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TWISTORS AND GAUGE FIELDS

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ABSTRACT. We describe briefly the basic ideas and results of the twistor theory. The main points: twistor representation of Minkowsky space, Penrose correspondence and its geometrical properties, twistor interpretation of linear massless fields, Yang-Mills fields (including instantons and monopoles) and Einstein-Hilbert equations.

KEY WORDS AND PHRASES. Twistors, Minkowsky space, linear massless fields, Yang-Mills equations, Einstein-Hilbert equations.
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1. INTRODUCTION.

The theory of twistors, first initiated by R. Penrose at the end of the sixties, now enjoys a period of rapid development. The increased interest in this theory is explained, mainly, by its applications to solving the fundamental nonlinear equations of theoretical physics, namely Yang-Mills and Einstein-Hilbert equations (YM- and EH-equations, for short). The twistor method together with the closely related inverse scattering method have now become the main methods available to construct classical solutions of these equations. In this review we describe briefly the basic ideas and results of the twistor theory. For a more detailed exposition and further references see [1-3].

Twistor model of Minkowsky space. The Minkowsky space-time M which is the basic space of relativistic field theory appears in the twistor theory in a rather unusual form. The twistor model of M is constructed in two steps, the first of which is the spinor model of Minkowsky space. Let $x = (x^0, x^1, x^2, x^3)$ denote the coordinates of a point $x \in M$ with a fixed orgin.

The mapping

$$x \longmapsto X = \begin{pmatrix} x^0 + x^1 & x^2 - ix^3 \\ x^2 + ix^3 & x^0 - x^1 \end{pmatrix} \tag{1.1}$$

assigns to every point $x \in M$ an Hermitian 2×2 -matrix X. The space of all such matrices is called the spinor model of Minkowski space and is denoted by the same letter M. The vector space $S = C^2$, where these matrices operate, is called the spinor space. The coordinates of spinors, i.e. vectors $z \in S$ are denoted by $z^A = (z^0, z^1)$; coordinates of complex conjugate spinors, i.e. vectors $w \in \bar{S}$, are denoted by $w^{A'}$ =

$(w^{0'}, w^{1'})$. Dual spinors, i.e. vectors $u \in S'$, $v \in \bar{S}'$, are denoted: $u_A = (u_0, u_1)$, $v_{A'} = (v_{0'}, v_{1'})$.

In spinor notation the mapping (1.1) has the form: $x = (x^a) \longmapsto X = (x^{AA'})$, where $A = 0,1$. Using the standard basis in the space of Hermitian 2×2 -matrices:

$$\sigma_0 = I , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(1.1) can be written in the form

$$x = (x^a) \longmapsto X = (x^{AA'}) = (\sum_{a=0}^{3} x^a \sigma_a^{AA'}) \tag{1.2}$$

where $\sigma_a^{AA'}$ are entries of the matrix σ_a, $a = 0,1,2,3$, in spinor notation. Physicists usually omit the "Σ" sign in formulas and sum over repeated indices.

The mapping (1.1) has the interesting property of transforming the Lorentz norm $|x|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$ of a vector $x \in M$ into $\det X$. The action of the Lorentz group $\bar{L}$ (the group of orthogonal transformations of M in Lorentz metric) corresponds to the inner action of the group $SL(2,C)$ (2×2 -matrices with determinant 1) on Hermitian matrices: $X \longrightarrow A \times A^*$, $A \in SL(2,C)$. The group $SL(2,C)$ is thus seen to be the two-fold covering of the identity component L_o of the Lorentz group (note that elements A and $-A$ of $SL(2,C)$ induce the same transformation of M). In other words, the spinor space is a space of the two-dimensional (fundamental) representation of the group $SL(2,C)$, the two-fold covering of the Lorentz group.

The mapping (1.1) can be extended to the complex Minkowski space CM, which is a space of points $z = (z^a) \in C^4$, by the following formula:

$$z = (z^a) \longrightarrow Z = (z^{AA'}) = (z^a \sigma_a^{AA'}).$$

The image of this mapping coincides with a space of all complex 2×2 -matrices which is called the spinor model of CM (and is denoted by the same letter), the norm $|z|^2 = (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2$ transforms into $\det Z$.

Let's turn now to a construction of the twistor model of Minkowski space. Assign to a matrix $Z \in CM$ the complex 4×2 -matrix $\begin{pmatrix} -iZ \\ I_2 \end{pmatrix}$ where I_2 is the identity 2×2 -matrix, and consider the two-dimensional plane in C^4 which is generated by the basis consisting of the two four-dimensional columns of this matrix. The space of all two-dimensional subspaces in C^4, i.e. the Grassmann manifold $G_2(C^4)$, is called the twistor model of the complex Minkowski space and is denoted by CM. The mapping constructed above assigns to every matrix $Z \in CM$ a point of the space CM. We shall write this mapping in coordinates. Denote coordinates in C^4 by $Z = (Z^\alpha) = (Z^0, Z^1, Z^2, Z^3)$ and consider a vector $Z \in C^4$ as a pair of spinors $Z = (\omega^A, \pi_{A'})$, i.e. $Z^0 = \omega^0$, $Z^1 = \omega^1$, $Z^2 = \pi_{0'}$, $Z^3 = \pi_{1'}$. The above mapping can be rewritten then in the form

$$Z = (z^{AA'}) \longmapsto \{(\omega^A, \pi_{A'}): \omega^A = -iz^{AA'}\pi_{A'},\quad A = 0,1\} \tag{1.3}$$

$\boxed{\text{Fig. 1}}$ => In other words, we assign to a matrix $(z^{AA'})$ the plane defined by a pair of linear equations: $\omega^A = -iz^{AA'}\pi_{A'}$. Since these equations are homogeneous in $(\omega^A, \pi_{A'})$, they also define a projective line in the 3-dimensional complex projective space CP^3. The space $T = C^4$ with coordinates $(\omega^A, \pi_{A'})$ and the corresponding space $PT = CP^3$ are called the spaces of twistors and projective twistors respectively.

The superposition mapping

$$(z^a) \longmapsto \{(\omega^A, \pi_{A'}): \omega^A = -iz^{AA'}\pi_{A'}\} \tag{1.4}$$

assigns to a point $z = (z^a)$ of the complex Minkowski space CM a two-dimensional plane in the twistor space T or a projective line in the space PT. The mapping (1.4) extends also to the conformal compactification CM of the complex Minkowski space CM, which is obtained by "adding" to CM a complex light cone "at infinity" (cf. [2]).

2. GEOMETRY OF TWISTORS.

We have defined the mapping (1.4) which assigns to every point of CM a projective line in PT . The associated correspondence between points of CM and PT is called the Penrose correspondence. We now consider geometric properties of this correspondence. These properties are formulated in diagrams where geometrical objects of CM occupy the left side and corresponding objects of PT the right side. We have:

{point of CM} $\longleftrightarrow$ {projective line of PT } .

The converse assertion:

$\boxed{\text{Fig. 2}}$ => {null complex 2-plane in CM ≡ α-plane} $\longleftrightarrow$ {point of PT ≡ bundle of projective lines passing through a fixed point of PT}

A null plane is by definition, a plane such that the distance (in the complex Lorentz metric) between any two of its points is zero. A null plane corresponding to a point of PT by the Penrose correspondence is called an α- plane. The dual assertion:

$\boxed{\text{Fig. 3}}$ => {null complex 2-plane in CM ≡ β-plane} $\longleftrightarrow$ {projective plane of PT ≡ point of PT* ≡ pencil of projective lines lying in a fixed projective plane}

The "intersection" of the last two diagrams gives:

$\boxed{\text{Fig. 4}}$ => {null line in CM ≡ complex light ray} $\longleftrightarrow$ {(0,2)-flag in PT ≡ (point of PT ; projective plane in PT including this point) ≡ pair of incident points of PT and PT*}

It follows from the last assertion:

{complex light cone in CM≡ bundle of complex light rays passing through a fixed point of CM} $\longleftrightarrow$ {projective line of PT ≡ (0,1,2) -flags in PT with a fixed projective line}

These are the main facts of the twistor geometry for the complex Minkowski space.

Let's now consider the real compactified Minkowski space M. Denote by N the quadric in T given by the equation: $\phi(Z) = |Z^0|^2 + |Z^1|^2 - |Z^2|^2 - |Z^3|^2 = 0$, and let PN be the associated projective quadric in PT. A restriction of the Penrose correspondence to M has the following properties:

$\boxed{\text{Fig. 5}}$ => {point of M} ⟷ {projective line of PT lying in PN}

{null line of M ≡ light ray of M} ⟷ {point of PN ≡ (point of PN; complex tangent plane to PN in this point)}

{light cone of M ≡ bundle of light rays passing through a fixed point of M} ⟷ {projective line of PN ≡ intersection of PN with a complex tangent plane to PN in any point of a fixed projective line}

The quadric N divides the twistor space T into the two subspaces - the space of positive twistors T^+ ($Z \in T^+ \Leftrightarrow \phi(Z) > 0$) and the space of the negative twistors T^- ($Z \in T^- \Leftrightarrow \phi(Z) < 0$). Denote by CM^+ (resp. CM^-) coincides with a space of points $z = x + iy \in CM$ such that $|y|^2 > 0$ and $y^0 > 0$ (resp. $|y|^2 > 0$, $y^0 < 0$). A restriction of the Penrose correspondence to these spaces gives:

{point of CM^+ (resp. CM^-)} ⟷ {projective line of PT lying in PT^+ (resp. PT^-)}

So, in the real case we have the following duality: points of M correspond to projective lines of PN; points of PN correspond to light rays M. Note that light rays which can intersect each other in M split into separate points of PN. This fact is of the great importance in the twistor theory.

The transformation group SU(2,2) (the group of unitary transformations of T with determinant 1 preserving the form $\phi(Z)$) preserves the quadric N, hence it induces transformations of the Minkowski space M which carry light cones again into light cones. In other words, the group SU(2,2) induces conformal transformations of M. Moreover, this group is the four-fold covering of the identity component of the conformal group of M (note that elements ±A, ±iA of SU(2,2) induce the same transformation of M). Hence, we can define the twistor space (by analogy with the spinors) as a space of the 4-dimensional (fundamental) representation of the group SU(2,2), the four-fold covering of the conformal group.

It is also interesting to consider the Penrose correspondence for the compactified Euclidean space E (we recall the Euclidean space E is a subspace of CM where the complex Lorentz metric $|z|^2$ coincides with Euclidean metric). We have

{point of E} ⟷ {projective line in PT, invariant under mapping $j: (Z^0, Z^1, Z^2, Z^3) \longmapsto (-\bar{Z}^1, \bar{Z}^0, -\bar{Z}^3, \bar{Z}^2)$}

It appears also that a restriction of the Penrose correspondence to E coincides with the natural bundle: $CP^3 \longrightarrow HP^1 = S^4 = E$, i.e. fibers of this bundle are exactly the j-invariant projective lines in PT or images of points of E under the Penrose correspondence.

It is useful to introduce the Klein model of CM along with the twistor model. The Klein model can be constructed as follows. We have defined the twistor model of CM as the Grassmann manifold $G_2(T)$ of two-dimensional subspaces in T. Every such subspace can be defined (up to a non-zero complex multiplier) by a byvector p in

$\Lambda^2 T$ given by byvectors $e_i \wedge e_j$, $i < j$, $i,j = 1,2,3,4$, where $\{e_i\}$ is a basis of T. Assign to a byvector p six complex numbers $\{p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}\}$ (defined up to proportionality) which are called the Plücker coordinates of the subspace. The evident condition $p \wedge p = 0$ is rewritten in the Plücker coordinates as

$$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0 \qquad (2.1)$$

Hence, we have assigned to every plane of $G_2(T)$ a point of the projective quadric Q given by (2.1) in CP^5 . This is a 1-1 correspondence and we call the quadric Q <u>the Klein model of Minkowski space</u> CM. In suitable coordinates it can be written in the form: $p_1^2 + p_2^2 + p_3^2 = q_1^2 + q_2^2 + q_3^2$. The basic objects of the twistor geometry have the following interpretation in terms of the Klein model:

{point of CM}	$\longleftrightarrow$	{point of Q }
{α-planes and β-planes of CM }	$\longleftrightarrow$	{straight generators (2-planes) of Q }
{complex light cone ≡ point of CM }	$\longleftrightarrow$	{intersection of the tangent space of Q in the corresponding point of Q with Q}
{point of M}	$\longleftrightarrow$	{point of the quadric: $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 = 0$ lying in Q}
{point of E}	$\longleftrightarrow$	{point of the real quadric: $x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - x_6^2 = 0$ lying in Q}

Fig. 6 =>

3. TWISTOR INTERPRETATION OF LM(LINEAR MASSLESS)-EQUATIONS.

In § 2 we have given a twistor interpretation of geometric objects of CM. How do relativistic fields or solutions of conformally-invariant equations on CM transform under the Penrose correspondence? According to the "twistor programme" of Penrose ([4]) relativistic fields are to be interpreted in terms of complex geometry of PT, i.e. in terms of holomorphic bundles, cohomologies with coefficients in such bundles and so on. In other words, relativistic equations are "coded" in the complex structure of the twistor space and in this sense equations "disappear" when we pass to twistors. We explain these heuristic considerations first in the case of Maxwell equations and, more generally, LM -equations.

The Maxwell equations on M can be written in the form: $dF = 0$, $d(*F) = 0$, where $F = F_{ab}dx^a \wedge dx^b$ is a 2-form defined by the tensor F_{ab} $(a,b = 0,1,2,3)$ of the electromagnetic field, and $*$ is the Hodge operator defined by the metric of M . If F has a potential, i.e. $F = dA$ for some 1-form A, the the equation $dF = 0$ is automatically satisfied, so Maxwell's equation reduce to $d(*F) = 0$. We can split the form F into a sum of its <u>self-dual and anti-self-dual</u> components: $F = F_+ + F_-$ where $F_\pm = \frac{1}{2}(iF \pm *F)$, $*F_\pm = \pm iF_\pm$. The Maxwell equations in terms of these components can be written in the form: $dF_\pm = 0$. (If F has a potential and is (anti)-self-dual, i.e. $F = F+$ (or F_-) then Maxwell's equations are automatically satisfied).

In order to obtain a twistor interpretation of the Maxwell equations we must apply the mappings (1.1) and (1.3) to F. The tensor F_{ab} transforms under (1.2) into the spinor $F_{AA'BB'} = F_{ab}\sigma^a_{AA'}\sigma^b_{BB'}$ where σ^a are Pauli matrices. Splitting this spinor into a sum of its self-dual and anti-self-dual components we obtain the spinor version of Maxwell's equations: $\frac{\partial}{\partial x_{AA'}} \phi_{AB} = 0$, $\frac{\partial}{\partial x_{AA'}} \Psi_{A'B'} = 0$, where ϕ_{AB}, $\Psi_{A'B'}$ are symmetric spinor functions on M. The first equation corresponds to the anti-self-dual Maxwell equation $dF_- = 0$, the second- to the self-dual equation $dF_+ = 0$. More generally, we call the following system LM-equations of spin s:

$$\frac{\partial}{\partial x_{AA'}} \phi_{AB\ldots L} = 0, \quad s = 0,-\tfrac{1}{2},-1,\ldots; \quad \frac{\partial}{\partial x_{AA'}} \Psi_{A'B'\ldots L'} = 0, \quad s = 0,\tfrac{1}{2},1,\ldots;$$

where $\phi_{AB\ldots L}$, $\Psi_{A'B'\ldots L'}$ are symmetric spinor functions on the spinor model of M with $2|s|$ spinor indices. Again the first equation is called anti-self-dual, the second- self-dual. The case $s = \pm 1$ corresponds to Maxwell equations, $s = \pm 2$ -linearized EH-equations, $s = 0$ -wave equation.

It is more convenient to begin with a twistor interpretation of holomorphic solutions of the above equations. Since every distribution solution of LM-equations can be represented as a jump of boundary values of holomorphic solutions in the future and past tube (proved for the wave equation in [5]), it is sufficient to consider the following system of LM-equations in the future tube:

$$\nabla^{AA'}\phi_{AB\ldots L}(z^{AA'}) = 0, \qquad s = 0,-\tfrac{1}{2},-1,\ldots, \tag{3.1}$$

$$\nabla^{AA'}\Psi_{A'B'\ldots L'}(z^{AA'}) = 0, \qquad s = 0,\tfrac{1}{2},1,\ldots \tag{3.2}$$

where $\nabla^{AA'} = \partial/\partial z_{AA'}$, $\phi_{AB\ldots L}$, $\Psi_{A'B'\ldots L'}$ are symmetric spinor functions (with $2|s|$ indices), holomorphic in the future tube. Note that the spinor model of CM^+ is a space of 2×2 -matrices with positive imaginary part.

A twistor interpretation of the anti-self-dual Maxwell equations (equations (3.1) for $s = 1$) is given by the theorem of Penrose ([6]) which asserts that there is a 1-1 correspondence

$$\left\{\begin{matrix}\text{anti-self-dual holomorphic}\\ \text{solutions of Maxwell}\\ \text{equations in } CM^+\end{matrix}\right\} \longleftrightarrow \{ H^1(PT^+, 0) \} \tag{3.3}$$

In Dolbeault's representation the cohomology group on the right side coincides with a space of smooth $\bar\partial$ -closed (0,1) -forms on PT^+ modulo $\bar\partial$ -exact forms. The self-dual Maxwell equations have the following interpretation:

$$\left\{\begin{matrix}\text{self-dual holomorphic}\\ \text{solutions of Maxwell}\\ \text{equations in } CM^+\end{matrix}\right\} \longleftrightarrow \{ H^1(PT^+, 0(-4)) \}$$

where $0(-4)$ is a sheaf of holomorphic functions of PT whose local sections are given by homogeneous functions of degree -4 in homogeneous coordinates of PT. The apparent asymmetry between self-dual and anti-self-dual cases can be removed if we replace PT^+ in self-dual case by the dual space $(PT^+)^*$. Then both formulations will become analogous.

We now explain the idea of the proof of above results in the self-dual case. Let f be an element of $H^1(PT^+, \mathcal{O}(-4))$. Since a point $z^{AA'} \in CM^+$ corresponds by (1.3) to a whole projective line in PT^+, the value $\Psi_{A'B'}(z^{AA'})$ of pre-images of f under (1.3) is calculated by averaging f along the projective line: $\omega^A = -iz^{AA'}\pi_{A'}$. This averaging is given by a fiber-integral of f along the indicated line with a standard kernel of the form $\pi_{A'}\cdot\pi_{B'}$ in homogeneous coordinates $[\pi_{0'},\pi_{1'}]$ on the line. For the same construction in the anti-self-dual case we need to consider the "normal derivatives" of an element $g \in H^1(PT^+,\mathcal{O})$. In other words, we have to go out into an infinitesimal neighbourhood of the line. The value $\phi_{AB}(z^{AA'})$ of pre-images of g under (1.3) is given by a fiber-integral of the "normal derivative" $\partial/\partial\omega^A \cdot \partial/\partial\omega^B\ (g)$.

The results formulated above for Maxwell equations can be generalized naturally to LM-equations. Namely, the following correspondence

$$\left\{\begin{array}{l}\text{holomorphic solutions of}\\ \text{LM-equations with spin } s\\ \text{in } CM^+\end{array}\right\} \longleftrightarrow \{H^1(PT^+, \mathcal{O}(-2s-2))\} \qquad (3.4)$$

is 1-1. The mapping (3.4) is constructed in the same way as in the Maxwell case.

There is also a direct twistor interpretation of solutions of LM-equations on M not using their representation as a jump of holomorphic solutions in $CM_\pm$. To obtain this interpretation one has to replace the cohomologies of PT^+ in (3.4) by tangent cohomologies of the quadric PN (cf. [7]). A characterization of solutions of LM-equations in terms of (0,2) -flags in PT or incident pairs of points of $PT \times PT^*$ can be given without splitting these solutions into their anti- and self-dual components.

In the case $s=0$ of a complex wave equation (d'Alembertian) we have a representation of solutions as fiber-integrals of elements of $H^1(PT^\pm, \mathcal{O}(-2))$ along projective lines. It is interesting to note that an analogue of this representation for the ultrahyperbolic equation was proved in 1938 by F. John. (Recall that the ultrahyperbolic operator coincides with the restriction of the d'Alembertian to the real subspace of CM where the complex Lorentz metric has signature (2,2).) At the same time this representation for solutions of the real wave equations and Laplacian, obtained by restriction of D'Alembertian to M and E respectively, was apparently unknown.

4. TWISTOR INTERPRETATION OF YM(YANG-MILLS) EQUATIONS.

The YM-equations are matrix analogues of Maxwell's equations. More precisely, consider a principal G -bundle $P \to M$ with connection over Minkovski space. If a Lie group G is a matrix group, then the connection is given by a matrix valued 1-form $A = A_a dx^a$ on M with values in the Lie algebra $\mathfrak{g}$ of G. We shall consider further the case $G = SU(N)$ related to the most important physical applications. In this case the coefficients A_a of the connection are anti-Hermitian $N \times N$ -matrices with zero trace. The connection A is called otherwise a matrix potential. The curvature of A is given by the covariant derivative $F = \nabla_A A = dA + \frac{1}{2}[A,A]$ and is a matrix-valued 2-form: $F = F_{ab}dx^a \wedge dx^b$, $F_{ab} = \partial_a A_b - \partial_b A_a + [A_a,A_b]$ where $\partial_a = \partial/\partial x^a$, $[A_a,A_b]$ is the commutator of matrices A_a, A_b. A connection A is called a YM(Yang-

Mills) -field if its curvature satisfies the YM -equation: $\nabla_A(*F) = 0 \iff d(*F) + \frac{1}{2}[A,*F] = 0$. A field A is (anti)-self-dual if $*F = (-)iF$. (Anti)-self-dual fields automatically satisfy the YM-equations (by the Bianchi identity).

Let us describe first a twistor interpretation of holomorphic anti-self-dual YM -fields on CM^+ . By the theorem of Ward ([8]) there is a 1-1 correspondence:

$$\left\{\begin{array}{l}\text{anti-self-dual holomorphic}\\ \text{YM -fields in } CM^+\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{holomorphic vector bundles}\\ \text{on } PT^+ \text{ holomorphically}\\ \text{trivial on projective}\\ \text{lines in } PT^+ \equiv \text{images of}\\ \text{points of } CM^+\end{array}\right\} \tag{4.1}$$

(One can check that in the scalar case this theorem coincides with (3.3)). The right side of (4.1) can be redefined as a space of $\bar\partial$ -closed matrix-valued (0,1) -forms modulo $\bar\partial$ - exact forms. (A matrix-valued (0,1) -form u is $\bar\partial$ -closed iff $\bar\partial u + \frac{1}{2}[u,u] = 0$ and is $\bar\partial$ -exact iff $u = v^{-1}\bar\partial v$ for some non-degenerate matrix-valued function v).

The idea of the proof is the following. We have a geometric criterion of anti-self-duality: a connection A is anti-self-dual $\iff$ its curvature F vanishes on α -planes. Let A be an anti-self-dual connection in a principal $SU(N)$ - bundle P on CM^+ and E -associated vector bundle. It is convenient to use the following diagram:

$$\begin{array}{ccc} & F^+ & \\ {\scriptstyle\mu}\swarrow & & \searrow{\scriptstyle\nu} \\ PT^+ & & CM^+ \end{array}$$

where F^+ is a space of (0,1) -flags in PT^+, i.e. pairs consisting of a point of PT^+ and a projective line in PT^+ including this point, and μ, ν are natural projections. Denote by E' and A' the trivial liftings of the bundle E and the connection A to F^+, and define the bundle E'' on PT^+ as follows. The fiber of E'' over a point $Z \in PT^+$ is given by A' -horizontal sections of E' over $\mu^{-1}(Z)$, i.e. by sections s' of E' over $\mu^{-1}(Z)$ such that $\nabla^\mu_{A'} s' = 0$ where $\nabla^\mu_{A'}$ is the fiber component of $\nabla_{A'}$ along the fibers of μ. (This definition is correct due to anti-self-duality of A). The bundle E'' is an image of the field A under (4.1). To prove that this bundle is holomorphic, we define an almost complex structure (Cauchy-Riemann operator) on E'' by $\bar\partial s \underset{\text{def}}{=} \nabla^{(0,1)}_{A'} s'$, where s'' is the section of E'' given by a section s' of the bundle E', and $\nabla^{(0,1)}_{A'}$ is the (0,1)-component of $\nabla_{A'}$. To show this definition to be correct it is sufficicent to check that $\bar\partial s''$ is again a section of E'' , i.e. that $\nabla^\mu_{A'}(\nabla^{(0,1)}_{A'} s') = 0$. But $\nabla^\mu_{A'}(\nabla^{(0,1)}_{A'} s') = \nabla^{(0,1)}_{A'}(\nabla^\mu_{A'} s') + [\nabla^\mu_{A'}, \nabla^{(0,1)}_{A'}]\, s' = 0$ because $\nabla^\mu_{A'} s' = 0$ by the given condition, and $[\nabla^\mu_{A'}, \nabla^{(0,1)}_{A'}] = 0$ due to anti-self-duality of A. One can show also that $\bar\partial^2 = 0$ (almost complex analogue of the Frobenius condition) hence by the Newlander-Nirenberg theorem ([9]) this almost complex structure defines, in fact, a complex structure on E''. By construction, the bundle E'' is holomorphically trivial on projective lines in PT^+ which are images of points of CM^+.

Conversely, let E'' be a holomorphic bundle on PT^+ trivial on projective lines and E' be its trivial lifting to F^+. Since the fiber of E over a point $z \in CM^+$ is given by holomorphic sections of E' over $\nu^{-1}(z)$. This defines the bundle E on CM^+. (This definition is correct due to compactness of $\nu^{-1}(z) \backsim CP^1$.) We shall construct a connection in E by means of parallel transport. It is sufficient to define it along complex light rays in CM^+. The parallel transport in E along a complex light ray is given by identifying fibers of E' over projective lines (corresponding to the different points of the ray with their common point (corresponding to the ray). The infinitesimal version of this definition defines a connection in E which is anti-self-dual by construction (its curvature is zero on α -planes corresponding to the points of PT^+).

The self-dual YM-fields have an analogous interpretation in terms of the dual space PT^*. An arbitrary YM-field cannot be decomposed into the sum of anti- and self-dual fields owing to nonlinearity of the YM-equations. However, general YM-fields also have a natural twistor interpretation in terms of (0,2)-flags in PT or incident pairs in $PT \times PT^*$ (cf. [10,11]).

The most interesting physical applications of the above results are related to the Euclidean case. The instantons are by definition (anti)-self-dual YM-fields on E which realize minima of the action functional: $S(F) = \int (F,*F)\, d^4x$. By the Ward's theorem there is a 1-1 correspondence between instantons and holomorphic vector bundles on PT which are holomorphically trivial on j-invariant projective lines (cf. §2) and have an additional quaternionic structure generated by j. With the help of this result there was given in [12] a description of the space of instantons in terms of quaternion matrices, satisfying some quadratic constraints. Considering physical applications of this result we note that it would be desirable to have (if possible) a more explicit description of the space of instantons because one has to integrate over this space for quantisation.

There is also one more important class of (anti)-self-dual solutions of YM-equations called monopoles. They are by definition (anti)-self-dual YM-fields $A = (A_a)$ on E not depending on "time" (i.e. coordinate x^0) and satisfying the boundary condition: $||A_0|| = 1 + k/r + 0\,(1/r)\ k \in Z)$ as $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \to \infty$. This condition realizes minima of the energy functional: $E(F) = \int (F,*F) dx^1 dx^2 dx^3$. It is natural to consider the monopoles as fields, so called Yang-Mills-Higgs fields, on the 3-dimensional Euclidean space E^3. The space $T(CP^1)$ (the tangent bundle space of the Riemann sphere) appears to be an analogue of the twistor space PT in this case. We have again an analogue of Ward's theorem ([13]) which states that there is a 1-1 correspondence between Yang-Mills-Higgs fields on E^3 and holomorphic vector vector bundles on $T(CP^1)$ which is holomorphically trivial on projective lines in $T(CP^1)$ and has an additional quaternionic structure. Hitchin [13] used this theorem to construct for every monopole some algebraic curve (called a spectral curve) thus reducing the problem of description of monopoles to the description of such curves. It is interesting to note that a similar construction appeared in the paper by K. Weirstrass in 1866 dedicated to the minimal surfaces in E^3. The twistor approach (with suitable modifications) can be applied also to the other important classes of solutions of YM-equations (e.g. to the vortices, that is, (anti)-self-dual YM-fields not depending on two variables).

5. TWISTOR INTERPRETATION OF EH(EINSTEIN-HILBERT)-EQUATIONS.

So far our considerations were related to the flat space-time CM. What can be said if this space is curved? Do our constructions continue to be valid in the presence of gravitation? It appears that under some essential restrictions the answer is affirmative.

Let M be a 4-dimensional complex manifold with a complex non-degenerate Riemann metric $g = g_{ab}dz^a da^b$. (Note that the metric g is non-Hermitian.) We can consider the Riemann curvature R of M as a linear operator on the space $\Lambda = \Lambda^2(M)$ of (holomorphic) 2-forms on M. Denote by Λ_+, Λ_- the subsapces of Λ consisting respectively of self-dual and anti-self-dual forms ($\Lambda_\pm$ is the eigenspace of the $*$-operator with the eigenvalues ± 1). Then $\Lambda = \Lambda_+ \oplus \Lambda_-$, so the operator R can be represented as a block matrix $R = \begin{pmatrix} A & B \\ B' & C \end{pmatrix}$ where B is defined by the traceless part of the Ricci tensor, $\frac{1}{4}\mathrm{tr}A = \frac{1}{4}\mathrm{tr}C$ is the scalar curvature, and $W_+ = A - 1/3\mathrm{tr}A$ and $W_- = C - 1/3\mathrm{tr}C$ are respectively self-dual and anti-self-dual components of Weyl tensor $W = W_+ \oplus W_-$. If the metric g satisfies to the complex EH-equations then the Ricci tensor and scalar curvature vanish and the operator R coincides with the direct sum $W_+ \oplus W_-$. A manifold M will be called self-dual (resp. anti-self-dual iff it satisfies the complex EH-equations and $W_- = 0$ (resp. $W_+ = 0$). (Anti)-self-dual manifolds have the following natural twistor interpretation. Let us call a conformal structure on M the class of conformally equivalent complex Riemann metrics on M. A manifold with such structure will be called conformal. The theorem of Penrose ([14]) asserts that there is a 1-1 correspondence:

$$\left\{\begin{array}{l}\text{anti-self-dual conformal}\\ \text{manifolds}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{deformations of complex}\\ \text{structures of domains in}\\ \text{PT ruled by projective}\\ \text{lines}\end{array}\right\} \tag{5.1}$$

Self-dual conformal manifolds have a similar interpretation in terms of PT^*.

The idea of the proof of (5.1) is the following. Let T be a 3-dimensional complex manifold which is a deformation of the domain of PT, ruled by projective lines. Then by the theorem of Kodaira ([15]) T has a 4-complex-parameter collection of rational curves (i.e. curves isomorphic to the Riemann sphere CP^1). Let M be a manifold of all these curves. The set of rational curves in T intersecting the line L_p corresponding to some point $p \in M$, is called a "light cone" with vertex p. This defines a conformal structure on M. The manifold M with this structure is anti-self-dual. In fact, let's call the set of rational curves in T passing through a fixed point of T an α-surface in M. We formulate the following geometric criterion of anti-self-duality: a space M is anti-self-dual <=> there is an α-surface passing through every point of M "in every null directon". This criterion is analogous to the criterion of anti-self-duality of YM-fields (Riemann curvature vanishes on null surfaces). As the constructed manifold M does have a sufficient number of αsurfaces it is anti-self-dual by the criterion. Conversely, an anti-self-dual manifold M has a 3-complex-parameter collection of α-surfaces and the manifold of these surfaces is identified with T.

To define a metric on M i.e. to obtain a complex solution of EH-equations, we have to introduce according to the "twistor programme" of Penrose some additional structure on T not belonging to the "complex geometry" of T. For instance let T be a sufficiently small neighbourhood of a rational curve L in PT. Then we can identify T with the normal bundle of L, and other rational curves in T with holomorphic sections of this bundle. Denote also by K_T the canonical line bundle of 3-forms on T (written in local coordinates in the form $f\ dz^1 \wedge dz^2 \wedge dz^3$). Then a restriction of K_T to L coincides with the standard bundle $O(-4)$ on CP^1. The theorem of Penrose ([14]) asserts that these data are sufficient for the construction of anti-self-dual solutions of EH-equations. More precisely, there is (locally) a 1-1 correspondence

$$\{\text{anti-self-dual manifolds}\} \longleftrightarrow \left\{\begin{array}{l}\text{holomorphic bundles } \pi:\ T \to CP^1 \\ \text{with 4-parameter collection of} \\ \text{sections and isomorphism} \\ K_T \approx \pi^* O(-4)\end{array}\right\}$$

Unfortunately, this result cannot be applied to the construction of real Lorentz solutions of EH-equations because real anti-self-dual manifolds always have even signature. Nevertheless, this theorem can be successfully applied to the construction of Euclidean anti-self-dual manifolds which are considered in quantum gravity. To this class belong ALE (asymptotically locally Euclidean)-manifolds introduced in [16] which have topology of $S^3/\Gamma \times R$ in a neighbourhood of infinity where Γ is a finite group of isometrics on S^3. The twistor interpretation of these spaces in kind of (10) was given in [17].

6. FINAL REMARKS.

We want to underline first a connection between Penrose transformation and another interesting mathematical result - theorem of Sato-Kawai-Kashiwara ([18]). By this theorem an arbitrary overdetermined system of pseudodifferential equations in general position can be transformed microlocally (i.e. locally in the cotangent bundle) by means of a canonical transformation of infinite order into a system of tangential Cauchy-Riemann equations. This transformation (which does not reduce in general to a coordinate transformation in the base space) preserves only analytic singularities of solutions. The Penrose transform also carries the field theory equations into systems of tangential Cauchy-Riemann equations on twistor manifolds and, moreover, it allows one to obtain explicit formulas for solutions.

There is a close connection between the twistor approach and the method of Riemann-Hilbert boundary problem (cf.[19]) which is also applied to solution of the self-dual YM-equations. The solution of these equations by the indicated method reduces to solving the factorisation problem for a rational matrix-valued function on the Riemann sphere CP^1 (which is equivalent to a trivialization of a holomorphic bundle on CP^1 defined by this matrix function) depending on three complex parameters. This is equivalent to a construction of a holomorphic vector bundle on PT holomorphically trivial on projective lines which are images of points on CM.

The Penrose transform is attached to 4-dimensional manifolds. Its existence on the group-theoretical language is due to the following local group isomorphisms: $SO(4) \approx SO(3) \times SO(3)$ (note that the group $SO(n)$ is simple for $n \geq 3$, $n \neq 4$), $SL(4,C) \approx O(6,C)$ $SU(2,2) \approx SO(4,2)$. All these isomorphisms are "fasten" to the dimension four.

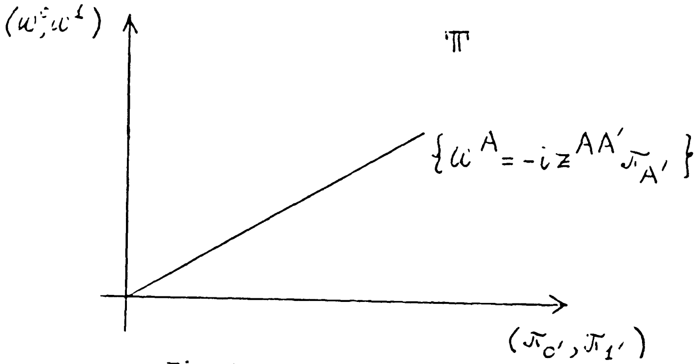

Fig.1

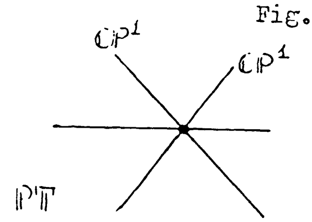

Fig.2

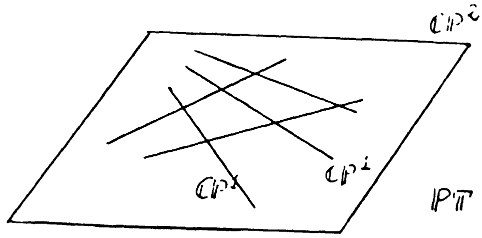

Fig.3

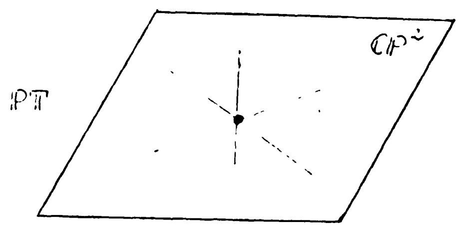

Fig.4

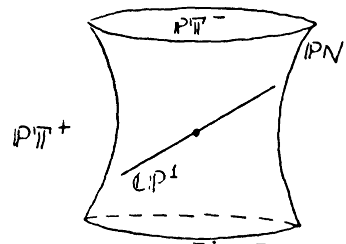

Fig.5

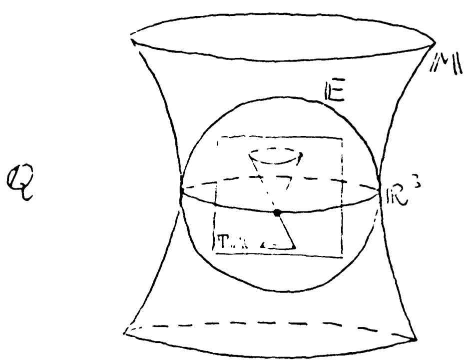

Fig.6

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