error estimate for approximations satisfying (3.3).

THEOREM 3.1. Let  $u(t)$  denote an arbitrary solution of (1.2), let  $v_a(t)$  be defined by (3.2), and assume that  $f$  satisfies (3.3). If

$$\lambda + \ln g(\lambda) \leq (\delta/M) \frac{1}{t} e^{\lambda t} \quad \text{for } 0 \leq \lambda \leq \ln(M/\delta), \tag{3.5}$$

then

$$\|u(t) - v_a(t)\| \leq (t + \max(1, 2(1-t))) \delta^t M^{1-t}. \tag{3.6}$$

Proof. As in Theorem 2.1 we have

$$\|u(t) - v_a(t)\| \leq \sup_{\lambda \geq 0} A(\lambda)M + \sup_{\lambda \geq 0} B(\lambda)\delta. \tag{3.7}$$

Look at it in details

$$\begin{aligned} \|u(t) - v_a(t)\| &= \left\| u_0 e^{-\int_0^t L(\tau) d\tau} - \frac{g^t}{g + \mu(t)} (e^{-\int_0^1 L(\tau) d\tau} + \psi) \right\| \\ &\leq \left\| e^{-\int_0^t L(\tau) d\tau} - \frac{g^t}{g + \mu(t)} e^{-\int_0^1 L(\tau) d\tau} \right\| \|u_0\| + \left\| \frac{g^t}{g + \mu(t)} \right\| \|\psi\|. \end{aligned} \tag{3.8}$$

Here

$$\begin{aligned} A &= \left\| e^{-\int_0^t \lambda(\tau) d\tau} - \frac{g^t}{g + \mu(t)} e^{-\int_0^1 \lambda(\tau) d\tau} \right\|, \\ B &= \frac{g^t}{g + \mu(t)}. \end{aligned} \tag{3.9}$$

By (2.6) we have

$$B \leq t \left( \frac{M}{\delta} \right)^{1-t}. \tag{3.10}$$

We have then  $A = |A_1 - A_2|$ ,  $A_1 = e^{-\int_0^t \lambda(\tau) d\tau}$ ,  $A_2 = (g^t/(g + \mu(t)))e^{-\int_0^1 \lambda(\tau) d\tau}$ . Let  $\int_0^1 \lambda(\tau) d\tau \geq k$ , then, for  $A_1 \geq A_2$  we have  $A \leq e^{-\int_0^t \lambda(\tau) d\tau} \leq e^{-t \int_0^1 \lambda(\tau) d\tau} \leq e^{-kt}$ ; and for  $A_1 < A_2$  we have  $A \leq (g^t/(g + \mu(t)))e^{-\int_0^1 \lambda(\tau) d\tau} \leq t(M/\delta)^{1-t} e^{-k} = t e^{k(1-t)} e^{-k} = t e^{-kt}$ , therefore

$$A \leq e^{-kt}. \tag{3.11}$$

Consider now case  $\int_0^1 \lambda(\tau) d\tau \leq k$ .

Let at first  $A_1 \geq A_2$ , then  $A = A_1 - A_2$ . We have then

$$\begin{aligned}
 A(g + \mu(t)) &= \mu(t)e^{-\int_0^t \lambda(\tau) d\tau} + ge^{-\int_0^t \lambda(\tau) d\tau} - g^t e^{-\int_0^1 \lambda(\tau) d\tau} \\
 &\leq \mu(t)e^{-t \int_0^1 \lambda(\tau) d\tau} + ge^{-t \int_0^1 \lambda(\tau) d\tau} - g^t e^{-\int_0^1 \lambda(\tau) d\tau} \\
 &= \mu(t)e^{-t \int_0^1 \lambda(\tau) d\tau} + ge^{-t \int_0^1 \lambda(\tau) d\tau} \left( g^{1-t} - e^{-(1-t) \int_0^1 \lambda(\tau) d\tau} \right) \\
 &\leq \mu(t)e^{-t \int_0^1 \lambda(\tau) d\tau} + ge^{-t \int_0^1 \lambda(\tau) d\tau} \left( \ln g + \int_0^1 \lambda(\tau) d\tau \right) (1-t) \\
 &\leq \mu(t)e^{-t \int_0^1 \lambda(\tau) d\tau} + ge^{-t \int_0^1 \lambda(\tau) d\tau} \frac{\delta}{M} \frac{1}{t} e^{-t \int_0^1 \lambda(\tau) d\tau} (1-t) \\
 &= \mu(t)e^{-t \int_0^1 \lambda(\tau) d\tau} + g^t \mu(t) 0 = \mu(t) \left( e^{-t \int_0^1 \lambda(\tau) d\tau} + g^t \right) \leq 2\mu(t)g^t, \\
 A &\leq 2\mu(t) \frac{g^t}{g + \mu(t)} \leq 2 \frac{\delta}{M} \frac{1-t}{t} t \left( \frac{M}{\delta} \right)^{1-t} = 2 \left( \frac{\delta}{M} \right)^t (1-t) = 2(1-t)e^{-kt}.
 \end{aligned} \tag{3.12}$$

Let now  $A_1 \leq A_2$ , then  $A = A_2 - A_1$ . We have then

$$\begin{aligned}
 A(g + \mu(t)) &= g^t e^{-\int_0^1 \lambda(\tau) d\tau} - ge^{-\int_0^t \lambda(\tau) d\tau} \mu(t) e^{-\int_0^t \lambda(\tau) d\tau} \\
 &= g^t e^{-\int_0^t \lambda(\tau) d\tau} \left( e^{-\int_t^1 \lambda(\tau) d\tau} - g^{1-t} \right) - \mu(t) e^{-\int_0^t \lambda(\tau) d\tau} \\
 &= g^t e^{-\int_0^t \lambda(\tau) d\tau} \left( e^{-\lambda(\xi)(1-t)} - g^{1-t} \right) - \mu(t) e^{-\int_0^t \lambda(\tau) d\tau} \\
 &\leq g^t e^{-\int_0^t \lambda(\tau) d\tau} (-\lambda) (-\lambda(\xi) - \ln g) (1-t) - \mu(t) e^{-\int_0^t \lambda(\tau) d\tau}.
 \end{aligned} \tag{3.13}$$

Here if  $-\lambda(\xi) - \ln g < 0$ , then  $A < 0$  and  $A_1 \geq A_2$ , that is, we have earlier observed case.

Let  $-\lambda(\xi) - \ln g \leq (\delta/M)(1/t)e^{\int_0^t \lambda(\tau) d\tau}$ ,  $\int_0^1 \lambda(\tau) d\tau \leq k = \ln(M/\delta)$ ,  $t \leq \xi \leq 1$ . Then

$$\begin{aligned}
 A(g + \mu(t)) &= g^t e^{-\int_0^t \lambda(\tau) d\tau} \leq \frac{\delta}{M} \frac{1}{t} e^{\int_0^t \lambda(\tau) d\tau} (1-t) - \mu(t) e^{-\int_0^t \lambda(\tau) d\tau} \leq g^t \mu(t), \\
 A &\leq \frac{g^t}{g + \mu(t)} \mu(t) = (1-t)e^{-kt}.
 \end{aligned} \tag{3.14}$$

Thus,

$$\begin{aligned}
 A &\leq 2(1-t) \left( \frac{\delta}{M} \right)^t, \\
 \|\mu(t) - v_a(t)\| &\leq 2(1-t)(\delta M)^t M + t(M\delta)^{1-t} \delta = (t + \max(1, 2(1-t))) \delta^t M^{1-t}.
 \end{aligned} \tag{3.15}$$

□

We next give the corresponding theorem for approximations satisfying (3.4).

**THEOREM 3.2.** Let  $u(t)$  denote an arbitrary solution of (1.2), let  $v_a(t)$  be defined by (3.2), and assume that  $f$  satisfies (3.4). If

$$-\int_0^1 \lambda(\tau) d\tau - \ln g(\lambda) \leq \frac{\delta}{M} \frac{\ln 2}{t} e^{\int_0^1 \lambda(\tau) d\tau} \quad \text{for } 0 \leq \lambda \leq \ln \frac{M}{\delta}, \quad (3.16)$$

then

$$\|u(t) - v_a(t)\| \leq (t + \max(1, 2(1-t))) \delta^t M^{1-t}. \quad (3.17)$$

*Remark 3.3.* The assumption (3.4) implies that  $-\lambda - \log f(\lambda)$  is nonnegative.

*Proof.* As in proof of Theorem 3.1 we get

$$\|u(t) - v_a(t)\| \leq \sup_{\lambda \geq 0} A(\lambda) M + \sup_{\lambda \geq 0} B(\lambda) \delta, \quad (3.18)$$

where  $A(\lambda)$  and  $B(\lambda)$  are the same.

$$B \leq t \left( \frac{M}{\delta} \right)^{1-t}. \quad (3.19)$$

If  $\int_0^1 \lambda(\tau) d\tau \geq k$ , then  $A \leq e^{-kt} = (\delta/M)^t$ .

If  $\int_0^1 \lambda(\tau) d\tau \geq k$ , then let at first  $A = A_1 - A_2$ , and so we have

$$\begin{aligned} A(g + \mu(t)) &= \mu(t) e^{-\int_0^t \lambda(\tau) d\tau} + g e^{-\int_0^t \lambda(\tau) d\tau} - g^t e^{-\int_0^t \lambda(\tau) d\tau} \\ &\leq \mu(t) e^{-t \int_0^1 \lambda(\tau) d\tau} + g^t e^{-t \int_0^1 \lambda(\tau) d\tau} \left( g^{1-t} - e^{-(1-t) \int_0^1 \lambda(\tau) d\tau} \right) \\ &\leq \mu(t) e^{-t \int_0^1 \lambda(\tau) d\tau}. \end{aligned} \quad (3.20)$$

As we have

$$0 \leq \int_0^1 \lambda(\tau) d\tau \leq \ln \frac{M}{\delta} = k, \quad (3.21)$$

so  $e^{\int_0^1 \lambda(\tau) d\tau} (\delta/M) \leq 1$ , and accounting condition of theorem

$$-\int_0^1 \lambda(\tau) d\tau - \ln g(\lambda) \leq \frac{\delta}{M} \frac{\ln 2}{t} e^{\int_0^1 \lambda(\tau) d\tau} \quad (3.22)$$

for  $\int_0^1 \lambda(\tau) d\tau \leq k$  we have  $-\int_0^1 \lambda(\tau) d\tau - \ln g(\lambda) \leq \ln 2/t$ , therefore

$$-t \int_0^1 \lambda(\tau) d\tau - t \ln g(\lambda) \leq \ln 2, \quad -t \int_0^1 \lambda(\tau) d\tau \leq \ln 2 + t \ln g, \quad e^{-t \int_0^1 \lambda(\tau) d\tau} \leq 2g^t, \quad (3.23)$$



and, hence,

$$A(g + \mu(t)) \leq \mu(t) \cdot 2g^t, \quad A \leq 2\mu(t) \frac{g^t}{g + \mu(t)} \leq 2(1-t) \left(\frac{\delta}{m}\right)^t. \quad (3.24)$$

Let now  $A = A_2 - A_1$ . Then

$$\begin{aligned} A(g + \mu(t)) &= g^t e^{-\int_0^1 \lambda(\tau) d\tau} - g e^{-\int_0^1 \lambda(\tau) d\tau} - \mu(t) e^{-\int_0^1 \lambda(\tau) d\tau} \\ &= g^t e^{-\int_0^1 \lambda(\tau) d\tau} (e^{-\int_t^1 \lambda(\tau) d\tau} - g^{1-t}) \\ &\quad - \mu(t) e^{-\int_0^1 \lambda(\tau) d\tau} g^t e^{-\int_0^1 \lambda(\tau) d\tau} (e^{-\int_t^1 \lambda(\xi) d\xi} - g^{1-t}) - \mu(t) e^{-\int_0^1 \lambda(\tau) d\tau} \\ &\leq g^t e^{-\int_0^1 \lambda(\tau) d\tau} (-\lambda(\xi) - \ln g) - \mu(t) e^{-\int_0^1 \lambda(\tau) d\tau} \\ &= g^t e^{-\int_0^1 \lambda(\tau) d\tau} \frac{\delta}{M} \frac{1}{t} e^{\int_0^1 \lambda(\tau) d\tau} (1-t) - \mu(t) e^{-\int_0^1 \lambda(\tau) d\tau} \\ &= \mu(t) (g^t - e^{-\int_0^1 \lambda(\tau) d\tau}) \leq \mu t g^t, \end{aligned} \quad (3.25)$$

then  $A \leq \mu(t) (g^t / (g + \mu(t))) \leq (1-t) (\delta/M)^t$ .

So, we have  $A \leq 2(1-t) (\delta/M)^t$  and

$$\|u - v_a\| \leq 2(1-t) \left(\frac{\delta}{M}\right)^t M + t \left(\frac{M}{\delta}\right)^{1-t} \delta = (t + \max 1, 2(1-t)) \delta^t M^{1-t}. \quad (3.26)$$

□

Thus we obtain the same error estimate as for case with operator independent of time [4]. It means that the method can be applied for more wide field of problems.

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Valentina Burmistrova: International Research and Exchanges Board (IREX), 48 A Gerogoly Street, 744000 Ashgabat, Turkmenistan

*E-mail addresses:* valeb@mail.ru; valeb@iatp.edu.tm

## Special Issue on Decision Support for Intermodal Transport

### Call for Papers

Intermodal transport refers to the movement of goods in a single loading unit which uses successive various modes of transport (road, rail, water) without handling the goods during mode transfers. Intermodal transport has become an important policy issue, mainly because it is considered to be one of the means to lower the congestion caused by single-mode road transport and to be more environmentally friendly than the single-mode road transport. Both considerations have been followed by an increase in attention toward intermodal freight transportation research.

Various intermodal freight transport decision problems are in demand of mathematical models of supporting them. As the intermodal transport system is more complex than a single-mode system, this fact offers interesting and challenging opportunities to modelers in applied mathematics. This special issue aims to fill in some gaps in the research agenda of decision-making in intermodal transport.

The mathematical models may be of the optimization type or of the evaluation type to gain an insight in intermodal operations. The mathematical models aim to support decisions on the strategic, tactical, and operational levels. The decision-makers belong to the various players in the intermodal transport world, namely, drayage operators, terminal operators, network operators, or intermodal operators.

Topics of relevance to this type of decision-making both in time horizon as in terms of operators are:

- Intermodal terminal design
- Infrastructure network configuration
- Location of terminals
- Cooperation between drayage companies
- Allocation of shippers/receivers to a terminal
- Pricing strategies
- Capacity levels of equipment and labour
- Operational routines and lay-out structure
- Redistribution of load units, railcars, barges, and so forth
- Scheduling of trips or jobs
- Allocation of capacity to jobs
- Loading orders
- Selection of routing and service

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### Lead Guest Editor

**Gerrit K. Janssens**, Transportation Research Institute (IMOB), Hasselt University, Agoralaan, Building D, 3590 Diepenbeek (Hasselt), Belgium; [Gerrit.Janssens@uhasselt.be](mailto:Gerrit.Janssens@uhasselt.be)

### Guest Editor

**Cathy Macharis**, Department of Mathematics, Operational Research, Statistics and Information for Systems (MOSI), Transport and Logistics Research Group, Management School, Vrije Universiteit Brussel, Pleinlaan 2, 1050 Brussel, Belgium; [Cathy.Macharis@vub.ac.be](mailto:Cathy.Macharis@vub.ac.be)