

# ON GENERALIZED DERIVATIVES FOR $C^{1,1}$ VECTOR OPTIMIZATION PROBLEMS

DAVIDE LA TORRE

*Received 19 September 2002 and in revised form 16 February 2003*

We introduce generalized definitions of Peano and Riemann directional derivatives in order to obtain second-order optimality conditions for vector optimization problems involving  $C^{1,1}$  data. We show that these conditions are stronger than those in literature obtained by means of second-order Clarke subdifferential.

## 1. $C^{1,1}$ vector functions and second-order subdifferentials

A function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is said to be locally lipschitzian (or of class  $C^{0,1}$ ) at  $x_0 \in \mathbb{R}^m$  when there exist constants  $K_{x_0}$  and  $\delta_{x_0}$  such that

$$\|f(x_1) - f(x_2)\| \leq K_{x_0} \|x_1 - x_2\|, \quad (1.1)$$

for all  $x_1, x_2 \in \mathbb{R}^m$ ,  $\|x_1 - x_0\| \leq \delta_{x_0}$ , and  $\|x_2 - x_0\| \leq \delta_{x_0}$ . For this type of functions, according to Rademacher theorem,  $f$  is differentiable almost everywhere (in the sense of Lebesgue measure). Then the first-order Clarke generalized Jacobian of  $f$  at  $x_0$ , denoted by  $\partial f(x_0)$ , exists and is given by

$$\partial f(x_0) := \text{clconv} \{ \lim f'(x_k) : x_k \rightarrow x_0, f'(x_k) \text{ exists} \}, \quad (1.2)$$

where  $\text{clconv}\{\dots\}$  stands for the closed convex hull of the set under the parentheses. Now, assume that  $f$  is a differentiable vector function from

$\mathbb{R}^m$  to  $\mathbb{R}^n$  whose directional derivative is of class  $C^{0,1}$  at  $x_0 \in \mathbb{R}^m$ . In this case, we say that  $f$  is of class  $C^{1,1}$  at  $x_0$ . The second-order Clarke subdifferential of  $f$  at  $x_0$ , denoted by  $\partial^2 f(x_0)$ , is defined as

$$\partial^2 f(x_0) := \text{clconv} \{ \lim f''(x_k) : x_k \rightarrow x_0, f''(x_k) \text{ exists} \}. \quad (1.3)$$

Thus  $\partial^2 f(x_0)$  is a subset of the finite-dimensional space  $L(\mathbb{R}^m; L(\mathbb{R}^m; \mathbb{R}^n))$  of linear operators from  $\mathbb{R}^m$  to the space  $L(\mathbb{R}^m; \mathbb{R}^n)$  of linear operators from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ . The elements of  $\partial^2 f(x_0)$  can therefore be viewed as a bilinear function on  $\mathbb{R}^m \times \mathbb{R}^m$  with values in  $\mathbb{R}^n$ . For the case  $n = 1$ , the term “generalized Hessian matrix” was used in [9] to denote the set  $\partial^2 f(x_0)$ . By the previous construction, the second-order subdifferential enjoys all properties of the generalized jacobian. For instance,  $\partial^2 f(x_0)$  is a nonempty convex compact set of the space  $L(\mathbb{R}^m; L(\mathbb{R}^m; \mathbb{R}^n))$  and the set-valued map  $x \mapsto \partial^2 f(x)$  is upper semicontinuous. Let  $u \in \mathbb{R}^m$ ; in the following we will denote by  $Lu$  the value of a linear operator  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  at the point  $u \in \mathbb{R}^m$  and by  $H(u, v)$  the value of a bilinear operator  $H : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  at the point  $(u, v) \in \mathbb{R}^m \times \mathbb{R}^m$ . So we will set

$$\begin{aligned} \partial f(x_0)(u) &= \{ Lu : L \in \partial f(x_0) \}, \\ \partial f(x_0)(u, v) &= \{ H(u, v) : H \in \partial^2 f(x_0) \}. \end{aligned} \quad (1.4)$$

Some important properties are listed here (see [8]).

(i) Mean value theorem: let  $f$  be of class  $C^{0,1}$  and  $a, b \in \mathbb{R}^m$ . Then

$$f(b) - f(a) \in \text{clconv} \{ \partial f(x)(b - a) : x \in [a, b] \}, \quad (1.5)$$

where  $[a, b] = \text{conv} \{ a, b \}$ .

(ii) Taylor expansion: let  $f$  be of class  $C^{1,1}$  and  $a, b \in \mathbb{R}^m$ . Then

$$f(b) - f(a) \in f'(a)(b - a) + \frac{1}{2} \text{clconv} \{ \partial^2 f(x)(b - a, b - a) : x \in [a, b] \}. \quad (1.6)$$

Several problems of applied mathematics including, for example, portfolio analysis, data classification, and so forth involve functions with no hope of being twice differentiable (see [4, 5, 13] and [Example 4.3](#)) but they can be approximated by  $C^{1,1}$  objects. Furthermore, the theory of  $C^{1,1}$  functions has revealed its importance in some penalty methods, as shown in the following examples.

*Example 1.1.* Let  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  be twice continuously differentiable on  $\mathbb{R}^m$  and consider  $f(x) = [g^+(x)]^2$ , where  $g^+(x) = \max \{ g(x), 0 \}$ . Then  $f$  is  $C^{1,1}$  on  $\mathbb{R}^m$ .

*Example 1.2.* Consider the minimization problem  $\min f_0(x)$  over all  $x \in \mathbb{R}^m$  such that  $f_1(x) \leq 0, \dots, f_k(x) \leq 0$ . Letting  $r$  denote a positive parameter, the augmented Lagrangian  $L_r$  [20] is defined on  $\mathbb{R}^m \times \mathbb{R}^k$  as

$$L_r(x, \mathbf{y}) = f_0(x) + \frac{1}{4r} \sum_{i=1}^k \{ [y_i + 2r f_i(x)]^+ \}^2 - y_i^2. \tag{1.7}$$

Upon setting  $\mathbf{y} = 0$  in the previous expression, we observe that

$$L_r(x, 0) = f_0(x) + r \sum_{i=1}^k [f_i^+(x)]^2 \tag{1.8}$$

is the ordinary penalized version of the minimization problem. The augmented Lagrangian  $L_r$  is differentiable everywhere on  $\mathbb{R}^m \times \mathbb{R}^k$  with

$$\begin{aligned} \nabla_x L_r(x, \mathbf{y}) &= \nabla f_0(x) + \sum_{j=1}^k [y_j + 2r f_j(x)]^+ \nabla f_j(x) \\ \frac{\partial L_r}{\partial y_i}(x, \mathbf{y}) &= \max \left\{ f_i(x), -\frac{y_i}{2r} \right\}, \quad i = 1, \dots, k, \end{aligned} \tag{1.9}$$

when  $f_i$  are  $C^2$  on  $\mathbb{R}^m$  and  $L_r$  is  $C^{1,1}$  on  $\mathbb{R}^{m+k}$ .

Optimality conditions for  $C^{1,1}$  scalar functions have been studied by many authors and numerical methods have been proposed too (see [7, 9, 10, 15, 17, 18, 21, 22, 23]). In [7], Guerraggio and Luc have given necessary and sufficient optimality conditions for vector optimization problems expressed by means of  $\partial^2 f(x)$ . In this paper, we introduce generalized Peano and Riemann directional derivatives for  $C^{1,1}$  vector functions and we study second-order optimality conditions for set-constrained optimization problems.

## 2. Generalized directional derivatives for $C^{1,1}$ vector functions

Let  $\Omega$  be an open subset of  $\mathbb{R}^m$  and let  $f : \Omega \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$  vector function. For such a function we define

$$\delta_2^d f(x; h) = f(x + 2hd) - 2f(x + hd) + f(x) \tag{2.1}$$

with  $x \in \Omega$ ,  $h \in \mathbb{R}$ , and  $d \in \mathbb{R}^m$ . The following result characterizes a function of class  $C^{1,1}$  in terms of  $\delta_2^d$ . It has been proved, for the scalar case, in [12, Theorem 2.1].

**PROPOSITION 2.1.** *Assume that the function  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  is bounded on a neighborhood of the point  $x_0 \in \Omega$ . Then  $f$  is of class  $C^{1,1}$  at  $x_0$  if and only if there exist neighborhoods  $U$  of  $x_0$  and  $V$  of  $0 \in \mathbb{R}^n$  such that  $\|\delta_2^d f(x; h)\|/h^2$  is bounded on  $U \times V \setminus \{0\}$ , for all  $d \in S^1 = \{d \in \mathbb{R}^m : \|d\| = 1\}$ .*

*Proof.* The proof follows from recalling that a vector function is of class  $C^{1,1}$  at  $x_0$  if and only if each component of  $f$  is of class  $C^{1,1}$  at  $x_0$ .  $\square$

The following definition extends to the vector case the well-known definition of the Riemann directional derivative [2].

**Definition 2.2.** The second Riemann directional derivative of  $f$  at  $x_0 \in \Omega$  in the direction  $d \in \mathbb{R}^m$  is defined as

$$f''_R(x_0; d) = \left\{ l = \lim_{k \rightarrow +\infty} \frac{\delta_2^d f(x_0; t_k)}{h_k^2}, t_k \downarrow 0 \right\}, \quad (2.2)$$

that is, the set of all cluster points of sequences  $\delta_2^d f(x_0; t_k)/t_k^2$ ,  $t_k \downarrow 0$ ,  $k \rightarrow +\infty$ .

If  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$ , then  $\sup_{l \in f''_R(x_0; d)} l$  ( $\inf_{l \in f''_R(x_0; d)} l$ ) coincides with the classical definition of the second upper Riemann directional derivative  $\overline{f''}_R(x_0; d)$  (second lower Riemann directional derivative  $\underline{f''}_R(x_0; d)$ ). If we define  $\Delta_2^d f(x; h) = f(x + hd) - 2f(x) + f(x - hd)$ , then one can introduce the corresponding Riemann directional derivatives. Riemann [19] introduced (for scalar functions) the notion of second-order directional derivative while he was studying the convergence of trigonometric series. For properties and applications of Riemann directional derivatives, see [1, 2, 8].

**Remark 2.3.** If we define  $\delta_2^{u,v} f(x; s, t) = f(x + su + tv) - f(x + su) - f(x + tv) + f(x)$  with  $x \in \Omega$ ,  $s, t \in \mathbb{R}$ , and  $u, v \in \mathbb{R}^m$ , then the set

$$f''_C(x_0; u, v) = \left\{ l = \lim_{k \rightarrow +\infty} \frac{\delta_2^{u,v} f(x_0; s_k, t_k)}{s_k t_k}, s_k, t_k \downarrow 0 \right\} \quad (2.3)$$

is an extension to the vector case of the definition according to Cominetti and Correa [6]. It is clear that  $f''_R(x_0; d) \subset f''_C(x_0; d, d)$ .

*Definition 2.4.* The second Peano directional derivative of  $f$  at  $x_0 \in \Omega$  in the direction  $d$  is defined as

$$f''_P(x_0; d) = \left\{ l = \lim_{k \rightarrow +\infty} 2 \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0) d}{t_k^2}, t_k \downarrow 0 \right\}, \quad (2.4)$$

that is, the set of all cluster points of sequences  $2(f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0) d) / t_k^2$ ,  $t_k \downarrow 0$ ,  $k \rightarrow +\infty$ .

If  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$ , then  $\sup_{l \in f''_P(x_0; d)} l$  ( $\inf_{l \in f''_P(x_0; d)} l$ ) coincides with the classical definition of the second upper Peano directional derivative  $\overline{f''}_P(x_0; d)$  (second lower Peano directional derivative  $\underline{f''}_P(x_0; d)$ ). Peano [16], studying the Taylor expansion formula for real functions, introduced a concept of a higher-order directional derivative of a function  $f$  at a point  $x_0$  known thereafter as the Peano directional derivative. A similar directional derivative (parabolic directional derivative) is also used in [3] in the scalar case. Furthermore, in [14] some optimality conditions for  $C^{1,1}$  vector optimization problems involving inequality constraints are given by using this type of generalized directional derivative. In this paper we study optimality conditions for set constraints.

It is trivial to prove that the previous sets are nonempty compact subsets of  $\mathbb{R}^n$ . The aim of the next sections is to investigate some properties of these generalized directional derivatives for  $C^{1,1}$  vector functions and then apply them in order to obtain second-order optimality conditions for vector optimization problems.

### 3. Preliminary properties

Let  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$  be a  $C^{1,1}$  scalar function at  $x_0 \in \Omega$ . The following theorem states an inequality between second upper Peano and Riemann directional derivatives.

**THEOREM 3.1** [11]. *Let  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}$  be a  $C^{1,1}$  scalar function at  $x_0 \in \Omega$ . Then  $\overline{f''}_P(x_0; d) \leq \overline{f''}_R(x_0; d)$ .*

Now, let  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$  vector function at  $x_0 \in \Omega$ .

**LEMMA 3.2.** *For all  $\zeta \in \mathbb{R}^n$ , there exists  $\tilde{r}_\zeta \in f''_R(x_0; d)$  such that  $\zeta(p - \tilde{r}_\zeta) \leq 0$ , for all  $p \in f''_P(x_0; d)$ .*

*Proof.* In fact, for all  $\zeta \in \mathbb{R}^n$  and eventually by extracting subsequences, we have

$$\overline{\zeta f''}_P(x_0; d) \leq \overline{\zeta f''}_R(x_0; d) = \zeta \tilde{r}_\zeta, \quad (3.1)$$

where  $\tilde{r}_\zeta \in f''_R(x_0; d)$ . So, for all  $p \in f''_P(x_0; d)$ , we have  $\zeta p \leq \overline{\zeta f''_P}(x_0; d) \leq \zeta \tilde{r}_\zeta$  and then the thesis follows.  $\square$

**THEOREM 3.3.** *Let  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$  vector function at  $x_0 \in \Omega$ . Then  $f''_P(x_0; d) \subset \text{conv } f''_R(x_0; d)$ .*

*Proof.* Ab absurdo, there exists  $p^* \in f''_P(x_0; d)$  such that  $p^* \notin \text{conv } f''_R(x_0; d)$ . Since  $\text{conv } f''_R(x_0; d)$  is a compact convex set of  $\mathbb{R}^n$ , then, by standard separation argument, there exists  $\zeta \in \mathbb{R}^n$ ,  $\zeta \neq 0$ , such that  $\zeta(p^* - r) > 0$ , for all  $r \in \text{conv } f''_R(x_0; d)$ . From **Lemma 3.2**, there exists  $\tilde{r}_\zeta \in f''_R(x_0; d)$  such that  $\zeta(p - \tilde{r}_\zeta) \leq 0$ , for all  $p \in f''_P(x_0; d)$ . So  $\zeta(p^* - \tilde{r}_\zeta) \leq 0$  and  $\zeta(p^* - \tilde{r}_\zeta) > 0$ .  $\square$

**THEOREM 3.4.** *Let  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$  vector function at  $x_0 \in \Omega$ . Then  $f''_R(x_0; d) \subset \partial^2 f(x_0)(d, d)$ .*

*Proof.* Let  $r \in f''_R(x_0; d)$ . Then

$$r = \lim_{k \rightarrow +\infty} \frac{f(x_0 + 2t_k d) - 2f(x_0 + t_k d) + f(x_0)}{t_k^2}, \tag{3.2}$$

where  $t_k \downarrow 0$ . By Taylor formula and the upper semicontinuity of the map  $x \mapsto \partial^2 f(x)$ , for all  $\epsilon > 0$ , there exists  $k_0(\epsilon) \in \mathbb{N}$  such that, for all  $k \geq k_0(\epsilon)$ ,

$$\frac{f(x_0 + 2t_k d) - 2f(x_0 + t_k d) + f(x_0)}{t_k^2} \in \partial^2 f(x_0)(d, d) + \epsilon B(0, 1), \tag{3.3}$$

where  $B(0, 1)$  is the closed unit ball in  $\mathbb{R}^n$ . So, taking the limit when  $k \rightarrow +\infty$  and  $\epsilon \rightarrow 0$ , we have  $f''_R(x_0; d) \subset \partial^2 f(x_0)(d, d)$ .  $\square$

**COROLLARY 3.5.** *Let  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$  vector function at  $x_0 \in \Omega$ . Then  $f''_P(x_0; d) \subset \partial^2 f(x_0)(d, d)$ .*

The following example shows that the inclusion is strict.

*Example 3.6.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}^2$ ,  $f(x) = (x^4 \sin(1/x) + x^4, x^4)$ . The function  $f$  is of class  $C^{1,1}$  at  $x_0 = 0$  and  $f''_P(0; d) = (0, 0) \in \partial^2 f(0)(d, d) = [-d^2, d^2] \times \{0\}$ .

**THEOREM 3.7.** *Let  $f : \Omega \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$  vector function at  $x_0 \in \Omega$ . Then*

$$f''_P(x_0; d) = \left\{ l = \lim_{k \rightarrow +\infty} 2 \frac{f(x_0 + t_k d_k) - f(x_0) - t_k \nabla f(x_0) d_k}{t_k^2}, t_k \downarrow 0, d_k \rightarrow d \right\}. \tag{3.4}$$

*Proof.* Let

$$\mathcal{A}(x_0; d) = \left\{ l = \lim_{k \rightarrow +\infty} 2 \frac{f(x_0 + t_k d_k) - f(x_0) - t_k \nabla f(x_0) d_k}{t_k^2}, t_k \downarrow 0, d_k \rightarrow d \right\}. \tag{3.5}$$

Then the inclusion  $f''_P(x_0; d) \subset \mathcal{A}$  follows immediately considering  $d_k = d$ . Vice versa, let  $l \in \mathcal{A}$ , there exist  $t_k \downarrow 0, d_k \rightarrow d$  such that

$$\begin{aligned} \frac{l}{2} &= \lim_{k \rightarrow +\infty} \frac{f(x_0 + t_k d_k) - f(x_0) - t_k \nabla f(x_0) d_k}{t_k^2} \\ &= \lim_{k \rightarrow +\infty} \frac{f(x_0 + t_k d_k) - f(x_0 + t_k d)}{t_k^2} \\ &\quad + \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0) d - t_k \nabla f(x_0) (d_k - d)}{t_k^2}. \end{aligned} \tag{3.6}$$

Taking eventually a subsequence

$$\begin{aligned} \lim_{k \rightarrow +\infty} \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0) d}{t_k^2} &= \frac{p}{2}, \quad p \in f''_P(x_0; d), \\ \lim_{k \rightarrow +\infty} \left| \frac{f_i(x_0 + t_k d_k) - f_i(x_0 + t_k d) - t_k \nabla f_i(x_0) (d_k - d)}{t_k^2} \right| \\ &= \lim_{k \rightarrow +\infty} \left| \frac{t_k \nabla f_i(\xi_k) (d_k - d) - t_k \nabla f_i(x_0) (d_k - d)}{t_k^2} \right| \\ &\leq \lim_{k \rightarrow +\infty} \frac{K \|d_k - d\| \|\xi_k - x_0\|}{t_k} = 0, \end{aligned} \tag{3.7}$$

where  $K$  is the Lipschitz modulus of the derivative  $\nabla f$  around  $x_0$ ,  $\xi_k \in [x_0 + t_k d_k, x_0 + t_k d]$  and  $(\xi_k - x_0)/t_k \rightarrow d$ . Therefore

$$\lim_{k \rightarrow +\infty} \frac{f(x_0 + t_k d_k) - f(x_0 + t_k d) - t_k \nabla f(x_0)(d_k - d)}{t_k^2} = 0 \quad (3.8)$$

and then  $p = l$ . □

#### 4. Set-constrained optimization problems

Let  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  be a  $C^{1,1}$  vector function and consider the following set-constrained optimization problem:

$$\min_{x \in X} f(x), \quad (4.1)$$

where  $X$  is a subset of  $\mathbb{R}^m$ . For such a problem, the following definition states the classical notion of local minimum point and local weak minimum point.

*Definition 4.1.* Let  $C$  be a pointed closed convex cone of  $\mathbb{R}^n$ . Then  $x_0 \in X$  is called a local minimum point of (4.1) if there exists a neighborhood  $N \cap X$  of  $x_0$  such that no  $x \in N \cap X$  satisfies  $f(x_0) - f(x) \in C \setminus \{0\}$ .

*Definition 4.2.* Let  $C$  be a pointed closed convex cone of  $\mathbb{R}^n$ . Then  $x_0 \in X$  is called a local weak minimum point of (4.1) if there exists a neighborhood  $N \cap X$  of  $x_0$  such that no  $x \in N \cap X$  satisfies  $f(x_0) - f(x) \in \text{int} C$ .

The following example shows the importance of the class of  $C^{1,1}$  vector functions in order to obtain smooth approximations of strongly non-differentiable optimization problems.

*Example 4.3.* In many applications in portfolio analysis, data classification, and approximation theory [4, 5, 13], one can be interested in solving optimization problems as

$$\min_{x \in X} (f(x), \text{supp}(x)), \quad (4.2)$$

where  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  is a smooth function,  $X$  is a subset of  $\mathbb{R}^m$ , and  $\text{supp}(x) = |\{i : x_i \neq 0\}|$  ( $|A|$  is the cardinality of the set  $A \subset \mathbb{N}$ ). For example, in portfolio optimization, the variables correspond to commodities to be bought, the function  $\text{supp}(x)$  specifies that not too many different types of commodities can be chosen, the function  $f(x)$  is a "measure of risk," and the constraints  $X$  prescribe levels of "performance." In data classification context, the function  $f$  is a measure of the margin between the



separating planes, and the minimization of  $\text{supp}(x)$  concerns the number of features of  $x$ , that is, the number of nonzero components of the vector, in order to keep the complexity of the classification rule low (this improves the understandability of the model results).

This type of optimization problems involves strongly nondifferentiable functions; one way to solve it is to replace the function  $\text{supp}(x)$  with an approximation  $\text{supp}_\alpha^*(x)$  as

$$\text{supp}_\alpha^*(x) = \sum_{i=1}^m [\max \{e^{-\alpha x_i}, e^{\alpha x_i}\}]^2. \tag{4.3}$$

Then the function  $x \rightarrow (f(x), \text{supp}_\alpha^*(x))$  is a  $C^{1,1}$  vector function, for all  $\alpha > 0$ .

*Definition 4.4.* Let  $A$  be a given subset of  $\mathbb{R}^n$  and  $x_0 \in \text{cl}A$ . The sets

$$\begin{aligned} WF(A, x_0) &= \{d \in \mathbb{R}^n : \exists t_k \downarrow 0, x_0 + t_k d \in X\}, \\ T(A, x_0) &= \{d \in \mathbb{R}^n : \exists t_k \downarrow 0, \exists d_k \rightarrow d, x_0 + t_k d_k \in X\} \end{aligned} \tag{4.4}$$

are called cone of weak feasible directions and tangent cone to  $A$  at  $x_0$ .

**THEOREM 4.5** (necessary optimality condition). *If  $x_0$  is a local weak minimum point, then, for all  $d \in \mathbb{R}^m$  such that  $-\nabla f(x_0)d \in C$  and  $d \in WF(X, x_0)$ ,*

$$f_p''(x_0; d) \notin -\text{int}C. \tag{4.5}$$

*Proof.* Ab absurdo, there exist  $d \in \mathbb{R}^n$ ,  $-\nabla f(x_0)d \in C$ , and  $d \in WF(X, x_0)$  such that  $f_p''(x_0; d) \subset -\text{int}C$ . Now, for all  $l \in f_p''(x_0; d)$ , we have

$$2 \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0)d}{t_k^2} = l + o(t_k^2), \tag{4.6}$$

where  $l \in f_p''(x_0; d) \subset -\text{int}C$  and there exists  $k_0$  such that for all  $k \geq k_0$  we have

$$2 \frac{f(x_0 + t_k d) - f(x_0) - t_k \nabla f(x_0)d}{t_k^2} \in -\text{int}C \tag{4.7}$$

and then

$$f(x_0 + t_k d) - f(x_0) \in t_k \nabla f(x_0) d - \text{int} C \subset -\text{int} C. \quad (4.8)$$

□

**COROLLARY 4.6.** *If  $x_0$  is a local weak minimum point, then, for all  $d \in \mathbb{R}^m$  such that  $-\nabla f(x_0)d \in C$  and  $d \in WF(X, x_0)$ , we have*

$$\partial^2 f(x_0)(d, d) \not\subset -\text{int} C. \quad (4.9)$$

*Example 4.7.* Let  $f : \mathbb{R} \rightarrow \mathbb{R}^2, f(x) = (x^4 \sin(1/x) - x^2/4, -x^2)$ , and  $C = \mathbb{R}_+^2$ . The point  $x_0 = 0$  is not a local weak minimum point. We have  $f_p''(0; d) = (-d^2/2, -2d^2) \in -\text{int} \mathbb{R}_+^2$  (the necessary condition is not satisfied) but  $\partial^2 f(0)(d, d) = [-3d^2/2, d^2/2] \times \{-2d^2\}$  (the necessary condition is satisfied).

**THEOREM 4.8** (sufficient optimality condition). *Let  $x_0 \in X$  and suppose that  $\nabla f(x_0)d \not\subset -\text{int} C$ , for all  $d \in T(X, x_0)$ . If for all  $d \neq 0, d \in T(X, x_0)$ , and  $\nabla f(x_0)d \in -(C \setminus \text{int} C)$  there exists a neighborhood  $U$  of  $d$  such that, for all  $v \in U(d), \nabla f(x_0)v \not\subset -\text{int} C$ , and  $f_p''(x_0; d) \subset \text{int} C$ , then  $x_0$  is a local minimum point.*

*Proof.* Ab absurdo, there exists  $x_k \in X, x_k \rightarrow x_0$  such that  $f(x_k) - f(x_0) \in -C \setminus \{0\}$ . If  $d_k = (x_k - x_0) / \|x_k - x_0\|$ , then, eventually by extracting subsequence,  $x_k = x_0 + t_k d_k$  and  $d_k \rightarrow d, \|d\| = 1$ . So  $d \in T(X, x_0)$ . Furthermore,

$$f(x_k) - f(x_0) = t_k \nabla f(x_0) d_k + o(t_k) \quad (4.10)$$

and then  $\nabla f(x_0)d \in -C$  since  $C$  is closed. So  $\nabla f(x_0)d \in -(C \setminus \text{int} C)$  and then there exists  $U(d)$  such that  $\nabla f(x_0)v \not\subset -\text{int} C$ , for all  $v \in U(d)$ .

Let  $p = \lim_{k \rightarrow +\infty} 2((f(x_k) - f(x_0) - t_k \nabla f(x_0) d_k) / t_k^2) \in \text{int} C$ , then  $p \in f_p''(x_0; d)$  and, since  $\text{int} C$  is open, there exists  $k_0$  such that, for all  $k \geq k_0$ ,

$$2 \frac{f(x_k) - f(x_0) - t_k \nabla f(x_0) d_k}{t_k^2} \in \text{int} C \quad (4.11)$$

and then

$$f(x_k) - f(x_0) \in t_k \nabla f(x_0) d_k + \text{int} C \subset (-\text{int} C)^c + \text{int} C \subset (-C)^c. \quad (4.12)$$

□

**COROLLARY 4.9.** *Let  $x_0 \in X$  and suppose that  $\nabla f(x_0)d \notin -\text{int}C$ , for all  $d \in T(X, x_0)$ . If for all  $d \neq 0$ ,  $d \in T(X, x_0)$ , and  $\nabla f(x_0)d \in -(C \setminus \text{int}C)$ , there exists a neighborhood  $U$  of  $d$  such that, for all  $v \in U$ ,  $\nabla f(x_0)v \notin -\text{int}C$ , and  $\partial^2 f(x_0)(d, d) \subset \text{int}C$ ,  $x_0$  is a local minimum point.*

## References

- [1] J. M. Ash, *Generalizations of the Riemann derivative*, Trans. Amer. Math. Soc. **126** (1967), 181–199.
- [2] ———, *Very generalized Riemann derivatives, generalized Riemann derivatives and associated summability methods*, Real Anal. Exchange **11** (1985), no. 1, 10–29.
- [3] A. Ben-Tal and J. Zowe, *Directional derivatives in nonsmooth optimization*, J. Optim. Theory Appl. **47** (1985), no. 4, 483–490.
- [4] D. Bienstock, *Computational study of a family of mixed-integer quadratic programming problems*, Math. Programming, Ser. A **74** (1996), no. 2, 121–140.
- [5] T.-J. Chang, N. Meade, J. E. Beasley, and Y. M. Sharaiha, *Heuristics for cardinality constrained portfolio optimization*, Comput. Oper. Res. **27** (2000), no. 13, 1271–1302.
- [6] R. Cominetti and R. Correa, *A generalized second-order derivative in nonsmooth optimization*, SIAM J. Control Optim. **28** (1990), no. 4, 789–809.
- [7] A. Guerraggio and D. T. Luc, *Optimality conditions for  $C^{1,1}$  vector optimization problems*, J. Optim. Theory Appl. **109** (2001), no. 3, 615–629.
- [8] A. Guerraggio and M. Rocca, *Derivate dirette di Riemann e di Peano*, Convessità e Calcolo Parallelo, Libreria Univ. Editrice, Verona, 1997, pp. 147–178 (Italian).
- [9] J.-B. Hiriart-Urruty, J.-J. Strodiot, and V. H. Nguyen, *Generalized Hessian matrix and second-order optimality conditions for problems with  $C^{1,1}$  data*, Appl. Math. Optim. **11** (1984), no. 1, 43–56.
- [10] D. Klatté and K. Tammer, *On second-order sufficient optimality conditions for  $C^{1,1}$ -optimization problems*, Optimization **19** (1988), no. 2, 169–179.
- [11] D. La Torre, *Characterizations of  $C^{1,1}$  functions, Taylor's formulae and optimality conditions*, Doctoral thesis, University of Milan, 2001.
- [12] D. La Torre and M. Rocca,  *$C^{1,1}$  functions and optimality conditions*, to appear in J. Comput. Anal. Appl.
- [13] D. La Torre and C. Vercellis,  *$C^{1,1}$  approximations of generalized support vector machines*, to appear in J. of Concrete and Applicable Math.
- [14] L. Liu, P. Neittaanmäki, and M. Křížek, *Second-order optimality conditions for nondominated solutions of multiobjective programming with  $C^{1,1}$  data*, Appl. Math. **45** (2000), no. 5, 381–397.
- [15] D. T. Luc, *Taylor's formula for  $C^{k,1}$  functions*, SIAM J. Optim. **5** (1995), no. 3, 659–669.
- [16] G. Peano, *Sulla formula di Taylor*, Atti Accad. Sci. Torino **27** (1891/92), 40–46 (Italian).
- [17] L. Q. Qi, *Superlinearly convergent approximate Newton methods for  $LC^1$  optimization problems*, Math. Programming, Ser. A **64** (1994), no. 3, 277–294.
- [18] ———,  *$LC^1$  functions and  $LC^1$  optimization*, Operations Research and Its Application, World Publishing, Beijing, 1996, pp. 4–13.

- [19] B. Riemann, *Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*, Ges. Werke **2** (1982), 227–271 (German).
- [20] R. T. Rockafellar, *Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization*, Math. Oper. Res. **6** (1981), no. 3, 424–436.
- [21] W. Sun, R. J. B. de Sampaio, and J. Yuan, *Two algorithms for  $LC^1$  unconstrained optimization*, J. Comput. Math. **18** (2000), no. 6, 621–632.
- [22] X. Q. Yang, *Second-order conditions in  $C^{1,1}$  optimization with applications*, Numer. Funct. Anal. Optim. **14** (1993), no. 5-6, 621–632.
- [23] X. Q. Yang and V. Jeyakumar, *Generalized second-order directional derivatives and optimization with  $C^{1,1}$  functions*, Optimization **26** (1992), no. 3-4, 165–185.

Davide La Torre: Department of Economics, University of Milan, Via Conservatorio 7, 20122 Milano, Italy

E-mail address: [davide.latorre@unimi.it](mailto:davide.latorre@unimi.it)

## Special Issue on Modeling Experimental Nonlinear Dynamics and Chaotic Scenarios

### Call for Papers

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the *Mathematical Problems in Engineering* aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at <http://www.hindawi.com/journals/mpe/>. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at <http://mts.hindawi.com/> according to the following timetable:

Manuscript Due	February 1, 2009
First Round of Reviews	May 1, 2009
Publication Date	August 1, 2009

### Guest Editors

**José Roberto Castilho Piqueira**, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; [piqueira@lac.usp.br](mailto:piqueira@lac.usp.br)

**Elbert E. Neher Macau**, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil ; [elbert@lac.inpe.br](mailto:elbert@lac.inpe.br)

**Celso Grebogi**, Department of Physics, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK; [grebogi@abdn.ac.uk](mailto:grebogi@abdn.ac.uk)