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K-COMPONENT DISCONJUGACY FOR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT: Disconjugacy of the kth component of the mth order system of nth order differential equations $Y^{(n)} = f(x, Y, Y', \ldots, Y^{(n-1)})$, (1.1), is defined, where $f(x, Y_1, \ldots, Y_n)$, $\frac{\partial f}{\partial y_{ij}}(x, Y_1, \ldots, Y_n)$: $(a,b) \times \mathbb{R}^{mn} \to \mathbb{R}^m$ are continuous. Given a a solution $Y_0(x)$ of (1.1), k-component disconjugacy of the variational equation $Z^{(n)} = \sum_{i=1}^{n} f_{Y_i}(x, Y_0(x), \ldots, Y_0^{(n-1)}(x))\, Z^{(i-1)}$, (1.2), is also studied. Conditions are given for continuous dependence and differentiability of solutions of (1.1) with respect to boundary conditions, and then intervals on which (1.1) is k-component disconjugate are characterized in terms of intervals on which (1.2) is k-component disconjugate.

KEY WORDS AND PHRASES. System of differential equations, variational equation, k-component disconjugacy (right disfocality), continuity and differentiability with respect to boundary conditions.
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1. INTRODUCTION.

In the past several years, a number of results have been proven concerning the disconjugacy of an nth order scalar ordinary differential equation when certain disconjugacy assumptions are made for the corresponding linear variational equation. In this paper we investigate similar concepts for systems of ordinary differential equations. In particular, we shall be concerned with solutions of boundary value problems for the mth order system of nth order differential equations

$$Y^{(n)} = f(x, Y, Y', \ldots, Y^{(n-1)}),$$

where we assume throughout:

(A) $f(x, Y_1, \ldots, Y_n)$: $(a,b) \times \mathbb{R}^{mn} \to \mathbb{R}^m$ is continuous;

(B) $\frac{\partial f}{\partial y_{ij}}(x, Y_1, \ldots, Y_n)$: $(a,b) \times \mathbb{R}^{mn} \to \mathbb{R}^m$, $1 \le j \le m$, $1 \le i \le n$, are continuous, where $Y_i = (y_{i1}, \ldots, y_{im})$; (Note: If $Y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, then $Y_{\hat{k}}$ will will denote the (m-1)-tuple $(y_1, \ldots, y_{k-1}, y_{k+1}, \ldots, y_m)$.)

(C) Solutions of (1) extend to (a,b);

and (D) If there exist a sequence of solutions $\{Y_r(x)\}$ of (1.1), a point $x_0 \in (a,b)$ a compact subinterval $[c,d] \subset (a,b)$, $M > 0$, and $1 \leq k \leq m$ such that $(Y_\mu)_{\hat{k}}^{(i-1)}(x_0) = (Y_\nu)_{\hat{k}}^{(i-1)}(x_0)$, $1 \leq i \leq n$, for all $\mu, \nu \in \mathbb{N}$, and $|y_{\mu k}(x)| \leq M$ on $[c,d]$, for all $\mu \in \mathbb{N}$, then there is a subsequence $\{Y_{r_j}(x)\}$ such that $\{y_{r_j k}^{(i-1)}(x)\}$ converges uniformly on each compact subinterval of (a,b), for $1 \leq i \leq n$.

Given a solution $Y_0(x)$ of (1.1), we will also be concerned with solutions of the linear mth order system of nth order equations called the variational equation along $Y_0(x)$ and given by

$$Z^{(n)} = \sum_{i=1}^{n} f_{Y_i}(x, Y_0(x), Y_0'(x), \ldots, Y_0^{(n-1)}(x))\, Z^{(i-1)}, \tag{1.2}$$

where f_{Y_i}, $1 \leq i \leq n$, denotes the $m \times m$ Jacobian matrix $\left[\frac{\partial f_k}{\partial y_{ij}}\right]$, $1 \leq k, j \leq m$.

Rather than with the disconjugacy of (1.1), we will be concerned with the disconjugacy of one of the components of the system (1.1). Motivation for our considerations here are papers by Peterson [1-2], Spencer [3], and Sukup [4].

DEFINITION. Let $1 \leq k \leq m$ be given. We say that (1.1) is <u>k-component disconjugate on (a,b)</u>, if given $2 \leq q \leq n$, points $a < x_1 < \ldots < x_q < b$, $\alpha \in (a,b)$, positive integers $m_1, \ldots, m_q$ partitioning n, and solutions $Y(x)$ and $Z(x)$ of (1.1) satisfying

$$Y_{\hat{k}}^{(i-1)}(\alpha) = Z_{\hat{k}}^{(i-1)}(\alpha),\ 1 \leq i \leq n,$$

and
$$y_k^{(i)}(x_j) = z_k^{(i)}(x_j),\ 0 \leq i \leq m_j - 1,\ 1 \leq j \leq q,$$

it follows that $y_k(x) \equiv z_k(x)$.

Given a solution $Y_0(x)$ of (1.1), k-component disconjugacy of (1,2) along $Y_0(x)$ is defined similarly.

In this paper, we first show that if the system (1.1) is k-component disconjugate, for some $1 \leq k \leq m$, then solutions of certain boundary value problems for (1.1) can be differentiated with respect to boundary conditions. The resulting partial derivatives as functions of x are solutions of related boundary value problems for the system (1.2). The main results of this paper appear in section 3, where we show that intervals on which (1.1) is k-component disconjugate can be characterized in terms of intervals on which (1.2) is k-component disconjugate. Then in our last section, we state without proof some analogues of the results in section 3 in terms of k-component right disfocality for the system (1.1).

2. CONTINUITY AND DIFFERENTIABILITY WITH RESPECT TO BOUNDARY CONDITIONS.

Our first result is concerned with the continuous dependence of solutions of (1.1) on boundary conditions. Its proof is a standard application of the Brouwer Invariance of Domain Theorem.

THEOREM 1. Assume that for some $1 \leq k \leq m$, the system (1.1) is k-component disconjugate on (a,b). Let $Y(x)$ be a solution of (1.1). Given $2 \leq q \leq n$, points $a < x_1 < \ldots < x_q < b$, $\alpha \in (a,b)$, positive integers $m_1, \ldots, m_q$ partitioning n, and $\varepsilon > 0$, there exists $\delta > 0$ such that $|t_j - x_j| < \delta$, $1 \leq j \leq q$, $\| Y_{\hat{k}}^{(i-1)}(\alpha) - (V_i)_{\hat{k}} \| < \delta$, $1 \leq i \leq n$, $|y_k^{(i)}(x_j) - c_{ij}| < \delta$, $0 \leq i \leq m_j - 1$, $1 \leq j \leq q$, imply there exists a unique solution $Z_\delta(x)$ of (1.1) satisfying $(Z_\delta)_{\hat{k}}^{(i-1)}(\alpha) = (V_i)_{\hat{k}}$, $1 \leq i \leq n$, $z_{\delta k}^{(i)}(t_j) = c_{ij}$, $0 \leq i \leq m_j - 1$, $1 \leq j \leq q$, and $\lim_{\delta \to 0^+} Z_\delta^{(i)}(x) = Y^{(i)}(x)$ uniformly on each compact subinterval of (a,b), for $0 \leq i \leq n-1$.

In addition to the continuous dependence in Theorem 1, connectedness properties have played an important role in establishing disconjugacy or disfocality results in the papers of Henderson [5], Peterson [2], and Sukup [4].

THEOREM 2. Assume that for some $1 \leq k \leq m$, the system (1.1) is k-component disconjugate on (a,b). Let $Y(x)$ be a solution of (1.1) and let q, $x_1, \ldots, x_q$, α, and $m_1, \ldots, m_q$ be as in Theorem 1. Then, for $1 \leq p \leq q$, the set $S_p = \{v_k^{(m_p-1)}(x_p) \mid V(x)$ is a solution of (1.1), $V_{\hat{k}}^{(i-1)}(\alpha) = Y_{\hat{k}}^{(i-1)}(\alpha)$, $1 \leq i \leq n$, $v_k^{(i)}(x_j) = y_k^{(i)}(x_j)$, $0 \leq i \leq m_j-1$, $1 \leq j \leq q$, $j \neq p$, and $v_k^{(i)}(x_p) = y_k^{(i)}(x_p)$, $0 \leq i \leq m_p - 2\}$ is an open interval.

PROOF. It follows immediately from Theorem 1 that S_p is open. It suffices to show that if $\tau_0 = \sup \{\tau \mid [y_k^{(m_p-1)}(x_p), \tau] \subset S_p\}$ and $\tau' > \tau_0$, then $\tau' \notin S_p$, and if $\sigma_0 = \inf \{\sigma \mid [\sigma, y_k^{(m_p-1)}(x_p)] \subset S_p\}$ and $\sigma' < \sigma_0$, then $\sigma' \notin S_p$. We will make the the argument for the first case. We suppose that there exists $\tau' > \tau_0$ and $\tau' \in S_p$. Then there is a solution $V(x)$ of (1.1) satisfying
$V_{\hat{k}}^{(i-1)}(\alpha) = Y_{\hat{k}}^{(i-1)}(\alpha)$, $1 \leq i \leq n$, $v_k^{(i)}(x_j) = y_k^{(i)}(x_j)$, $0 \leq i \leq m_j-1$, $1 \leq j \leq q$, $j \neq p$, $v_k^{(i)}(x_p) = y_k^{(i)}(x_p)$, $0 \leq i \leq m_p-2$, and $v_k^{(m_p-1)}(x_p) = \tau'$.

Now, from the definition of τ_0, there exists a strictly monotone increasing sequence $\{\tau_\mu\} \subset S_p$ such that $y_k^{(m_p-1)}(x_p) < \tau_\mu < \tau_0$ and $\tau_\mu \uparrow \tau_0$. Let $\{W_\mu\}$ be the corresponding sequence of solutions (1.1) satisfying $(W_\mu)_{\hat{k}}^{(i-1)}(\alpha) = Y_{\hat{k}}^{(i-1)}(\alpha)$, $1 \leq i \leq n$, $w_{\mu k}^{(i)}(x_j) = y_k^{(i)}(x_j)$, $0 \leq i \leq m_j-1$, $1 \leq j \leq q$, $j \neq p$, $w_{\mu k}^{(i)}(x_p) = y_k^{(i)}(x_p)$, $0 \leq i \leq m_p - 2$, and $w_{\mu k}^{(m_p-1)}(x_p) = \tau_\mu$. Now if for some $\varepsilon > 0$, $\{w_{\mu k}^{(m_p-1)}(x)\}$ is uniformly bounded on $[x_p, x_p + \varepsilon]$, it follows from the boundary conditions that $\{w_{\mu k}(x)\}$ is uniformly bounded on $[x_p, x_p + \varepsilon]$. By condition (D), there exists a subsequence $\{W_{\mu_j}(x)\}$ such that $\{w_{\mu_j k}^{(i-1)}(x)\}$ converges uniformly on compact subintervals of (a,b), for $1 \leq i \leq n$. In particular, this convergence is uniform on any compact subinterval containing α, and consequently, the subsequence $\{W_{\mu_j}(x)\}$ converges uniformly to a solution $W_0(x)$ of (1.1) on compact subintervals of (a,b). Thus, it follows that $\tau_0 \in S_p$, which is contradictory to the fact that S_p is open. Hence, given $\varepsilon > 0$, $\{w_{\mu k}^{(m_p-1)}(x)\}$ is not uniformly bounded on $[x_p, x_p + \varepsilon]$.

Since, for each $\mu \in N$, $y_k^{(m_p-1)}(x_p) < w_{\mu k}^{(m_p-1)}(x_p) < v_k^{(m_p-1)}(x_p)$, it follows that there exists a sequence $\{\delta_j\}$ with $\delta_j \downarrow 0$ such that, either

(i) $w_{\mu_j k}^{(m_p-1)}(x_p+\delta_j) = y_k^{(m_p-1)}(x_p+\delta_j)$ and $y_k^{(m_p-1)}(x) < w_{\mu_j k}^{(m_p-1)}(x) < v_k^{(m_p-1)}(x)$, on $(x_p, x_p + \delta_j)$, for all j,

or (ii) $w_{\mu_j k}^{(m_p-1)}(x_p+\delta_j) = v_k^{(m_p-1)}(x_p+\delta_j)$ and $y_k^{(m_p-1)}(x) < w_{\mu_j k}^{(m_p-1)}(x) < v_k^{(m_p-1)}(x)$, on $(x_p, x_p + \delta_j)$, for all j. We then have by continuity that

$\lim_{j\to\infty} w_{\mu_j k}^{(i)}(x_p + \delta_j) = y_k^{(i)}(x_p)$, $0 \le i \le m_p-2$,

and hence by Theorem 1, $\{w_{\mu_j k}^{(i)}(x)\}$ converges to $y_k^{(i)}(x)$ uniformly on each compact subinterval of (a,b), for $0 \le i \le n-1$; a contradiction. This completes the proof.

Our next result deals with differentiation of solutions of (1.1) with respect to boundary conditions in the presence of k-component disconjugacy. The proof follows along the lines of those given in Henderson [5-6] and Peterson [1] and we will omit it here.

THEOREM 3. Let $1 \le k \le m$ be given and assume that (1.1) and the variational equation (1.2) along all solutions $Y(x)$ of (1.1) is k-component disconjugate on (a,b). Let $Y(x)$ be a solution of (1.1), and let $2 \le q \le n$, points $a < x_1 < \dots < x_q < b$, $\alpha \in (a,b)$, and positive integers $m_1, \dots, m_q$ partitioning n be given. For $1 \le p \le q$, let S_p be as in Theorem 2, and for each $s \in S_p$, let $V(x,s)$ denote the corresponding solution of (1.1) where $v_k^{(m_p-1)}(x_p,s) = s$. Then $\frac{\partial V}{\partial s}(x,s)$ exists and $Z(x,s) \equiv \frac{\partial V}{\partial s}(x,s)$ is the solution of (1.2) along $V(x,s)$ and satisfies the boundary conditions

$$Z_{\hat{k}}^{(i-1)}(\alpha) = 0,\ 1 \le i \le n,$$
$$z_k^{(i)}(x_j) = 0,\ 0 \le i \le m_j - 1,\ 1 \le j \le q,\ j \ne p,$$
$$z_k^{i}(x_p) = 0,\ 0 \le i \le m_p-2,$$
$$z_k^{(m_p-1)}(x_p) = 1.$$

3. INTERVALS OF K-COMPONENT DISCONJUGACY.

In this section, we determine subintervals of (a,b) on which (1.1) is k-component disconjugate in terms of subintervals on which (1.2) is k-component disconjugate. Our arguments for this characterization are much like those in Peterson [2] and Spencer [3].

For notational purposes, given $\alpha \in (a,b)$, let $Y(x; \alpha, V_1, \dots, V_n)$ denote the solution of the initial value problem for (1.1) satisfying $Y^{(i-1)}(\alpha) = V_i = (v_{i1}, \dots, v_{im})$, $1 \le i \le n$. Then, under our assumptions (A) - (D), for each

$1 \le \mu \le n$ and $1 \le \nu \le m$, $U_{\mu\nu}(x;\alpha, V_1,\ldots, V_n) \equiv \frac{\partial Y}{\partial v_{\mu\nu}}(x;\alpha, V_1, \ldots, V_n)$ exists and is a solution of (1.2) along $Y(x;\alpha, V_1, \ldots, V_n)$ satisfying $U_{\mu\nu}^{(i-1)}(\alpha) = 0$, $= 0$, $1 \le i \le n$, $i \ne \mu$, $U_{\mu\nu}^{(\mu-1)}(\alpha) = e_\nu = (\delta_{1\nu}, \ldots, \delta_{m\nu})$.

DEFINITIONS. Let $1 \le k \le m$ be given and let $t \in (a,b)$.

(i) Define $\eta_1^k(t) = \inf \{t_1 \in (t,b) \mid$ (1.1) is not k-component disconjugate on $[t,t_1]\}$. If (1.1) is k-component disconjugate on $[t,b)$, we set $\eta_1^k(t) = b$.

(ii) Given a solution $Y_0(x)$ of (1.1), define $\eta_1^k(t; Y_0(x)) = \inf\{t_1 \in (t,b) \mid$ (1.2) is not k-component disconjugate along $Y_0(x)$ on $[t,t_1]\}$.

The main result of this paper is that $\eta_1^k(t) = \inf_{Y_0(x)} \{\eta_1^k(t; Y_0(x))\}$ which will be established in two parts. Similar to the argument in Spencer [3], we first prove that $\eta_1^k(t) \le \inf_{Y_0(x)} \{\eta_1^k(t; Y_0(x))\}$. The proof of the final theorem of the section shows that strict inequality is not possible, hence the equality will be established.

THEOREM 4. Let $1 \le k \le m$ be given. Then $\eta_1^k(t) \le \inf_{Y_0(x)} \{\eta_1^k(t; Y_0(x))\}$.

PROOF. Let $\tau = \inf_{Y_0(x)} \{\eta_1^k(t; Y_0(x))\}$ and let $\varepsilon > 0$ be given. Then on the interval $[t, \tau+\varepsilon)$, there exist $2 \le q \le n$, points $t \le x_1 < \ldots < x_q < \tau+\varepsilon$, $\alpha \in [t,\tau+\varepsilon)$, positive integers $m_1, \ldots, m_q$ partitioning n, and a non-trivial solution $Z(x; Y_0(x))$ of the variational equation (1.2) along a solution $Y_0(x)$ of (1.1), such that $Z_{\hat{k}}^{(i-1)}(\alpha; Y_0(x)) = 0$, $1 \le i \le n$, and $z_k^{(i)}(x_j; Y_0(x)) = 0$, $0 \le i \le m_j-1$, $1 \le j \le q$.

By disconjugacy arguments similar to those in Henderson [7], Muldowney [8], and Peterson [9], it follows that there is a solution $W(x; Y_0(x))$ of (1.2) along $Y_0(x)$ and points $t \le t_1 < \ldots < t_n < \tau+\varepsilon$ such that $W_{\hat{k}}^{(i-1)}(\alpha; Y_0(x)) = 0$, $1 \le i \le n$, $w_k(x; Y_0(x))$ has a simple zero at $x = t_i$, $1 \le i \le n-1$, and has an odd order zero at $x = t_n$. Now for suitable constants C_{ik}, $1 \le i \le n$, $W(x; Y_0(x)) = \sum_{i=1}^n C_{ik} U_{ik}(x;\alpha, V_1, \ldots, V_n)$, where $Y_0^{(i-1)}(\alpha) = V_i$, $1 \le i \le n$. For $h \ne 0$, consider now the difference quotient

$$\frac{1}{h}\{Y(x;\alpha,(V_1)_{\hat{k}},v_{1k}+hC_{1k},\ldots,(V_n)_{\hat{k}},v_{nk}+hC_{nk})- Y(x;\alpha, V_1, \ldots, V_n)\} =$$

$$\frac{1}{h}\begin{bmatrix} y_1(x;\alpha,(V_1)_{\hat{k}},v_{1k}+hC_{1k},\ldots,(V_n)_{\hat{k}},v_{nk}+hC_{nk})-y_1(x;\alpha, V_1,\ldots, V_n) \\ \vdots \\ y_m(x;\alpha,(V_1)_{\hat{k}},v_{1k}+hC_{1k},\ldots,(V_n)_{\hat{k}},v_{nk}+hC_{nk})-y_m(x;\alpha, V_1,\ldots, V_n) \end{bmatrix}. \quad (3.1)$$

By adding and subtracting, to the jth component, $1 \le j \le m$, of the quotient, terms of the form $y_j(x;\alpha, V_1,\ldots,(V_s)_{\hat{k}}, v_{sk} + hC_{sk},\ldots, (V_n)_{\hat{k}}, v_{nk} + hC_{nk})$, we obtain

$$\frac{1}{h}\{ Y(x;\alpha,(V_1)_{\hat{k}},v_{1k}+hC_{1k},\ldots,(V_n)_{\hat{k}},v_{nk}+hC_{nk}) - Y(x;\alpha,V_1,\ldots,V_n)\} =$$

$$C_{1k}\begin{bmatrix} u_{1k1}(x;\alpha,(V_1)_{\hat{k}},v_{1k}+\xi_{1k1},(V_2)_{\hat{k}},v_{2k}+hC_{2k},\ldots,(V_n)_{\hat{k}},v_{nk}+hC_{nk}) \\ \vdots \\ u_{1km}(x;\alpha,(V_1)_{\hat{k}},v_{1k}+\xi_{1km},(V_2)_{\hat{k}},v_{2k}+hC_{2k},\ldots,(V_n)_{\hat{k}},v_{nk}+hC_{nk}) \end{bmatrix}$$

$$+ \ldots +$$

$$C_{nk}\begin{bmatrix} u_{nk1}(x;\alpha,V_1,\ldots,V_{n-1},(V_n)_{\hat{k}},v_{nk}+\xi_{nk1}) \\ \vdots \\ u_{nkm}(x;\alpha,V_1,\ldots,V_{n-1},(V_n)_{\hat{k}},v_{nk}+\xi_{nkm}) \end{bmatrix}, \text{ where for each}$$

$1 \le \nu \le m$, $\xi_{\mu k\nu}$ is between 0 and $hC_{\mu k}$. Hence, as $h \to 0$, the difference quotient (3.1) converges uniformly on compact subintervals to

$$C_{1k}\begin{bmatrix} u_{1k1}(x,\alpha,V_1,\ldots,V_n) \\ \vdots \\ u_{1km}(x;\alpha,V_1,\ldots,V_n) \end{bmatrix} + \ldots + C_{nk}\begin{bmatrix} u_{nk1}(x,\alpha,V_1,\ldots,V_n) \\ \vdots \\ u_{nkm}(x;\alpha,V_1,\ldots,V_n) \end{bmatrix} =$$

$\sum_{i=1}^{n} C_{ik}\, U_{ik}(x;\alpha,V_1,\ldots,V_n)$. Thus, for h sufficiently small, the difference $P(x) \equiv Y(x;\alpha,(V_1)_{\hat{k}},v_{1k}+hC_{1k},\ldots,(V)_{\hat{k}},v_{nk}+hC_{nk}) - Y(x;\alpha,V_1,\ldots,V_n)$ satisfies the conditions $P_{\hat{k}}^{(i-1)}(\alpha) = 0$, $1 \le i \le n$, and $p_k(\sigma_i) = 0$, $1 \le i \le n$, for some $t \le \sigma_1 < \sigma_2 < \ldots < \sigma_n < \tau+\varepsilon$.

It follows that $\eta_1^k(t) < \tau+\varepsilon$, and since $\varepsilon > 0$ was arbitrary, we have $\eta_1^k(t) \le \inf_{Y_0(x)} \{\eta_1^k(t;\, Y_0(x))\}$.

THEOREM 5. Let $1 \le k \le m$ be given. Then $\eta_1^k(t) = \inf_{Y_0(x)} \{\eta_1^k(t;\, Y_0(x))\}$.

PROOF. Let $\sigma = \inf_{Y_0(x)} \{\eta_1^k(t;\, Y_0(x))\}$ and assume that $\eta_1^k(t) < \sigma$. On the set $\{(m_1,\ldots,m_q)\}$, where $m_1,\ldots,m_q$ are positive integers partitioning n, $1 \le q \le n$, we define a lexicographical ordering by $(n_1,\ldots,n_q) > (m_1,\ldots,m_p)$, if $n_1 > m_1$, or if there exists $s \in \{1,\ldots,q-1\}$ such that $n_i = m_i$, $1 \le i \le s$, and $n_{s+1} > m_{s+1}$.

Since we are assuming that $\eta_1^k(t) < \sigma$, there exist a last tuple $(m_1,\ldots,m_q)$, points $t \le x_1 < \ldots < x_q < \sigma$, $\alpha \in [t,\sigma)$, and distinct solutions $Y(x)$ and $W(x)$ of (1.1) such that $Y_{\hat{k}}^{(i-1)}(\alpha) = W_{\hat{k}}^{(i-1)}(\alpha), 1 \le i \le n$, and $y_k^{(i)}(x_j) = w_k^{(i)}(x_j)$, $0 \le i \le m_j-1$, $1 \le j \le q$. $(m_1,\ldots,m_q)$ is the last tuple for such distinct solutions, hence $y_k^{(m_1)}(x_1) \ne w_k^{(m_1)}(x_1)$. That being the case, it follows from the argument used in the proof of Theorem 2, that the set $S = \{v_k^{(m_1)}(x_1) \mid V(x)$ is a solution of (1.1), $v_{\hat{k}}^{(i-1)}(\alpha) = Y_{\hat{k}}^{(i-1)}(\alpha)$, $1 \le i \le n$, $v_k^{(i)}(x_j) = y_k^{(i)}(x_j)$, $0 \le i \le m_j-1$, $1 \le j \le q-1$, $v_k^{(i)}(x_q) = y_k^{(i)}(x_q)$, $0 \le i \le m_q-2\}$ is an open interval.

If for each $s \in S$, we let $V(x,s)$ denote the corresponding solution of (1.1), then there are distinct $s,s' \in S$ such that $Y(x) = V(x,s)$ and $W(x) = V(x,s')$. From the connectedness of S and Theorem 3, we conclude that there exists an $\bar{s} \in S$ which is between s and s' such that, for the kth component,

$$0 = v_k^{(m_q-1)}(x_q,s) - v_k^{(m_q-1)}(x_q,s') = (s-s')\frac{\partial v_k^{(m_q-1)}}{\partial s}(x_q,\bar{s}).$$

If we set $Z(x,\bar{s}) = \frac{\partial V}{\partial s}(x,\bar{s})$, then $Z(x,\bar{s})$ is the solution of (1.2) along $V(x,\bar{s})$ and satisfies $z_k^{(i-1)}(\alpha,\bar{s}) = 0$, $1 \leq i \leq n$, $z_k^{(i)}(x_j,\bar{s}) = 0$, $0 \leq i \leq m_j-1$, $1 \leq j \leq q-1$, $z_k^{(i)}(x_q,\bar{s}) = 0$, $0 \leq i \leq m_q-2$, and $z_k^{(m_1)}(x_1,\bar{s}) = 1$. But we also have above that $z_k^{(m_q-1)}(x_q,\bar{s}) = 0$, which contradicts the disconjugacy of (1.2) on $[t,\sigma)$. Thus, our assumption is false and $n_1^k(t) = \inf_{Y_0(x)} \{n_1^k(t;\, Y_0(x))\}$.

4. RIGHT DISFOCALITY AND INTERVALS OF RIGHT DISFOCALITY.

In this section we present analogues of the results of section 3 in terms of what we call k-component right disfocality. Much of our notation is that used by Muldowney [10].

DEFINITIONS. Let $\tau = (t_1, \ldots, t_n)$ be an n-tuple of points from (a,b). We say that a function $y(x)$ <u>has n zeros at τ</u> provided $y(t_i) = 0$, $1 \leq i \leq n$, and $y^{(j)}(t_i) = 0$, $0 \leq j \leq m-1$, if t_i occurs m times in τ. A <u>partition</u> $(\tau_1; \ldots; \tau_\ell)$ of τ is obtained by inserting $\ell-1$ semicolons. Let $m_i = |\tau_i|$ be the number of components of τ_i. We say that $(\tau_1; \ldots; \tau_\ell)$ <u>is an increasing partition of τ</u> provided $t_1 \leq t_2 \leq \ldots \leq t_n$, and if $t \in \tau_i$, $s \in \tau_j$ with $i < j$, then either $t < s$ or $t = s$ and $i + m \leq j$, where m is the multiplicity of t in τ_i.

We say the system (1.1) is <u>k-component right</u> $(m_1; \ldots; m_\ell)$ <u>disfocal</u> on (a,b), $0 \leq m_j \leq n-j+1$, if given solutions $Y(x)$, $Z(x)$ of (1.1) such that their difference $W(x) \equiv Y(x) - Z(x)$ satisfies $w_k^{(i-1)}(\alpha) = 0$, $1 \leq i \leq n$, some $\alpha \in (a,b)$, and $w_k^{(j-1)}(x)$ has m_j zeros at τ_j, $1 \leq j \leq \ell$, where $(\tau_1; \ldots; \tau_\ell)$ is an increasing partition of n points in (a,b) with $m_j = |\tau_j|$, then it follows that $w_k(x) \equiv 0$.

For a sequence of integers $\{n_j\}_{j=1}^{\ell}$ satisfying

$$n = n_1 > n_2 > \ldots > n_\ell \geq 1, \tag{4.1}$$

let $\{m_j\}_{j=1}^{\ell}$ be a sequence of nonnegative integers satisfying

$$m_1 + \ldots + m_\ell = n,\ m_2 + \ldots + m_\ell \leq n_2,\ \ldots,\ m_{\ell-1} + m_\ell \leq n_{\ell-1},\ m_\ell \leq n_\ell. \tag{4.2}$$

For a sequence $\{n_j\}_{j=1}^{\infty}$ satisfying (4.1), <u>define</u> $\beta^k(t) = \sup\{t_1 > t \mid$ (1.1) is k-component right $(m_1; \ldots; m_\ell)$ disfocal on $[t,t_1]$, for all sequences $\{m_j\}_{j=1}^{\ell}$ satisfying (4.2)$\}$. Given a solution $Y(x)$ of (1.1), $\beta^k(t;\, Y(x))$ is defined similarly for the variational equation (1.2) along $Y(x)$.

In much the same manner as Peterson [11] has proven for scalar equations, $\beta^k(t)$ can be characterized in terms of $\beta^k(t;\, Y(x))$ as stated in the following theorem.

THEOREM 6. Let $1 \leq k \leq m$ be given and let $c \in (a,b)$. Then, either (i) there is a solution $Y(x)$ of (1.1) such that the variational equation (1.2) along $Y(x)$ has a nontrivial solution $Z(x)$ satisfying the conditions

$$z_k^{(i-1)}(\alpha) = 0 \qquad , 1 \leq i \leq n,$$

$$z_k^{(i-1)}(c) = 0 \qquad , 1 \leq i \leq n-j,$$

$$z_k^{(i-1)}(d) = 0 \qquad , n-j+1 \leq i \leq n+1,$$

where $d = \beta^k(c; Y(x))$, $\alpha \in [c,d]$, and j satisfies $n_{n-j+2} < j \leq n_{n-j+1}$, where $1 \leq n-j+1 \leq \ell$, $(n_{\ell+1} = 0)$, or

(ii) $$\beta^k(c) = \inf_{Y(x)} \{\beta^k(c; Y(x))\}.$$

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