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ON A GENERALIZATION OF HAUSDORFF SPACE

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ABSTRACT. Here, a new separation axiom as a generalization of that of Hausdorff is introduced. Its simple consequences and relations with some other known separation axioms are studied. That a non-indiscrete topological group satisfies this axiom is shown.

KEY WORDS AND PHRASES. Separation Axiom, Hausdorff space.

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1. INTRODUCTION. Five well known separation axioms are introduced and these significances are studies in literature [1,2,3,4]. In addition to this, other separation axioms are formulated and their consequences with interrelations were discussed by several investigators. In this connection the papers of C. E. Aull [5] and A. Wilansky [6] are informative and of much interest.

Here a new separation axiom, which may be taken as a generalization of the Hausdorff axiom is stated and then its relations with T_0, T_1, T_2 separation axioms and also with other separation axioms KC, US [6]. After that simple consequences of the above axioms are studied. Finally non-indiscrete topological groups always imply as H-separation axiom.

DEFINITION. Let $(X,\mathcal{T})$ be a topological space. In a non singletone space, for every $x \in X$ there is a $y \in X$ such that $x \in G$, $y \in H$ and $G \cap H = \phi$ for some $G,H \in \mathcal{T}$. Then the space is called H-space and also every singletone space is H-space.

REMARK 1. It is clear that every Hausdorff space is H-space. But converse is not necessarily true by the following example.

EXAMPLE 1. Consider $X = \{1,2,3,4\}$ and $\mathcal{T} = \{\phi, X, \{1,2\}, \{3,4\}\}$ there $(X,\mathcal{T})$ is a H-space but $(X,\mathcal{T})$ is not a Hausdorff space. The space is also non-T_0 space.

REMARK 2. Example 1 and the following example show that a H-space and T-space are independent of each other.

EXAMPLE 2. Consider $X = \{1,2,3,4\}$ and $\mathcal{T} = \{\phi, X, \{1\}, \{1,2\}, \{1,2,3\}\}$, then $(X,\mathcal{T})$ is T_0-space but it is not a H-space.

REMARK 3. The following example shows that in the property of being H-space is non-hereditary property.

EXAMPLE 3. Consider $X = \{1,2,3,4,5\}$ and $\mathcal{T} = \{\phi, X, \{1,2,3\}, \{4,5\}\}$ then $(X,\mathcal{T})$ is H-space. Now consider the sub-space $\{1,2,3\}$ which is not a H-space.

REMARK 4. A T_0-space which is also H-space is not necessarily a T_1-space (by the following example).

EXAMPLE 4. Consider $X = \{1,2,3,4\}$ and $\mathcal{T} = \{\phi, X, \{1\}, \{1,2\}, \{3\}, \{3,4\}, \{1,3\}, \{1,3,4\}, \{1,2,3\}$. Now it is clearly a T_0-space and also a H-space. But $(X,\mathcal{T})$ is not a T_1-space.

REMARK 5. Example 1 and the following example shows that a H-space and a T_1-space are independent of each other.

EXAMPLE 5. Consider R is the set of all real numbers with cofinite topology. It is clear that the space is T_1 but it is not H-space.

REMARK 6. A T_1-space which is also H-space is not necessarily a T_2-space (by the following example).

EXAMPLE 6. Let us consider $X = \{1,2,3,4,...\}$ and the topology $\mathcal{T}$ is cofintite topology. Now let $X^* = \{0,1,2,3,...\}$ and $\mathcal{T}^* = \{G, G \cup \{0\} : G \in \mathcal{T}\}$.

Then clearly $(X^*,\mathcal{T}^*)$ is a topological space and it is clear that the space is T_1-space as well as H-space. But the space is not a T_2-space.

DEFINITION [6]. A topological space is called KC-space if every compact set is closed.

REMARK 7. Example 1 and the following example shows that a H-space and a KC-space are independent of each other.

EXAMPLE 7. Let us consider R^+ be the set of all positive real numbers with co-countable topology. It is clear that the space is KC-space. But it is not a H-space.

REMARK 8. A KC-space which is also H-space is not necessarily a T_2-space (by the following example).

EXAMPLE 8. Consider R^+ be the set of all positive real numbers with co-countable topology $\mathcal{T}$. Now let $\bar{R}$ be the set of all non-negative real numbers and $\bar{\mathcal{T}} = \{G, G \cup \{0\}: G \in \mathcal{T}\}$. Then clearly $(\bar{R},\bar{\mathcal{T}})$ is a topological space and it is clear that the space is KC-space as well as H-space. But the space is not a T_2-space.

DEFINITION [6]. A topological space is called a US-space if every convergent sequence has exactly one limit to which it converges.

REMARK 9. (a) Remark 5 and Remark 7 shows that US-space and H-space are independent of each other. Since $T_2 \Rightarrow KC \Rightarrow US \Rightarrow T_1$ (forom [6]).

(b) From the above example 8 it is clear that a US-space which is also H-space is not necessarily a T_2-space.

RESULT 1. Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two topological spaces. If a non-constant function $f: X \to Y$ is continuous and Y is T_2-space. Then X is a H-space.

PROOF. Since $f: X \to Y$ is a non-constant function, so for every $x \in X$ there is a $y \in X$ such that $f(x) \neq f(y)$. Since $f(x), f(y) \in Y$ and Y is T_2-space . Hence there are $U,V \in \mathcal{T}_2$ such that $f(x) \in U$, $f(y) \in V$ and $U \cap V = \phi$. Then $f^{-1}(U)$ and $f^{-1}(V)$ are mutually disjoint non-empty open in X [since f is continuous].

$x \in f^{-1}(U)$ and $y \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(U) = \phi$. Hence X is H-space.

RESULT 2. Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two topological spaces. If $(X,\mathcal{T}_1)$ is a H-space. Then the product space $X \times Y$ is also a H-space.

PROOF. Let (x,y) be any point in $X \times Y$. Since X is a H-space, then there is a $x_1 \in X$ such that $x \in V_1$, $x_1 \in V_2$ and $V_1 \cap V_2 = \phi$ for some $V_1, V_2 \in \mathcal{T}_1$. And if $y \in U \in \mathcal{T}_2$, then $(x,y) \in V_1 \times U$. $(x_1,y) \in V_2 \times U$ and $(V_1 \times U) \cap (V_2 \times U) = \phi$ (since $V_1 \cap V_2 = \phi$). Hence $X \times Y$ is a H-space.

RESULT 3. Let $(X,\mathcal{T})$ be a non-indiscrete topological group. Then $(X,\mathcal{T})$ is H-space.

PROOF. Let $x \in X$ and V be a non-empty proper open set in X. Case I: Let $x \in V$, since V be a non-empty proper open set in X, so there is a $y \in X$ such that $y \bar{\in} V$. Let $A = x^{-1}V$. Then A is a open neighborhood of e(identity). Let $B = A \cap A^{-1}$. Then B is a open neighborhood of e and $B = B^{-1}$. Let $U = yB$. Then U is a open neighborhood of y. We claim that $x \bar{\in} U$. For suppose $x \in U$. Then $x \in yB$ so $x = yb$ for some $b \in B$. Then $x^{-1} = b^{-1}y^{-1}$. But $b^{-1} \in B^{-1} = B$. So $x^{-1} \in B^{-1}y^{-1} = By^{-1}$. Now $B \subset A$, then $x^{-1} \in By^{-1} \subset Ay^{-1} = x^{-1}Vy^{-1}$. Then $e \in Vy^{-1}$. So $y \in V$ - a contradiction. So $x \bar{\in} U$. Hence we get, for every $x \in X$, there is $y \in X$ such that $x \in V$, $y \in U$ and $x \bar{\in} U$, $y \bar{\in} V$ for some $V,U \in \mathcal{T}$. Let V' be the complement of V, so V' is closed and $x \bar{\in} V'$, $y \in V'$. Since every topological group is regular, so there are $U_1, V_1 \in \mathcal{T}$ such that $x \in V_1$, $V' \subset U_1$ and $V_1 \cap U_1 = \phi$. Then $x \in V_1$ and $y \in U_1$ such that $V_1 \cap U_1 = \phi$ for some $V_1, U_1 \in \mathcal{T}$. Hence $(X,\mathcal{T})$ is H-space.

Case II: If $x \bar{\in} V$ then $x \in V'$ (complement of V). Since V is open in X so V' is closed in X. Since V is non-empty so there is a $y \in V$, so $y \bar{\in} V'$. Since every topological group is regular space. So there are $V_1, V_2 \in \mathcal{T}$ such that $y \in V_1$ and $V' \subset V_2$ such that $V_1 \cap V_2 = \phi$. Hence $x \in V_2$, $y \in V_1$ and $V_1 \cap V_2 = \phi$. Hence it is H-space.

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