

# GLOBAL SOLUTIONS OF A STRONGLY COUPLED REACTION-DIFFUSION SYSTEM WITH DIFFERENT DIFFUSION COEFFICIENTS

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We deal with a mathematical model for a four-component chemical reaction-diffusion process. The model is described by a system of strongly coupled reaction-diffusion equations with different diffusion rates. The existence of the global solution of this reaction-diffusion system in unbounded domain is proved by using semigroup theory and estimates on the growth of solutions.

## 1. Introduction

In this paper, we prove the existence of a global solution in an unbounded domain of the reaction-diffusion system

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= a_1 \Delta u_1 - f_1(u_1, u_2, v_1, v_2), & x \in \mathbb{R}^n, t > 0, \\ \frac{\partial u_2}{\partial t} &= a_2 \Delta u_2 - f_2(u_1, u_2, v_1, v_2), & x \in \mathbb{R}^n, t > 0, \\ \frac{\partial v_1}{\partial t} &= b_1 \Delta u_1 + d_1 \Delta v_1 + f_3(u_1, u_2, v_1, v_2), & x \in \mathbb{R}^n, t > 0, \\ \frac{\partial v_2}{\partial t} &= b_2 \Delta u_2 + d_2 \Delta v_2 + f_4(u_1, u_2, v_1, v_2), & x \in \mathbb{R}^n, t > 0,\end{aligned}\tag{RDS1}$$

with the initial conditions

$$u_i(x, 0) = u_i^0(x), \quad v_i(x, 0) = v_i^0(x) \quad (i = 1, 2), \quad x \in \mathbb{R}^n.\tag{IC1}$$

Here  $f_1(u_1, u_2, v_1, v_2) = m[k_2 u_1^m v_2^r - k_1 u_2^r v_1^m]$ ,  $f_2(u_1, u_2, v_1, v_2) = -r[k_2 u_1^m v_2^r - k_1 u_2^r v_1^m]$ ,  $k_1 \geq 0, k_2 \geq 0, m \geq 0, r \geq 1, f_3 = \rho f_1$ , and  $f_4 = \rho f_2, \rho > 0$ .

The constants  $a_i, b_i$  ( $i = 1, 2$ ) are such that  $a_i > 0, b_i \neq 0$  ( $i = 1, 2$ ), and  $4a_i d_i > b_i^2$  ( $i = 1, 2$ ) which reflects the parabolicity of the system.  $\Delta$  is the Laplace operator in  $\mathbb{R}^n$ . Moreover we assume that the functions  $u_i^0$  ( $i = 1, 2$ ) and  $v_i^0$  ( $i = 1, 2$ ) are uniformly bounded, continuous, and nonnegative. This reaction-diffusion system is a mathematical model for

a chemical reaction of the form



$u_1, u_2, v_1,$  and  $v_2$  represent the concentrations of  $\bar{A}, \bar{B}, A,$  and  $B,$  respectively (see [3]).

We remark that the system

$$\begin{aligned} \frac{\partial u}{\partial t} &= a\Delta u - uh(v), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial v}{\partial t} &= b\Delta u + d\Delta v + uh(v), \quad x \in \Omega, \quad t > 0, \end{aligned} \quad (\text{RDS2})$$

with the initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega \quad (\text{IC2})$$

on a bounded domain  $\Omega \subset \mathbb{R}^n$  with Neumann boundary conditions,  $b > 0, a \neq d, v_0 \geq (b/(a-d))u_0 \geq 0,$  and  $h(s),$  a differentiable nonnegative function on  $\mathbb{R},$  has been studied by Kirane [4]. The existence of global solutions for system (RDS2) on unbounded domains has been studied by Badraoui [1]. The existence of global solutions in  $\mathbb{R}^n$  for (RDS2) with  $h(s) = v^m$  has been studied by Collet and Xin [2].

The quasilinear system of reaction-diffusion equations

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot (a(u)\nabla u) - uh(u)v, \quad x \in \Omega, \quad t > 0, \\ \frac{\partial v}{\partial t} &= \nabla \cdot (b(v)\nabla v) + uh(u)v - \lambda v, \quad x \in \Omega, \quad t > 0, \end{aligned} \quad (\text{RDS3})$$

with the initial conditions

$$u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \quad (\text{IC3})$$

and with Neumann or Dirichlet boundary conditions, is studied by Kirane [5] where in particular the existence of a globally bounded solution is shown. Also he has discussed large time behavior of the solution.

Our aim is to investigate the existence of a global solution for system (RDS1)–(IC1) in an unbounded domain.

Throughout this paper the following notations are used.

- (1)  $\|\cdot\|$  is the supremum norm on  $\mathbb{R}^n,$  that is,  $\|u\| = \sup_{x \in \mathbb{R}^n} |u(x)|.$
- (2)  $C_{\text{ub}}(\mathbb{R}^n)$  is the space of uniformly bounded continuous functions on  $\mathbb{R}^n$  equipped with the supnorm.
- (3) For any  $f \in C_{\text{ub}}(\mathbb{R}^n), \int f = \int_{\mathbb{R}^n} f(x)dx$  if the integral exists.
- (4) For  $f \in L^p(\mathbb{R}^n)$  ( $p \geq 1$ ),  $\|f\|_p = (\int |f|^p)^{1/p}.$

## 2. Existence of a local solution

We convert system (RDS1)–(IC1) to an abstract first-order system in the Banach space  $X = (C_{\text{ub}}(\mathbb{R}^n))^4$  of the form

$$\begin{aligned} \frac{d}{dt}(u(t)) &= Au(t) + F(u(t)), \quad t > 0, \\ u(0) &= u_0 \in X, \end{aligned} \quad (2.1)$$

where  $u(t) = (u_1(t), u_2(t), v_1(t), v_2(t))^T$ . The operator  $A$  is defined as

$$A = \begin{pmatrix} a_1\Delta & 0 & 0 & 0 \\ 0 & a_2\Delta & 0 & 0 \\ b_1\Delta & 0 & d_1\Delta & 0 \\ 0 & b_2\Delta & 0 & d_2\Delta \end{pmatrix} \quad (2.2)$$

with domain  $D(A) = \{u = (u_1, u_2, v_1, v_2)^T \in X, (\Delta u_1, \Delta u_2, \Delta v_1, \Delta v_2)^T \in X\}$ .

Moreover the function  $F$  is defined as

$$F(u(t)) = (-f_1(u(t)), -f_2(u(t)), f_3(u(t)), f_4(u(t)))^T, \quad (2.3)$$

where

$$f_i(u(t)) = f_i(u_1(t), u_2(t), v_1(t), v_2(t)), \quad i = 1, 2, 3, 4. \quad (2.4)$$

Note that for  $\lambda > 0$  the operator  $\lambda\Delta$  generates an analytic semigroup  $G(t)$  in the space  $X$  given by

$$(G(t)u)(x) = \frac{1}{(4\pi\lambda t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{(x-y)^2}{4\lambda t}\right) u(y) dy, \quad t > 0, x \in \mathbb{R}^n. \quad (2.5)$$

Let  $S_1(t)$ ,  $S_2(t)$ ,  $S_3(t)$ , and  $S_4(t)$  be the semigroups generated by  $a_1\Delta$ ,  $a_2\Delta$ ,  $d_1\Delta$ , and  $d_2\Delta$ , respectively. Then one can show that  $A$  generates an analytic semigroup  $S(t)$  given by

$$S(t) = \begin{pmatrix} S_1(t) & 0 & 0 & 0 \\ 0 & S_2(t) & 0 & 0 \\ S_5(t) & 0 & S_3(t) & 0 \\ 0 & S_6(t) & 0 & S_4(t) \end{pmatrix}, \quad (2.6)$$

where

$$S_5(t) = \frac{b_1}{a_1 - d_1} (S_1(t) - S_3(t)), \quad (2.7)$$

$$S_6(t) = \frac{b_2}{a_2 - d_2} (S_2(t) - S_4(t)). \quad (2.8)$$

Assume that  $F$  is locally Lipschitz in  $u$  in the space  $X$ . Then there exist classical solutions on maximal existence interval  $[0, T_0]$  (see [6]).

### 3. Existence of global solutions

For proving the existence of a global solution we assume that the solutions are nonnegative.

**THEOREM 3.1.** *Consider the reaction-diffusion system (RDS1) with nonnegative initial conditions  $(u_1^0(x), u_2^0(x), v_1^0(x), v_2^0(x)) \in (C_{\text{ub}}(\mathbb{R}^n))^4$ ,  $a_i > 0$ ,  $d_i > a_i$ , and  $b_i < 0$ . Then there exist global in-time classical solutions such that*

$$(u_1, u_2, v_1, v_2) \in (C([0, \infty); C_{\text{ub}}(\mathbb{R}^n)) \cap C^1((0, \infty); C_{\text{ub}}(\mathbb{R}^n)))^4. \quad (3.1)$$

**LEMMA 3.2.** *Let  $(u_1, u_2, v_1, v_2)$  be a classical solution of (RDS1). Define the functionals*

$$F^i(u_i, v_i) = (\alpha_i + u_i + u_i^2)e^{\epsilon_i v_i} \quad (i = 1, 2) \text{ with } \epsilon_i > 0, \alpha_i > 0. \quad (3.2)$$

*Then for any smooth nonnegative function  $\psi = \psi(x, t)$  ( $x \in \mathbb{R}^n$ ) with exponential spatial decay at infinity,*

$$\begin{aligned} \frac{d}{dt} \int \psi F^i &= \int (\psi_t + d_i \Delta \psi) F^i + \int ((d_i - a_i) F_1^i - b_i F_2^i) \nabla \psi \nabla u_i \\ &\quad - \int \psi [(a_i F_{11}^i + b_i F_{12}^i) |\nabla u_i|^2 + ((a_i + d_i) F_{12}^i + b_i F_{22}^i) \nabla u_i \nabla v_i \\ &\quad + d_i F_{22}^i (\nabla v_i)^2] + \int \psi (\rho F_2^i f_i - F_1^i f_i), \quad i = 1, 2, \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} F_1^i &= \frac{\partial F^i}{\partial u_i}, & F_2^i &= \frac{\partial F^i}{\partial v_i}, & F_{11}^i &= \frac{\partial^2 F^i}{\partial u_i^2}, & F_{22}^i &= \frac{\partial^2 F^i}{\partial v_i^2}, \\ & & & & F_{12}^i &= \frac{\partial^2 F^i}{\partial u_i \partial v_i}, & & & i = 1, 2. \end{aligned} \quad (3.4)$$

*Proof.* For  $i = 1, 2$ , we have

$$\begin{aligned}
\frac{d}{dt} \int \psi F^i &= \int \psi_t F^i + \int \psi \left( \frac{\partial F^i}{\partial u_i} \frac{\partial u_i}{\partial t} + \frac{\partial F^i}{\partial v_i} \frac{\partial v_i}{\partial t} \right) \\
&= \int \psi_t F^i + \int \psi \left( F_1^i \frac{\partial u_i}{\partial t} + F_2^i \frac{\partial v_i}{\partial t} \right) \\
&= \int \psi_t F^i + \int \psi (F_1^i (a_i \Delta u_i - f_i) + F_2^i (b_i \Delta u_i + d_i \Delta v_i + \rho f_i)) \\
&= \int \psi_t F^i + a_i \int \psi F_1^i \Delta u_i + b_i \int \psi F_2^i \Delta u_i + d_i \int \psi F_2^i \Delta v_i \\
&\quad - \int \psi F_1^i f_i + \rho \int \psi F_2^i f_i.
\end{aligned} \tag{3.5}$$

However,

$$\begin{aligned}
\int \psi F_1^i \Delta u_i &= \int F_1^i \psi \Delta u_i \\
&= - \int \nabla (F_1^i \psi) \nabla u_i \\
&= - \int (F_1^i \nabla \psi + \psi \nabla F_1^i) \nabla u_i \\
&= - \int F_1^i \nabla \psi \nabla u_i - \int \psi \nabla F_1^i \nabla u_i \\
&= - \int F_1^i \nabla \psi \nabla u_i - \int \psi (F_{11}^i \nabla u_i + F_{12}^i \nabla v_i) \nabla u_i,
\end{aligned} \tag{3.6}$$

that is,

$$\int \psi F_1^i \Delta u_i = - \int F_1^i \nabla \psi \nabla u_i - \int \psi F_{11}^i |\nabla u_i|^2 - \int \psi F_{12}^i \nabla u_i \nabla v_i. \tag{3.7}$$

Similarly

$$\begin{aligned}
\int \psi F_2^i \Delta u_i &= - \int F_2^i \nabla \psi \nabla v_i - \int \psi F_{12}^i |\nabla u_i|^2 - \int \psi F_{22}^i \nabla u_i \nabla v_i, \\
\int \psi F_2^i \Delta v_i &= - \int F_2^i \nabla \psi \nabla v_i - \int \psi F_{22}^i |\nabla v_i|^2 - \int \psi F_{12}^i \nabla u_i \nabla v_i.
\end{aligned} \tag{3.8}$$

Also

$$\int F^i \Delta \psi = - \int \nabla F^i \nabla \psi = - \int F_1^i \nabla u_i \nabla \psi - \int F_2^i \nabla v_i \nabla \psi. \tag{3.9}$$

Using (3.5)–(3.9) we get

$$\begin{aligned}
\frac{d}{dt} \int \psi F^i &= \int (\psi_t + d_i \Delta \psi) F^i - a_i \int F_1^i \nabla \psi \nabla u_i - a_i \int \psi F_{11}^i |\nabla u_i|^2 \\
&\quad - a_i \int \psi F_{12}^i \nabla u_i \nabla v_i - b_i \int F_2^i \nabla \psi \nabla u_i - b_i \int \psi F_{12}^i |\nabla u_i|^2 \\
&\quad - b_i \int \psi F_{22}^i \nabla u_i \nabla v_i + d_i \int F_1^i \nabla u_i \nabla \psi - d \int \psi F_{22}^i |\nabla v_i|^2 \\
&\quad - d_i \int \psi F_{12}^i \nabla v_i \cdot \nabla v_i - \int \psi F_1^i f_i + \rho \int \psi F_2^i f_i \\
&= \int (\psi_t + d_i \Delta \psi) F^i + \int (d_i F_1^i - a_i F_1^i - b_i F_2^i) \nabla \psi \nabla u_i \\
&\quad - \int (a_i F_{11}^i + b_i F_{12}^i) |\nabla u_i|^2 - \int \psi (a_i F_{12}^i + b F_{22}^i + d_i F_{12}^i) \nabla u_i \cdot \nabla v_i \\
&\quad - d_i \int \psi F_{22}^i |\nabla v_i|^2 - \int \psi F_1^i f_i + \rho \int \psi F_2^i f_i \\
&= \int (\psi_t + d_i \nabla \psi) F^i + \int ((d_i - a_i) F_1^i - b_i F_2^i) \nabla \psi \nabla u_i \\
&\quad - \int \psi [(a_i F_{11}^i + b_i F_{12}^i) |\nabla u_i|^2 + ((a_i + d_i) F_{12}^i + b_i F_{22}^i) \nabla u_i \nabla v_i \\
&\quad \quad + d_i F_{22}^i |\nabla v_i|^2] + \int \psi (\rho F_2^i f_i - F_1^i f_i). \quad \square
\end{aligned} \tag{3.10}$$

LEMMA 3.3. *There exist four positive constants  $\alpha_i = \alpha_i(a_i, b, d_i, \|u_i^0\|)$  ( $i = 1, 2$ ) and  $\epsilon_i = \epsilon_i(a_i, b_i, d_i, \|u_i^0\|)$  ( $i = 1, 2$ ) such that*

$$\begin{aligned}
\frac{d}{dt} \int \psi F^i &\leq \int (\psi_t + d_i \nabla \psi) F^i + \int ((d_i - a_i) F_1^i - b_i F_2^i) \nabla \psi \nabla u_i \\
&\quad - \frac{1}{2} \int \psi \left[ \frac{a_i}{2} F_{11}^i |\nabla u_i|^2 + d_i F_{22}^i |\nabla v_i|^2 \right] - \frac{1}{2} \int \psi F_1^i f_i, \quad i = 1, 2.
\end{aligned} \tag{3.11}$$

*Proof.* For any  $(u_i, v_i) \in [0, \|u_i^0\|] \times \mathbb{R}^+$  ( $i = 1, 2$ ), we choose  $\alpha_i$  and  $\epsilon_i$  in Lemma 3.2 such that

$$\rho F_2^i \leq \frac{1}{2} F_1^i, \quad i = 1, 2, \tag{3.12}$$

$$(a_i + d_i)^2 (F_{12}^i)^2 + b_i^2 (F_{22}^i)^2 + b_i (2a_i + d_i) (F_{12}^i) (F_{22}^i) - a_i d_i F_{11}^i F_{22}^i \leq 0, \quad i = 1, 2, \tag{3.13}$$

$$\frac{(F_1^i)^2}{F_{11}^i} \leq F^i, \quad i = 1, 2, \tag{3.14}$$

$$F_{12}^i \leq \frac{a_i}{2|b_i|} F_{11}^i, \quad i = 1, 2. \tag{3.15}$$

We verify these conditions as follows

$$\begin{aligned}
F^i &= (\alpha_i + u_i + u_i^2) e^{\epsilon_i v_i}, \quad i = 1, 2, \\
F_1^i &= (1 + 2u_i) e^{\epsilon_i v_i}, \quad i = 1, 2, \\
F_{11}^i &= 2e^{\epsilon_i v_i}, \quad i = 1, 2, \\
F_2^i &= \epsilon_i (\alpha_i + u_i + u_i^2) e^{\epsilon_i v_i}, \quad i = 1, 2, \\
F_{22}^i &= \epsilon_i^2 (\alpha_i + u_i + u_i^2) e^{\epsilon_i v_i}, \quad i = 1, 2, \\
F_{12} &= F_{21}^i = \epsilon_i (1 + 2u_i) e^{\epsilon_i v_i}, \quad i = 1, 2.
\end{aligned} \tag{3.16}$$

Denote

$$\begin{aligned}
\epsilon_i^{(1)} &= \frac{1}{2\rho(\alpha_i + \|u_i^0\| + \|u_i^0\|^2)}, \\
\epsilon_i^{(2)} &= \frac{1}{b_i(\alpha_i + \|u_i^0\| + \|u_i^0\|^2)}, \\
\epsilon_i^{(3)} &= \frac{1}{|b_i| (1 + 2\|u_i^0\|)}, \\
\alpha_i^{(1)} &= \frac{(a_i + d_i)^2 (1 + 2\|u_i^0\|)^2 + (2a_i + d_i)(1 + 2\|u_i^0\|)}{2a_i d_i}, \\
\alpha_i^{(2)} &= \frac{1 + 2\|u_i^0\|^2 + 2\|u_i^0\|}{2} = \frac{(1 + \|u_i^0\|)^2 + \|u_i^0\|^2}{2}.
\end{aligned} \tag{3.17}$$

If we choose  $\epsilon_i$  ( $i = 1, 2$ ) such that  $\epsilon_i \leq \epsilon_i^{(1)}$  ( $i = 1, 2$ ) then condition (3.12) is satisfied.

Condition (3.13) is satisfied if

$$\begin{aligned}
&(a_i + d_i)^2 \epsilon_i^2 (1 + 2u_i)^2 e^{2\epsilon_i v_i} + b_i^2 \epsilon_i^4 (\alpha_i + u_i + u_i^2)^2 e^{2\epsilon_i v_i} + b_i (2a_i + d_i) \epsilon_i^3 (1 + 2u_i) \\
&\quad \times (\alpha_i + u_i + u_i^2)^2 e^{2\epsilon_i v_i} - 2a_i d_i \epsilon_i^2 (\alpha_i + u_i + u_i^2) e^{2\epsilon_i v_i} \leq 0, \quad i = 1, 2,
\end{aligned} \tag{3.18}$$

that is,

$$\begin{aligned}
&(a_i + d_i)^2 (1 + 2u_i)^2 + b_i^2 \epsilon_i^2 (\alpha_i + u_i + u_i^2)^2 \\
&\quad + b_i (2a_i + d_i) (1 + 2u_i) (\alpha_i + u_i + u_i^2) \epsilon_i - 2a_i d_i (\alpha_i + u_i + u_i^2) \leq 0.
\end{aligned} \tag{3.19}$$

If we choose  $\epsilon_i$  ( $i = 1, 2$ ) and  $\alpha_i$  ( $i = 1, 2$ ) such that  $\epsilon_i \leq \epsilon_i^{(2)}$  ( $i = 1, 2$ ) and  $\alpha_i \leq \alpha_i^{(1)}$  ( $i = 1, 2$ ), then (3.19) is satisfied. In other words, condition (3.13) is satisfied.

Condition (3.14) is satisfied if

$$\frac{(1 + 2u_i)^2}{2} \leq (\alpha_i + u_i + u_i^2), \tag{3.20}$$

that is,

$$\alpha_i > \frac{1 + 2u_i^2 + 2u_i}{2}. \tag{3.21}$$

If we choose  $\alpha_i$  ( $i = 1, 2$ ) such that  $\alpha_i \leq \alpha_i^{(2)}$  ( $i = 1, 2$ ), then (3.21) is satisfied. Hence condition (3.14) is satisfied.

Similarly we can show that condition (3.15) is satisfied if  $\epsilon_i$  ( $i = 1, 2$ ) is chosen such that  $\epsilon_i \leq \epsilon_i^{(3)}$ .

Now select

$$\begin{aligned} \alpha_i &\geq \max(\alpha_i^{(1)}, \alpha_i^{(2)}) \quad (i = 1, 2), \\ \epsilon_i &\leq \min(\epsilon_i^{(1)}, \epsilon_i^{(2)}, \epsilon_i^{(3)}) \quad (i = 1, 2). \end{aligned} \tag{3.22}$$

Then conditions (3.12)–(3.15) are satisfied.

Then, from (3.13) and (3.15), we get

$$\begin{aligned} &(a_i F_{11}^i + b_i F_{12}^i) |\nabla u_i|^2 + ((a_i + d_i) F_{12}^i + b_i F_{22}^i) \nabla u_i \nabla v_i + d_i F_{22}^i |\nabla v_i|^2 \\ &\geq \frac{1}{2} [(a_i F_{11}^i + b_i F_{12}^i) |\nabla u_i|^2 + d_i F_{22}^i |\nabla v_i|^2] \\ &\geq \frac{1}{2} \left[ \frac{a_i}{2} F_{11}^i |\nabla u_i|^2 + d_i F_{22}^i |\nabla v_i|^2 \right]. \end{aligned} \tag{3.23}$$

From (3.12), we get

$$\int \psi (\rho F_2^i f_i - F_1^i f_i) \leq -\frac{1}{2} \int \psi F_1^i f_i, \quad i = 1, 2. \tag{3.24}$$

From (3.3), (3.23), (3.24), we get (3.11).

The proof of Lemma 3.3 is completed.  $\square$

**THEOREM 3.4.** *If  $\alpha_i, \epsilon_i$  ( $i = 1, 2$ ) satisfy (3.22), then there exist a test function  $\psi$  and real positive constants  $\beta_i$  ( $i = 1, 2$ ) and  $\sigma_i$  such that*

$$\int \psi F_i \leq \beta_i e^{\sigma_i t}, \quad \forall t > 0, \quad i = 1, 2. \tag{3.25}$$

*Proof.* We define the test function  $\psi(x)$  as

$$\psi(x) = \frac{1}{(1 + |x - x_0|^2)^n}, \quad x \in \mathbb{R}^n, \tag{3.26}$$

and  $x_0 \in \mathbb{R}^n$  is an arbitrary point.

Then  $\psi$  is a smooth function with exponential decay at infinity and satisfies  $|\Delta \psi| \leq K_1 \psi, |\nabla \psi| \leq K_2 \psi$ . Let  $K = \max(K_1, K_2)$  for some positive constant  $K$ .



Then from (3.11), we obtain

$$\begin{aligned}
\frac{d}{dt} \int \psi F^i &\leq K d_i \int \psi F^i + K \left( (d_i - a_i) + \frac{1}{2} |b_i| \right) \int F_1^i \psi |\nabla u_i| \\
&\quad - \frac{a_i}{4} \int \psi F_{11}^i |\nabla u|^2 - \frac{d_i}{2} \int \psi F_{22}^i |\nabla v_i|^2 - \frac{1}{2} \int \psi F_1^i f_i \\
&\leq K d_i \int \psi F^i + \frac{K^2}{a_i} \left( (d_i - a_i) + \frac{1}{2} |b_i| \right)^2 \int \psi \frac{(F_1^i)^2}{F_{11}^i} \\
&\leq \left[ K d_i + \frac{K^2}{a_i} \left( (d_i - a_i) + \frac{1}{2} |b_i|^2 \right)^2 \right] \int \psi F^i, \quad i = 1, 2.
\end{aligned} \tag{3.27}$$

Let

$$\begin{aligned}
\sigma_i &= K d_i + \frac{K^2}{a_i} \left( |d_i - a_i| + \frac{1}{2} |b_i|^2 \right)^2, \quad i = 1, 2, \\
\beta_i &\geq \left( \alpha_i + \|u_i^0\| + \|u_i^0\|^2 \right) e^{\epsilon \|v_i^0\|} \|\psi\|_1, \quad i = 1, 2.
\end{aligned} \tag{3.28}$$

Then

$$\frac{d}{dt} \int \psi F^i \leq \sigma_i \int \psi F^i \quad \text{for } i = 1, 2, \tag{3.29}$$

which implies (3.25). □

LEMMA 3.5. For any unit cube  $Q$  and any finite  $p \geq 1$ ,

$$\int_Q |v_i|^p dx \leq 2^n \frac{\beta_i}{\alpha_i \epsilon_i^p} e^{\sigma_i t} (p+1)^{p+1} \quad \text{for } i = 1, 2. \tag{3.30}$$

*Proof.* Using the results in Theorem 3.4, for any nonnegative integer  $p$  we have

$$\beta_i e^{\sigma_i t} \geq \int \psi F^i \geq \alpha_i \int \psi e^{\epsilon_i v_i} \geq \alpha_i \epsilon_i^p \int_Q \psi \frac{v_i^p}{p!}, \quad t > 0, \quad i = 1, 2. \tag{3.31}$$

By taking  $x_0$  at the center of  $Q$ , we get

$$\beta_i e^{\sigma_i t} \geq \frac{\alpha_i \epsilon_i^p}{p!} \int_Q \frac{v_i^p}{2^n} \geq \frac{\alpha_i \epsilon_i^p}{2^n (p+1)^{p+1}} \int_Q v_i^p, \quad i = 1, 2. \tag{3.32}$$

This implies (3.30). □

LEMMA 3.6. *There exist constants  $c_i = c_i(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t)$ ,  $i = 1, 2$  such that*

$$\begin{aligned} G(t-s) * u_i^r(x, s) v_{3-i}^m(x, s) &= \frac{1}{(4\pi\lambda(t-s))^{n/2}} \int e^{-|x-y|^2/4\lambda(t-s)} u_i^r(y, s) v_{3-i}^m(y, s) dy \\ &\leq c_i((t-s)^{n/2q} + (t-s)^{-n/2p}), \quad i = 1, 2, \end{aligned} \quad (3.33)$$

for any  $p > \max\{1, n/2\}$ ,  $1/p + 1/q = 1$ . Here  $G(t)$  is the semigroup generated by the operator  $\lambda\Delta$ , ( $\lambda > 0$ ) on the space  $C_{\text{ub}}(\mathbb{R}^n)$ .

*Proof.* Let  $\{Q_j\}$ ,  $j = 0, 1, 2, \dots$ , be the tilling of  $\mathbb{R}^n$  by unit cubes  $Q_j$ 's such that  $x$  is at the center of  $Q_0$ . Then

$$\int e^{-(x-y)^2/4\lambda(t-s)} u_i^r(y, s) v_{3-i}^m(y, s) dy = \sum_j \int_{Q_j} e^{-(x-y)^2/4\lambda(t-s)} u_i^r(y, s) v_{3-i}^m(y, s) dy. \quad (3.34)$$

For  $y \in Q_j$  we have the inequality

$$e^{-(x-y)^2/8\lambda(t-s)} \leq \sup_{y \in Q_j} e^{-(x-y)^2/8\lambda(t-s)} = e^{-\text{dist}^2(x, Q_j)/8\lambda(t-s)}. \quad (3.35)$$

Also there exists a positive constant  $c(n)$  such that if  $y \in Q_j$ ,  $j \neq 0$ , we have

$$c(n) \text{dist}^2(x, Q_j) \geq (x-y)^2. \quad (3.36)$$

Let  $I_1 = \int e^{-(x-y)^2/8\lambda(t-s)} u_i^r(y, s) v_{3-i}^m(y, s) dy$ . Then applying Hölder's inequality with  $p \geq n/2$  and its conjugate  $q$ , we get

$$\begin{aligned} I_1 &\leq \left( \int_{Q_j} e^{-q(x-y)^2/8\lambda(t-s)} \right)^{1/q} \left( \int_{Q_j} u_i^{rp}(y, s) v_{3-i}^{mp}(y, s) dy \right)^{1/p} \\ &\leq \left( \int_{Q_j} e^{-q \text{dist}^2(x, Q_j)/8\lambda(t-s)} \right)^{1/q} \left( \int_{Q_j} u_i^{rp}(y, s) v_{3-i}^{mp}(y, s) dy \right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q} (t-s)^{n/2q}}{q} \left( \int_{Q_j} u_i^{rp}(y, s) v_{3-i}^{mp}(y, s) dy \right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q} (t-s)^{n/2q}}{q} \left( \int_{Q_j} |u_i|^{rp} |v_{3-i}|^{mp} dy \right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q} (t-s)^{n/2q}}{q} \|u_i^0\|^r \left( \int_{Q_j} |v_{3-i}|^{mp} dy \right)^{1/p} \\ &\leq \frac{(8\pi\lambda)^{n/2q} (t-s)^{n/2q}}{q} \|u_i^0\|^r \left( 2^{mp} \frac{\beta_i}{\alpha_i \epsilon_i^{mp}} e^{\sigma t} (mp+1)^{mp+1} \right)^{1/p} \\ &= \frac{(8\pi\lambda)^{n/2q} (t-s)^{n/2q}}{q} \|u_i^0\|^r 2^m \left( \frac{\beta_i}{\alpha_i} \right)^{1/p} e^{\sigma t/p} \epsilon_i^{-m} (mp+1)^{m+1/p}. \end{aligned} \quad (3.37)$$

That is

$$\int e^{-(x-y)^2/8\lambda(t-s)} u_i^r(y,s) v_{3-i}^m(y,s) dy \leq c_i^1(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) (t-s)^{n/2q}, \quad i = 1, 2, \tag{3.38}$$

where

$$c_i^1(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) = \frac{(8\pi\lambda)^{n/2q}}{q} \|u_i^0\|^r 2^m \left(\frac{\beta_i}{\alpha_i}\right)^{1/p} e^{\sigma t/p} \epsilon_i^{-m} (mp+1)^{m+1/p}, \quad i = 1, 2. \tag{3.39}$$

Now we have

$$\begin{aligned} I_2 &= G(t-s) * u_i^r(x,s) v_{3-i}^m(x,s) \\ &\leq \frac{1}{(4\pi\lambda(t-s))^{n/2}} \sum_{Q_j} e^{-\text{dist}^2(x,Q_j)/8\lambda(t-s)} \int e^{-|x-y|^2/8\lambda(t-s)} u_i^r(y,s) v_{3-i}^m(y,s) dy \\ &\leq \frac{1}{(4\pi\lambda(t-s))^{n/2}} c_i^1(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) (t-s)^{n/2q} \sum_{Q_j} e^{-\text{dist}^2(x,Q_j)/8\lambda(t-s)} \\ &\leq c_i(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) (t-s)^{-n/2p} \left( \int e^{-(x-y)^2/8\lambda c(n)(t-s)} dy + 1 \right) \\ &\leq c_i(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) (t-s)^{-n/2p} (8\pi\lambda c(n)(t-s)^{n/2} + 1) \\ &\leq c_i(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) ((t-s)^{n/2q} + (t-s)^{-n/2p}), \end{aligned} \tag{3.40}$$

where  $c_i(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) = c_i^1(n, \lambda, \|u_i^0\|, \|v_{3-i}^0\|, t) (1/(4\pi\lambda)^{n/2})$ . □

The integral formula for  $u(t)$  is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds. \tag{3.41}$$

From (3.41) we can deduce that

$$u_i(x,t) = S_i(t)u_i^0 - \int_0^t S_i(t-s)f_i(u(s))ds, \quad i = 1, 2, \tag{3.42}$$

$$v_1(x,t) = S_5(t)u_1^0 + S_3(t)v_1^0 - \int_0^t S_5(t-s)f_1(u(s))ds + \rho \int_0^t S_3(t-s)f_1(u(s))ds, \tag{3.43}$$

$$v_2(x,t) = S_6(t)u_2^0 + S_4(t)v_2^0 - \int_0^t S_6(t-s)f_1(u(s))ds + \rho \int_0^t S_4(t-s)f_1(u(s))ds. \tag{3.44}$$

From (2.7) and (3.43), we get

$$\begin{aligned}
v_1(x, t) &= \frac{b_1}{a_1 - d_1} (S_1(t) - S_3(t)) u_1^0 + S_3(t) v_1^0 \\
&\quad - \int_0^t \frac{b_1}{a_1 - d_1} (S_1(t-s) - S_3(t-s)) f_1(u(s)) ds \\
&\quad + \rho \int_0^t S_3(t-s) f_1(u(s)) ds \\
&= \frac{b_1}{a_1 - d_1} S_1(t) u_1^0 + S_3(t) \left( v_1^0 - \frac{b_1}{a_1 - d_1} u_1^0 \right) - \frac{b_1}{a_1 - d_1} \int_0^t S_1(t-s) f_1(u(s)) ds \\
&\quad + \left( \rho + \frac{b_1}{a_1 - d_1} \right) \int_0^t S_3(t-s) f_1(u(s)) ds.
\end{aligned} \tag{3.45}$$

Hence

$$\begin{aligned}
v_1(x, t) &= \frac{b_1}{a_1 - d_1} S_1(t) u_1^0 + S_3(t) \left( v_1^0 - \frac{b_1}{a_1 - d_1} u_1^0 \right) \\
&\quad + \frac{b_1}{a_1 - d_1} \int_0^t S_1(t-s) (mk_1 u_2^r(s) v_1^m(s)) ds \\
&\quad - \frac{b_1}{a_1 - d_1} \int_0^t S_1(t-s) (mk_2 u_1^m(s) v_2^r(s)) ds \\
&\quad - \left( \rho + \frac{b_1}{a_1 - d_1} \right) \int_0^t S_3(t-s) (mk_1 u_2^r(s) v_1^m(s)) ds \\
&\quad + \left( \rho + \frac{b_1}{a_1 - d_1} \right) \int_0^t S_3(t-s) (mk_2 u_1^m(s) v_2^r(s)) ds,
\end{aligned} \tag{3.46}$$

that is,

$$\begin{aligned}
|v_1(x, t)| &= \frac{b_1}{a_1 - d_1} |S_1(t) u_1^0| + \left| S_3(t) \left( v_1^0 - \frac{b_1}{a_1 - d_1} u_1^0 \right) \right| \\
&\quad + \frac{b_1}{a_1 - d_1} \left| \int_0^t S_1(t-s) (mk_1 u_2^r(s) v_1^m(s)) ds \right| \\
&\quad + \frac{b_1}{a_1 - d_1} \left| \int_0^t S_1(t-s) (mk_2 u_1^m(s) v_2^r(s)) ds \right| \\
&\quad + \left( \rho + \frac{b_1}{a_1 - d_1} \right) \left| \int_0^t S_3(t-s) (mk_1 u_2^r(s) v_1^m(s)) ds \right| \\
&\quad + \left( \rho + \frac{b_1}{a_1 - d_1} \right) \left| \int_0^t S_3(t-s) (mk_2 u_1^m(s) v_2^r(s)) ds \right|.
\end{aligned} \tag{3.47}$$

Using relation (3.33) and the property that  $S_1(t)$  and  $S_3(t)$  are contraction semigroups, we get

$$\begin{aligned}
\|v_1(x, t)\| &\leq \frac{b_1}{a_1 - d_1} \|u_1^0\| + \|v_1^0\| \\
&+ \frac{mb_1}{a_1 - d_1} (k_1 c_1(n, a_1, \|u_2^0\|, \|v_1^0\|, t) + k_2 c_2(n, a_1, \|u_1^0\|, \|v_2^0\|, t)) \\
&\times \left( \int_0^t ((t-s)^{n/2q} + (t-s)^{-n/2p}) ds \right) \\
&+ \left( \rho + \frac{mb_1}{a_1 - d_1} \right) (k_1 c_3(n, d_1, \|u_2^0\|, \|v_1^0\|, t) + k_2 c_4(n, d_1, \|u_1^0\|, \|v_1^0\|, t)) \\
&\times \left( \int_0^t ((t-s)^{n/2q} + (t-s)^{-n/2p}) ds \right), \\
\|v_1(x, t)\| &\leq \frac{b_1}{a_1 - d_1} \|u_1^0\| + \|v_1^0\| + \frac{mb_1}{a_1 - d_1} (k_1 c_1 + k_2 c_2) \left| \int_0^t ((t-s)^{n/2q} + (t-s)^{-n/2p}) ds \right| \\
&+ m \left( \rho + \frac{b_1}{a_1 - d_1} \right) (k_1 c_3 + k_2 c_4) \left| \int_0^t ((t-s)^{n/2q} + (t-s)^{-n/2p}) ds \right|.
\end{aligned} \tag{3.48}$$

If  $p > n/2$ , we get

$$\begin{aligned}
\|v_1(x, t)\| &\leq \frac{b_1}{a_1 - d_1} \|u_1^0\| + \|v_1^0\| + \frac{mb_1}{a_1 - d_1} (k_1 c_1 + k_2 c_2) \left( \frac{2qt^{n/2q+1}}{n+2q} + \frac{2pt^{1-n/2p}}{2p-n} \right) \\
&+ m \left( \rho + \frac{b_1}{a_1 - d_1} \right) (k_1 c_3 + k_2 c_4) \left( \frac{2qt^{n/2q+1}}{2q+n} + \frac{2pt^{1-n/2p}}{2p-n} \right), \quad t > 0.
\end{aligned} \tag{3.49}$$

Similarly

$$\begin{aligned}
\|v_2(x, t)\| &\leq \frac{b_2}{a_2 - d_2} \|u_1^0\| + \|v_1^0\| + \frac{rb_2}{a_2 - d_2} (k_1 c_5 + k_2 c_6) \left( \frac{2qt^{n/2q+1}}{n+2q} + \frac{2pt^{1-n/2p}}{2p-n} \right) \\
&+ r \left( \rho + \frac{b_2}{a_2 - d_2} \right) (k_1 c_7 + k_2 c_8) \left( \frac{2qt^{n/2q+1}}{2q+n} + \frac{2pt^{1-n/2p}}{2p-n} \right), \quad t > 0.
\end{aligned} \tag{3.50}$$

Similarly one can show that

$$\|u_1(x, t)\| \leq \|u_1^0\| + m(k_1 c_9 + k_2 c_{10}) \left( \frac{2qt^{n/2q+1}}{2q+n} + \frac{2pt^{1-n/2p}}{2p-n} \right), \quad t > 0, \tag{3.51}$$

$$\|u_2(x, t)\| \leq \|u_2^0\| + m(k_1 c_{11} + k_2 c_{12}) \left( \frac{2qt^{n/2q+1}}{2q+n} + \frac{2pt^{1-n/2p}}{2p-n} \right), \quad t > 0. \tag{3.52}$$

Here  $c_1 - c_{12}$  are constants.

From estimates (3.49)–(3.52) and the standard parabolic regularity theory we get the existence of global classical solution  $(u_1, u_2, v_1, v_2) \in (C([0, \infty), C_{\text{ub}}) \cap C^1((0, \infty), C_{\text{ub}}))^4$ .

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