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THE SEMIGROUP OF NONEMPTY FINITE SUBSETS OF INTEGERS

REUBEN SPAKE

Department of Mathematics,
University of California
Davis, California 95616 U.S.A.

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ABSTRACT. Let Z be the additive group of integers and $\mathcal{S}$ the semigroup consisting of all nonempty finite subsets of Z with respect to the operation defined by

$$A + B = \{a+b : a \in A, b \in B\}, \quad A, B \in \mathcal{S}.$$

For $X \in \mathcal{S}$, define A_X to be the basis of $\langle X-\min(X)\rangle$ and B_X the basis of $\langle\max(X)-X\rangle$. In the greatest semilattice decomposition of $\mathcal{S}$, let $\mathcal{a}(X)$ denote the archimedean component containing X and define $\mathcal{a}_0(X) = \{Y \in \mathcal{a}(X) : \min(Y) = 0\}$. In this paper we examine the structure of $\mathcal{S}$ and determine its greatest semilattice decomposition. In particular, we show that for $X, Y \in \mathcal{S}$, $\mathcal{a}(X) = \mathcal{a}(Y)$ if and only if $A_X = A_Y$ and $B_X = B_Y$. Furthermore, if $X \in \mathcal{S}$ is a non-singleton, then the idempotent-free $\mathcal{a}(X)$ is isomorphic to the direct product of the (idempotent-free) power joined subsemigroup $\mathcal{a}_0(X)$ and the group Z.

KEY WORDS AND PHRASES. Greatest semilattice decomposition, archimedean component.
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1. INTRODUCTION.

Let Z be the group of integers and $\mathcal{S}$ the semigroup consisting of all nonempty finite subsets of Z with respect to the operation defined by

$$A + B = \{a+b : a \in A, b \in B\}, \quad A, B \in \mathcal{S}.$$

The semigroup $\mathcal{S}$ is clearly commutative and is a subsemigroup of the power semigroup of the group of integers, (the semigroup of all nonempty subsets of Z). In this paper we will determine the greatest semilattice decomposition of $\mathcal{S}$ and describe the structure of the archimedean components in this decomposition. As we will soon see, there is a surprisingly simple necessary and sufficient condition for two elements to be in the same component.

For $X = \{x_1,\dots,x_n\} \in \mathcal{S}$, where $x_1 < \dots < x_n$, define $\min(X) = x_1$,

$\max(X) = x_n$, and $\gcd(X)$ to be the greatest (non-negative) common divisor of the integers $x_1,\ldots,x_n$, (where $\gcd(0) = 0$, $\gcd(X \cup \{0\}) = \gcd(X)$). A singleton element of $\mathcal{S}$ will be identified with the integer it contains. Let Z_+ be the set of positive integers and define $[a,b] = \{x \in Z : a \leq x \leq b\}$ if $a, b \in Z$ with $a \leq b$. For $U \in \mathcal{S}$, let $\langle U \rangle$ denote the semigroup generated by the set U, and for $m \in Z_+$ define mU, $m{*}U$, and Z_m as follows:

$$mU = \underbrace{U + \ldots + U}_{m}, \quad m{*}U = \{mu : u \in U\}, \quad \text{and} \quad Z_m = Z/\langle -m,m \rangle.$$

It will also be convenient to define $-U = \{-u : u \in U\}$.

In the greatest semilattice decomposition of $\mathcal{S}$, let $\mathcal{A}(A)$ denote the archimedean component containing A. As usual, define the partial order $\leq$ on the (lower) semilattice as: $\mathcal{A}(A) \leq \mathcal{A}(B)$ if and only if $nA = B + C$ for some $C \in \mathcal{S}$ and $n \in Z_+$ (equivalently: $X + Y \in \mathcal{A}(A)$ for some (all) $X \in \mathcal{A}(A)$ and $Y \in \mathcal{A}(B)$). We refer the reader to Clifford and Preston [2] and Petrich [3] for more on the greatest semilattice decomposition of a commutative semigroup. Observe that since 0 is the only idempotent and indeed the identity, $\mathcal{A}(A)$ is idempotent-free if A is a non-singleton, ($\mathcal{A}(0)$ consists of all the singletons in $\mathcal{S}$ and in fact $\mathcal{A}(0) \cong Z$). Furthermore, if follows that the subgroups of $\mathcal{S}$ are of the form $\{\{gx\} : x \in Z\}$, where g is a non-negative integer. Finally, note that $\mathcal{S}$ is clearly countable, but this of course does not imply that there are also infinitely many archimedean components. However, as will soon be shown, there are in fact infinitely many components.

2. GREATEST SEMILATTICE DECOMPOSITION.

For $X \in \mathcal{S}$, define A_X to be the basis of $\langle X-\min(X) \rangle$ and B_X the basis of $\langle \max(X)-X \rangle$. Note that $A_X = B_X = \{0\}$ if and only if X is a singleton. Also observe that A_X is a finite set with at most $a + 1$ elements, where a is the least positive integer in A_X (if $A_X \neq \{0\}$), and similarly for B_X. Since $\gcd(X-\min(X)) = \gcd(X-\max(X))$, it follows that in general $\gcd(A_X) = \gcd(B_X)$.

Given sets A and B, it is clearly not always possible to find an X such that $A_X = A$ and $B_X = B$. However, we do have a positive result. First we need the following lemma.

LEMMA 2.1. *Let* S *be a positive integer semigroup with respect to addition. The following are equivalent.*

(i) S *contains* m *such that* $x \geq m$ *implies* $x \in S$.

(ii) $\gcd(S) = 1$.

(iii) *If* ℓ *is the least element of* S, *then* S *contains*

$c_0,\dots,c_{\ell-1}$ *such that* $c_i \equiv i \pmod{\ell}$ *for* $i \in [0,\ell-1]$.

PROOF. Clearly (i) implies (ii), since if $m, m+1 \in S$, then $\gcd(S) = 1$. Next suppose $\gcd(S) = 1$ and let $B = \{b_1,\dots,b_n\}$ be a basis with $b_1 < \dots < b_n$. If $b_1 = 1$, then evidently (iii) follows. Thus assume $b_1 > 1$. This implies $n > 1$ and hence there exist $x_1,\dots,x_n$ such that $\sum_{i=1}^{n} x_i b_i = 1$. Choose $y_i > 0$ such that $y_i \equiv x_i \pmod{b_1}$ for $i \in [1,n]$. Let $c_0 = b_1$ and for $i \in [1, b_1-1]$ define $c_i = i \sum_{j=1}^{n} y_j b_j$. Note that $c_i \in S$. Furthermore, $c_i \equiv i \pmod{b_1}$; since, $c_0 \equiv 0 \pmod{b_1}$ and for $i \in [1, b_1-1]$:

$$c_i = i\left(\sum_{j=1}^{n} x_j\, b_j + \sum_{j=1}^{n} (y_j - x_j) b_j\right) \equiv i \pmod{b_1}.$$

Therefore (ii) implies (iii). Finally, suppose (iii) holds. Let $m = \max\{c_0,\dots,c_{\ell-1}\}$ and $x \geq m$. There exists an $i \in [0, \ell-1]$ such that $x \equiv i \pmod{\ell}$. Thus $x = c_i + k\ell$ for some $k \in Z$. However, since $x \geq c_i$ this implies $k \geq 0$ and hence $x \in S$. This completes the proof.

PROPOSITION 2.2. *Let* $A = \{a_1,\dots,a_n\}$ *and* $B = \{b_1,\dots,b_m\}$ *be elements of* $\mathcal{S}$ *satisfying*

(i) $a_1 = b_1 = 0$, $a_1 < \dots < a_n$, $b_1 < \dots < b_m$,

(ii) $\gcd(A) = \gcd(B)$,

(iii) $a_i \notin \langle a_1,\dots,a_{i-1}\rangle$, $b_j \notin \langle b_1,\dots,b_{j-1}\rangle$ *for* $i \in [2,n]$, $j \in [2,m]$.

Then there exists an r *such that* $X = A \cup (r-B)$ *is an element of* $\mathcal{S}$ *with* $A_X = A$ *and* $B_X = B$.

PROOF. For the case where $\gcd(A) = 0$, $X = \{r\}$ is an element with $A_X = A$ and $B_X = B$, since necessarily $A = B = \{0\}$. Thus we assume $\gcd(A) > 0$. Let A_1 and B_1 be such that $A = g^*A_1$ and $B = g^*B_1$, where $g = \gcd(A)$. Since $\gcd(A_1) = \gcd(B_1) = 1$, there exists a positive integer q such that $s \in \langle A_1\rangle$ and $s \in \langle B_1\rangle$ for all $s \geq q$. Let $p = q + \max\{\max(A_1), \max(B_1)\}$. Then $p-a \in \langle B_1\rangle$ and $p-b \in \langle A_1\rangle$ for all $a \in A_1$, $b \in B_1$. Hence, if $r = gp$, then $r-a \in \langle B\rangle$ and $r-b \in \langle A\rangle$ for all $a \in A$, $b \in B$. Since $r > \max\{a_n, b_m\}$ it follows that $X = A \cup (r-B) \subset \langle A\rangle$ and $\max(X)-X = B \cup (r-A) \subset \langle B\rangle$. By the definition of A and B, evidently $A_X = A$ and $B_X = B$.

The next result is the key theorem which gives a necessary and sufficient condition for two elements of $\mathcal{S}$ to be in the same archimedean component.

THEOREM 2.3. <u>For</u> $X, Y \in \mathcal{S}$, $\mathcal{A}(X) = \mathcal{A}(Y)$ <u>if and only if</u> $A_X = A_Y$ <u>and</u> $B_X = B_Y$.

PROOF. Suppose $X, Y \in \mathcal{S}$ with $A_X = A_Y$ and $B_X = B_Y$. Without loss of generality, assume $\min(X) = \min(Y) = 0$ and $\max(X) = \max(Y)$. If $\gcd(A_X) = 0$, then X and Y are singletons and thus $\mathcal{A}(X) = \mathcal{A}(Y)$. So assume $\gcd(A_X) > 0$. Let U and V be such that $X = g*U$ and $Y = g*V$, where $g = \gcd(A_X)$. Note that $\gcd(A_U) = 1$. Let a and b be the least positive integers in A_U and B_U, respectively. Define $A_i = \{x \in \langle A_U\rangle : x \equiv i \pmod a\}$ and $B_j = \{x \in \langle B_U\rangle : x \equiv j \pmod b\}$ for $i \in [0, a-1]$, $j \in [0, b-1]$. Also, define $c_i = \min(A_i)$, $d_i = \min(B_i)$, $c = \max\{c_i : i \in [0, a-1]\}$, and $d = \max\{d_i : i \in [0, b-1]\}$. Choose $m, r \in Z_+$ such that

(i) $\max\{c,d\} + \max(U) \leq (m+1)\min\{a,b\}$,

(ii) $c_i \in rU$ for all $i \in [0, a-1]$,

(iii) $d_i \in r(\max(U)-U)$ for all $i \in [0, b-1]$.

Finally, let $n = m+r$.

By the definition of n, evidently

$$\bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1, ma]$$

$$\subset \bigcup_{j=0}^{m} \bigcup_{i=0}^{a-1} \{c_i + ja\} \subset nU$$

and similarly

$$\bigcup_{i=0}^{b-1} \{x \in B_i : x < d-b\} \cup [d-b+1, mb] \subset n(\max(U)-U).$$

Also, observe that $c - a \notin sU$ and $d - b \notin s(\max(U)-U)$ for all $s \in Z_+$ (by definition). Since $c + \max(U) \leq (m+1)a$ and $d + \max(U) \leq (m+1)b$, it follows that for all $p \geq 0$

$$\bigcup_{i=0}^{p} [c-a+1 + i\max(U), ma + i\max(U)]$$

$$= [c-a+1, ma + p\max(U)] \subset (n+p)U$$

and similarly

$$[d-b+1,\ mb + p\max(U)] \subset (n+p)(\max(U)-U).$$

Thus, for all $q \geq n$

$$[c-a+1,\ ma + (q-n)\max(U)] \cup [n\max(U) - mb,\ q\max(U) + b-d-1] \subseteq qU.$$

In particular,

$$[c-a+1,\ ma + n\max(U)] \cup [n\max(U) - mb,\ 2n\max(U) + b-d-1] = [c-a+1,\ 2n\max(U) + b-d-1] \subseteq 2nU.$$

It is clear that if $u \in qU$ with $u < c-a$ and $q \geq n$, then

$$u \in \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\}.$$

Likewise if $u \in qU$ with $u > q\max(U) + b-d$ and $q \geq n$, then

$$u \in \bigcup_{i=0}^{b-1} \{q\max(U) - x : x \in B_i,\ x < d-b\}.$$

Hence

$$2nU = \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1,\ 2n\max(U) + b-d-1] \cup \bigcup_{i=0}^{b-1} \{2n\max(U)-x : x \in B_i,\ x < d-b\} = 2nV.$$

Therefore, $2nX = 2nY$ and $\mathfrak{a}(X) = \mathfrak{a}(Y)$.

Conversely, suppose $\mathfrak{a}(X) = \mathfrak{a}(Y)$. Then there exist $S, T \in \mathbf{S}$ and $s, t \in Z_+$ such that

$$s(X-\min(X)) = Y-\min(Y)+S \text{ and}$$
$$t(Y-\min(Y)) = X-\min(X) + T.$$

Since necessarily $\min(S) = \min(T) = 0$, it follows that

$$A_Y \subseteq Y-\min(Y) + S \subseteq \langle A_X\rangle$$

and similarly $A_X \subseteq \langle A_Y\rangle$. Consequently, $\langle A_X\rangle = \langle A_Y\rangle$ and hence by the definition of A_X and A_Y we have $A_X = A_Y$. Similarly it is easy to show $B_X = B_Y$ and this completes the proof.

Perhaps a brief example will help illustrate the simplicity of the condition given in Theorem 2.3. Let $W = \{-10, -8, 22, 55, 57\}$, $X = \{3, 5, 29, 68, 69\}$, and $Y = \{4, 6, 69, 85, 86\}$. Then

$$W\text{-min}(W) = \{0, 2, 32, 65, 67\}, \max(W)\text{-}W = \{0, 2, 35, 65, 67\},$$
$$X\text{-min}(X) = \{0, 2, 26, 65, 66\}, \max(X)\text{-}X = \{0, 1, 40, 64, 66\},$$
$$Y\text{-min}(Y) = \{0, 2, 65, 81, 82\}, \max(Y)\text{-}Y = \{0, 1, 17, 80, 82\}.$$

Hence, $A_W = A_X = A_Y = \{0, 2, 65\}$, $B_W = \{0, 2, 35\}$, and $B_X = B_Y = \{0,1\}$. Therefore $\mathcal{A}(X) = \mathcal{A}(Y)$ and $\mathcal{A}(W) \neq \mathcal{A}(X)$. Actually, $\mathcal{A}(X) \leq \mathcal{A}(W)$ by our next theorem.

Using Theorem 2.3 we can determine when two archimedean components are related with respect to the order on the semilattice.

THEOREM 2.4. The following are equivalent.

(i) $\mathcal{A}(X) \leq \mathcal{A}(Y)$.

(ii) $A_Y \subseteq \langle A_X \rangle$ and $B_Y \subseteq \langle B_X \rangle$.

(iii) $A_{X+Y} = A_X$ and $B_{X+Y} = B_X$.

PROOF. Suppose $\mathcal{A}(X) \leq \mathcal{A}(Y)$. There exist $U \in \mathcal{S}$ and $n \in Z_+$ such that

$$n(X\text{-min}(X)) = Y\text{-min}(Y) + U.$$

Since $\min(U) = 0$,

$$A_Y \subseteq Y\text{-min}(Y) + U \subseteq \langle A_X \rangle.$$

Similarly $B_Y \subseteq \langle B_X \rangle$. Suppose next that assertion (ii) holds. Then

$$Y\text{-min}(Y) \subseteq \langle A_Y \rangle \subseteq \langle A_X \rangle$$

and thus

$$X + Y - \min(X+Y) = A_X \cup X_1$$

where $X_1 \subseteq \langle A_X \rangle$. Hence $A_{X+Y} = A_X$. Likewise $B_{X+Y} = B_X$. Finally, if (iii) holds, then by Theorem 2.3 $X+Y \in \mathcal{A}(X)$; that is, $\mathcal{A}(X) \leq \mathcal{A}(Y)$ and the proof is complete.

Observe that clearly $A_Y \subseteq A_X$ and $B_Y \subseteq B_X$ is a sufficient condition for $\mathcal{A}(X) \leq \mathcal{A}(Y)$. However, it is not a necessary condition (see Spake [4]). Since A_Y and B_Y are finite sets, it is relatively easy to determine when $\mathcal{A}(X) \leq \mathcal{A}(Y)$ via Theorem 2.4 (ii). Also, as the trivial case of Theorem 2.4, we have $\mathcal{A}(0,1) \leq \mathcal{A}(X) \leq \mathcal{A}(0)$ for all $X \in \mathcal{S}$ and hence $\mathcal{A}(0,1)$ is an ideal of $\mathcal{S}$.

Define $\mathcal{A}_0(X) = \{Y \in \mathcal{A}(X) : \min(Y) = 0\}$ and note that $\mathcal{A}_0(X)$ is a subsemigroup of $\mathcal{A}(X)$. Moreover, since elements of $\mathcal{A}(X)$ can be uniquely expressed in the form $U + a$, where $U \in \mathcal{A}_0(X)$ and $a \in Z$, evidently $\mathcal{A}(X) \cong \mathcal{A}_0(X) \oplus Z$. Recalling the proof of Theorem 2.3, apparently if X is a non-singleton, then $\mathcal{A}_0(X)$ is power joined. We therefore immediately have

THEOREM 2.5. _The idempotent-free archimedean component_ $\mathcal{A}(X)$, _where_ X _is a non-singleton, is isomorphic to the direct product of the idempotent-free power joined subsemigroup_ $\mathcal{A}_0(X)$ _and the group_ Z.

We complete this section with a brief summary of the greatest semilattice decomposition of $\mathcal{S}$. Let

$$W = \{((a_1,\ldots,a_n;\ b_1,\ldots,b_m)) : a_i,\ b_j \in Z,\ 0 = a_1<\ldots<a_n,\ 0 = b_1<\ldots<b_m,$$
$$\gcd(a_1,\ldots,a_n) = \gcd(b_1,\ldots,b_m),$$
$$a_i \notin \langle a_1,\ldots,a_{i-1}\rangle \text{ and } b_j \notin \langle b_1,\ldots,b_{j-1}\rangle,$$
$$\text{for } i \in [2,n],\ j \in [2,m]\}.$$

Define a partial order $\leq$ on W as follows:

$$((a_1,\ldots,a_n;\ b_1,\ldots,b_m)) \leq ((c_1,\ldots,c_p;\ d_1,\ldots,d_q)) \text{ if and only if}$$
$$\{c_1,\ldots,c_p\} \subseteq \langle a_1,\ldots,a_n\rangle \text{ and } \{d_1,\ldots,d_q\} \subseteq \langle b_1,\ldots,b_m\rangle.$$

Also, define the map $\Phi : \mathcal{S} \to W$ by $\Phi(X) = ((a_1,\ldots,a_n;\ b_1,\ldots,b_m))$ where $\{a_1,\ldots,a_n\} = A_X$ and $\{b_1,\ldots,b_m\} = B_X$. Using our preceding results we have the following theorem.

THEOREM 2.6. _The map_ Φ _is the greatest semilattice homomorphism of_ $\mathcal{S}$, _with_ W _being the greatest semilattice homomorphic image. Moreover, if_ $\Phi(X) = ((a_1,\ldots,a_n;\ b_1,\ldots,b_m))$ _and_ $\Phi(Y) = ((c_1,\ldots,c_p;\ d_1,\ldots,d_q))$, _then_ $\Phi(X) \leq \Phi(Y)$ _if and only if_ $\{c_1,\ldots,c_p\} \subseteq \langle a_1,\ldots,a_n\rangle$ _and_ $\{d_1,\ldots,d_q\} \subseteq \langle b_1,\ldots,b_m\rangle$.

We further define two congruences δ and ζ on $\mathcal{S}$ as follows:

$X\ \delta\ Y$ if and only if $X = Y + z$ for some $z \in Z$,

$X\ \zeta\ Y$ if and only if $\Phi(X) = \Phi(Y)$ and $\min(X) = \min(Y)$.

Observe that $\mathcal{S}/\delta$ is isomorphic to the subsemigroup $\mathcal{H}$ of $\mathcal{S}$ consisting of X satisfying $\min(X) = 0$ and $\mathcal{H}$ is the semilattice W of $\mathcal{A}_0(A)$'s. Also, $\mathcal{S}/\zeta$ is isomorphic to the direct product $\mathcal{K}$ of W and Z. Next, recall the definition of spined product: if $g_i : S_i \to T$ is a homomorphism of S_i onto T $(i = 1, 2)$, then the spined product of S_1 and S_2 with respect to g_1 and g_2 is $\{(x,y): x \in S_1,\ y \in S_2,\ g_1(x) = g_2(y)\}$ in which $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$.

Using our results we have

THEOREM 2.7. <u>The semigroup</u> $\mathfrak{S}$ <u>is isomorphic to the spined product of</u> $\mathfrak{H}$ <u>and</u> $\mathfrak{K}$ <u>with respect to</u> $\mathfrak{H} \to W$ <u>and</u> $\mathfrak{K} \to W$.

3. STRUCTURE OF THE COMPONENTS.

The structure of $\mathfrak{A}(0)$ is clear, since $\mathfrak{A}(0) \cong Z$. In this section we investigate the structure of $\mathfrak{A}(X)$ when X is a non-singleton. We begin with a general result from Theorem 2.3.

PROPOSITION 3.1. <u>For</u> $X, Y \in \mathfrak{S}$, $Y \in \mathfrak{A}(X)$ <u>if and only if</u>

(i) $Y-\min(Y) = A_X \cup X_1$, <u>where</u> $X_1 \subseteq \langle A_X \rangle$; <u>and</u>

(ii) $\max(Y)-Y = B_X \cup X_2$, <u>where</u> $X_2 \subseteq \langle B_X \rangle$.

For $X = \{x_1,\dots,x_n\} \in \mathfrak{S}$, where $n > 1$ and $x_1 < \dots < x_n$, define $id(X) = x_2 - x_1$ and $fd(X) = x_n - x_{n-1}$. Notice that $id(X)$ and $fd(X)$ are the least positive integers in A_X and B_X, respectively. Recalling the proof of Theorem 2.3, we evidently have

THEOREM 3.2. <u>Let</u> X <u>be a non-singleton and</u> U <u>be such that</u> $X-\min(X) = g*U$, <u>where</u> $g = \gcd(A_X)$. <u>Define</u> $A_i = \{x \in \langle A_U \rangle : x \equiv i \pmod a\}$ <u>and</u> $B_j = \{x \in \langle B_U \rangle : x \equiv j \pmod b\}$ <u>for</u> $i \in [0, a-1]$, $j \in [0, b-1]$, <u>where</u> $a = id(U)$ <u>and</u> $b = fd(U)$. <u>Let</u> $c = \max\{\min(A_i) : i \in [0, a-1]\}$ <u>and</u> $d = \max\{\min(B_i) : i \in [0, b-1]\}$. <u>Then</u> $Y \in \mathfrak{A}(X)$ <u>if and only if there exist</u> $V \in \mathfrak{S}$ <u>and</u> $n_0 \in Z_+$ <u>such that</u> $Y-\min(Y) = g*V$ <u>and for all</u> $n \geq n_0$

$$nV = \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1, n\max(V) + b-d-1]$$
$$\cup \bigcup_{i=0}^{b-1} \{n\max(V) - x : x \in B_i,\ x < d-b\}.$$

Next we reproduce several definitions and facts from Tamura [5] that we will need in the following development. We direct the reader to [5] for a more complete discussion of the notions which follow. Let T be an additively denoted commutative idempotent-free archimedean semigroup. Define a congruence ρ_b on T, for fixed b, as

$$x\ \rho_b\ y \text{ if and only if } nb + x = mb + y \text{ for some } n, m \in Z_+.$$

Then $T/\rho_b = G_b$ is a group called the structure group of T determined by the standard element b. Also, define a compatible partial order $<$ on T as follows:

$$x \underset{b}{<} y \text{ if and only if } x = nb + y \text{ for some } n \in Z_+.$$

Then $T = \bigcup_{\lambda \in G_b} T_\lambda$, equivalently $T/\rho_b = \{T_\lambda\}$, $\lambda \in G_b$, where each T_λ is a lower semilattice with respect to $\underset{b}{<}$. In fact, for each $\lambda \in G_b$, T_λ forms a discrete tree without smallest element with respect to $\underset{b}{<}$, (a discrete tree, with respect to $\underset{b}{<}$, is a lower semilattice such that for any $c \underset{b}{<} d$ the set $\{x : c \underset{b}{<} x \underset{b}{<} d\}$ is a finite chain). Finally, we define a relation η on T as follows:

$$x \,\eta\, y \text{ if and only if } nb + x = nb + y \text{ for some } n \in Z_+.$$

The relation η is the smallest cancellative congruence on T.

We continue our development with the following theorem.

THEOREM 3.3. <u>Let</u> $A \in \mathcal{S}$ <u>be a non-singleton with</u> $\min(A) = 0$ <u>and</u> $g = \gcd(A)$. <u>The structure group of</u> $\mathcal{A}_0(A)$ <u>determined by the standard element</u> A <u>is</u> Z_m, <u>where</u> $m = \max(A)/g$. <u>Moreover</u>, $\mathcal{A}_0(A) = \bigcup_{i=0}^{m-1} \mathcal{A}_i$ <u>where</u> $\mathcal{A}_i = \{X \in \mathcal{A}_0(A) : \max(X)/g \equiv i \pmod m\}$ <u>is a discrete tree without smallest element with respect to</u> $\underset{A}{<}$. <u>Furthermore, the structure group of</u> $\mathcal{A}(A)$ <u>determined by the standard element</u> A <u>is</u> $Z \oplus Z_m$.

PROOF. Let $U, V \in \mathcal{A}_0(A)$ and C, U_1, V_1 be such that $A = g{*}C$, $U = g{*}U_1$, and $V = g{*}V_1$, where $g = \gcd(A)$. For $i \in [0, a-1]$, $j \in [0, b-1]$, where $a = id(C)$ and $b = fd(C)$, define $A_i = \{x \in \langle A_C\rangle : x \equiv i \pmod a\}$ and $B_j = \{x \in \langle B_C\rangle : x \equiv j \pmod b\}$. Also, let $c = \max\{\min(A_i) : i \in [0, a-1]\}$ and $d = \max\{\min(B_i) : i \in [0, b-1]\}$.

Suppose $\max(U_1) \equiv \max(V_1) \pmod m$, where $m = \max(C)$. Without loss of generality, assume $\max(U_1) = \max(V_1) + km$ with $k \geq 0$. There exists $p \geq c+d + \max(U_1)$ such that

$$pC = \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1,\ pm+b-d-1]$$
$$\cup \bigcup_{i=0}^{b-1} \{pm - x : x \in B_i,\ x < d-b\}.$$

Since $U_1 \in \mathcal{O}_0(C)$, it follows that

$$U_1 \subseteq \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1, \max(U_1) + b-d-1]$$
$$\cup \bigcup_{i=0}^{b-1} \{\max(U_1) - x : x \in B_i, x < d-b\},$$

and similarly for V_1. Hence

$$U_1 + pC = \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1, pm + \max(U_1) + b-d-1]$$
$$\cup \bigcup_{i=0}^{b-1} \{pm + \max(U_1) - x : x \in B_i, x < d-b\}$$
$$= \bigcup_{i=0}^{a-1} \{x \in A_i : x < c-a\} \cup [c-a+1, (p+k)m + \max(V_1) + b-d-1]$$
$$\cup \bigcup_{i=0}^{b-1} \{(p+k)m + \max(V_1) - x : x \in B_i, x < d-b\}$$
$$= V_1 + (p+k)C.$$

Consequently, $U + pA = V + (p+k)A$.

Conversely, if $U + rA = V + sA$ for some $r, s \in Z_+$, then

$$\max(U) + rgm = \max(V) + sgm.$$

Since $g|\max(U)$ and $g|\max(V)$, evidently $\max(U)/g \equiv \max(V)/g \pmod m$. By Proposition 3.1, if $t \geq \max\{\max(A_C) + d-b+1, \max(B_C) + c-a+1\}$ and $t \in Z_+$, then $X = A_C \cup (t - B_C) \in \mathcal{O}_0(C)$ with $\max(X) = t$. It follows that for each $i \in Z_m$, there exists $X \in \mathcal{O}_0(A)$ with $\max(X)/g \equiv i \pmod m$. Therefore, the structure group of $\mathcal{O}_0(A)$ determined by the standard element A is Z_m.

Using the above, it is clear that for $X, Y \in \mathcal{O}(A)$,

$X + rA = Y + sA$ for some, $r, s \in Z_+$ if and only if $\min(X) = \min(Y)$ and $(\max(X)-\min(X))/g \equiv (\max(Y)-\min(Y))/g \pmod m$.

This completes the proof.

We conclude this paper with two related propositions.

PROPOSITION 3.4. *Let* X *be a non-singleton. The homomorphism* $h : \mathcal{A}_0(X) \to Z_+$ *defined by* $h(U) = \max(U)$ *is the greatest cancellative homomorphism. That is, the relation* η *on* $\mathcal{A}_0(X)$ *defined by*

$$U \;\eta\; V \text{ if and only if } \max(U) = \max(V)$$

is the smallest cancellative congruence. Furthermore, the relation σ *on* $\mathcal{A}(X)$ *defined by*

$$U \;\sigma\; V \text{ if and only if } \min(U) = \min(V) \text{ and } \max(U) = \max(V)$$

is the smallest cancellative congruence. The semigroups $\mathcal{A}_0(X)/\eta$ *and* $\mathcal{A}(X)/\sigma$ *are* $\mathfrak{N}$-*semigroups*.

PROPOSITION 3.5. *Let* X *be a non-singleton and* U *be such that* $X - \min(X) = g{*}U$, *where* $g = \gcd(A_X)$. *For* $i \in [0, a-1]$, $j \in [0, b-1]$, *where* $a = id(U)$ *and* $b = fd(U)$, *define* c_i *and* d_j *to be the least integers in* $\langle A_U \rangle$ *and* $\langle B_U \rangle$, *respectively, such that* $c_i \equiv i \pmod a$ *and* $d_j \equiv j \pmod b$. *Let* $c = \max\{c_i : i \in [0, a-1]\}$, $d = \max\{d_i : i \in [0, b-1]\}$, $m = \max\{\max(A_U), \max(B_U)\}$, *and* $p = \max\{\max(A_U) + d-b+1, \max(B_U) + c-a+1\}$. *Then the greatest cancellative homomorphic image of* $\mathcal{A}_0(X)$ *is isomorphic to the following positive integer semigroup*:

$$\begin{aligned}\mathcal{C} = \{r \in [m, p-2] : \;& \text{for all } x \in A_U,\ y \in B_U,\ \text{if } r-x \equiv j \pmod b,\\ & \text{for some } j \in [0, b-1],\ \text{then } r \geq x+d_j \text{ and}\\ & \text{if } r-y \equiv i \pmod a,\ \text{for some } i \in [0, a-1],\\ & \text{then } r \geq y + c_i\}\\ \cup \{r \in Z : r \geq p\},&\end{aligned}$$

(*where if* $[m, p-2]$ *is not defined then* $\mathcal{C} = \{r \in Z : r \geq p\}$).

PROOF. First, observe that $\mathcal{A}_0(X)$ and $\mathcal{A}_0(U)$ have isomorphic greatest cancellative homomorphic images, since $\mathcal{A}_0(X) \cong \mathcal{A}_0(U)$. Let $V \in \mathcal{A}_0(U)$ and $t = \max(V)$. Since $A_U \subseteq V$ and $B_U \subseteq t - V$, it follows that $t \geq \max\{\max(A_U), \max(B_U)\}$. Moreover, using Proposition 3.1, $t - x \in \langle B_U \rangle$ and $t - y \in \langle A_U \rangle$ for all $x \in A_U$, $y \in B_U$. Thus, by the definition of c_i and d_j, $t \in \mathcal{C}$. Furthermore, if $r \in \mathcal{C}$ then evidently $A_U \cup (r-B_U) \in \mathcal{A}_0(U)$. Consequently, the proof is complete by Proposition 3.4.

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