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RELATIVE INTEGRAL BASIS FOR ALGEBRAIC NUMBER FIELDS

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ABSTRACT. At first conditions are given for existence of a relative integral basis for $O_K \cong O_k^{n-1} \oplus I$ with $[K;k] = n$. Then the construction of the ideal I in $O_K \cong O_k^{n-1} \oplus I$ is given for proof of existence of a relative integral basis for $O_{K_4}(\sqrt{m_1}, \sqrt{m_2})/O_k(\sqrt{m_3})$. Finally existence and construction of the relative integral basis for $O_{K_6}(\sqrt[3]{n}, \sqrt{-3})/O_{k_3}(\sqrt[3]{n}), O_{k_6}(\sqrt[3]{n}, \sqrt{-3})/O_{k_2}(\sqrt{-3})$ for some values of n are given.

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1. INTRODUCTION

Throughout this article the following notation will be used:

Q:	field of rational numbers
Z:	rational integers
k,K ($Q \subseteq k \subseteq K$):	algebraic number fields
disc(x):	discriminant of element x
$D_{K/k}$:	relative different of extension K/k
$O_{k_i} = O_i$	ring of integers of k_i
h_k :	class number for k
P.I.:	principal ideal
$N_{K/k}a$:	relative norm of an ideal a in K for extension K/k.

2. FINITELY GENERATED MODULES

In [1, p. 24] it was shown that if M is a finitely generated module over a Dedekind ring R then

$$M \cong R^m \oplus A \oplus I \ , \tag{2.1}$$

where I is an ideal in R, A is a torsion-submodule and m is a positive integer.

Now for extension K/k with $[K;k] = n$, by (2.1) we have

$$O_K \cong O_k^{n-1} \oplus I \tag{2.2}$$

so by this we have:

THEOREM 2.3. In the extension K/k for $[K;k] = n$, O_K has relative integral basis over $O_k \iff I = P.I.$

ILLUSTRATION 2.4. Let $k_1 = Q(\sqrt{2})$, $k_2 = Q(\sqrt{-7})$. Does a relative integral basis of $O(\sqrt{2},\sqrt{-7})/O_3 = O(\sqrt{-14})$ exist? see also [2].

SOLUTION. By (2.2), a "relative integral basis" exists <==> I = P.I., otherwise not.

We will construct an ideal I in O_{k_3} where $O_K \cong O_{k_3}^{2-1} \oplus I$. Since $(d_{K_1}, d_{K_2}) = (2\cdot 4, -7) = 1$, then using a theorem given in [3, p. 218],

$$O_K = \left[1,\sqrt{2}\right] \times \left[1, \frac{1+\sqrt{-7}}{2}\right]\cdot z = \left[1,\sqrt{2}, \frac{1+\sqrt{-7}}{2}, \frac{\sqrt{2}+\sqrt{-14}}{2}\right]\cdot z ,$$

$$O_K = \left[1,\sqrt{-14}, \frac{1+\sqrt{-7}}{2}, \frac{\sqrt{2}+\sqrt{-14}}{2}\right]\cdot z = \left[1,\sqrt{-14}\right] \oplus R ,$$

where R is an O_k-module,

$$R = \left[\frac{1+\sqrt{-7}}{2} + s + t\sqrt{-14}, \frac{\sqrt{2}+\sqrt{-14}}{2} + u + v\sqrt{-14}\right] , \tag{2.5}$$

$\sqrt{-14}\ R \subseteq R$.

$$R \cong \sqrt{-14}\ R = \left[\frac{\sqrt{-14} + -7\sqrt{2}}{2} + s\sqrt{-14} + -14t,\ -7 + \sqrt{-7} + u\sqrt{-14} + -14v\right] \tag{2.6}$$

We take (2.5) and (2.6) proportional; then

$$\begin{cases} 1 + \sqrt{-7} + 2s + 2t\sqrt{-14} = -7 + \sqrt{-7} + u\sqrt{-14} + -14\cdot v \\ \qquad 2s + 2t\sqrt{-14} = -8 + u\sqrt{-14} + -14v \\ u = 2t,\quad s = -8 + -14v ,\quad \text{for}\quad u = v = t = 0 ,\quad s = -4 , \end{cases}$$

$$\begin{cases} \sqrt{-14} + -7\sqrt{2} + 2s\sqrt{-14} - 28t = (\sqrt{2} + \sqrt{-7} + 2u + \sqrt{-14})\cdot -7 \\ \qquad -28t + \sqrt{-14}\ (1+2s) = -14u + -7\sqrt{-14}\ (1 + 2v) \\ u = 2t,\quad 1+2s = -7(1+2v) ,\quad \text{for}\quad u = t = v = 0 ,\quad s = -4 . \end{cases}$$

Then,

$$R = \left[\frac{1+\sqrt{-7}}{2} - 4 + 0 , \frac{\sqrt{2}+\sqrt{-14}}{2}\right] = \left[\frac{-7 + \sqrt{-7}}{2}, \frac{\sqrt{2} + \sqrt{-14}}{2}\right]$$

$$\sqrt{-14}\ R = \left[\frac{-7\sqrt{2} - 7\sqrt{-14}}{2}, \frac{2\sqrt{-7} - 14}{2}\right] = \left[\frac{17(\sqrt{2}+\sqrt{-14}}{2}, -7 + \sqrt{-7}\right],$$

$R = \dfrac{\sqrt{-14}\ (1+\sqrt{-7})}{2\sqrt{2}} \cdot [2,\sqrt{-14}]$, $R \cong I = [2,\sqrt{-14}]$ is an ideal in $O_k = [1,\sqrt{-14}]\cdot z$.

Since $I = [2,\sqrt{-14}]$ is not P.I. in O_3, then O_K does not have a relative integral over O_3. The ideal $I = [2,\sqrt{-14}]$ is unique (up to equivalence of ideals).

The method of the previous theorem is only good for the case n = 2 since for $n \geq 3$, computation of an ideal in $O_k \cong O_k^{n-1} \oplus I$ is too difficult. Thus we need a relation such as the following between I and one of the invariants in the extension K/k.

THEOREM 2.7. If C is the class of ideals in k containing $d_{K/k}$ and $C_{K/k}$ is a class containing I, then

$$C = C_{K/k}^2 .$$

Now we will give the "criterion for existence of a relative integral basis", for the extension K/k. See Norkiewicz et al. [1,4,5,6].

THEOREM 2.8. Let [K:k] = n, h_k = odd, then O_K has a "relative integral basis" over O_k <=> $d_{K/k}$ (relative discriminant) is a principal ideal. For more details see [1, p. 359].

PROOF. =>: If O_K/O_k has a relative integral basis, I = P.I. Therefore by Theorem 2.7 $d_{K/k}$ is P.I.

<=: $d_{K/k}$ = P.I., so every ideal in the class of $C^2_{K/k}$ is P.I. Therefore I^2= P.I., since $(2,h_k) = 1$. Then I = P.I. and by (2.2), O_K has a relative integral basis over O_k.

COROLLARY 2.9. If O_k = P.I.D., then $h_k = 1$ is odd and $d_{K/k}$ = P.I. Thus by Theorem 2.8 for every finite extension of k where the ring of integers is its P.I.D., a relative integral basis exists.

ILLUSTRATION 2.10. Let $k_3 = Q(\sqrt[3]{213})$ and $k = Q(\sqrt{-3})$, $K_6 = Q(\sqrt{-3},\sqrt[3]{213})$, $h_3 = 21$. Does a relative basis of O_{K_6/k_3} exist or not?

We know that for $n=ab^2$, $(a,b) = 1$, $ab \neq 1$ in $k_3 = Q(\sqrt[3]{n})$,

$$O_3 = \left[1, \sqrt[3]{ab^2}, \sqrt[3]{a^2b}\right]\cdot z \quad \text{for} \quad a \not\equiv \pm b \mod 9 \text{ , and}$$

$$O_3 = \left[\theta_0 = \frac{1+\sqrt[3]{ab^2}+\sqrt[3]{a^2b}}{3}, \sqrt[3]{ab^2}, \sqrt[3]{a^2b}\right]\cdot z \quad \text{for} \quad a \equiv b \mod 9 \text{ .}$$

We call these two cases respectively Type I and Type II. $3 = (\sqrt{-3})^2$ in k_2. In Type I, $3 = 3^6_{11}$, $3^2_{11} = 3_1$, $3^3_{11} = \sqrt{-3}$, so $3 = 3^3_1$ and we define $f_0 = 3ab$. In Type II, $3 = 3^2_{11}3^2_{12}3^2_{13}$, $3^2_{11} = 3_2$, $3_{12}\cdot 3_{13} = 3_1$, $3_{11}\cdot 3_{12}\cdot 3_{13} = \sqrt{-3}$, so $3 = 3^2_1\cdot 3_2$ and we define $f_0 = ab$. $D_{6/3} = 3_{11}$, $d_{6/3} = 3^2_{11} = 3_1$, $d_{6/3} = 3_1 = (-3,\sqrt[3]{213})$. See [5, p. 221]. By Theorem 2.5, since h_3 is odd and $d_{6/3} = (-3,\sqrt[3]{213}-6) = (-343,\sqrt[3]{213}-6) = (\sqrt[3]{213}-6)$ is a P.I., so a relative integral basis exists.

Incidentally in (3.1) we will prove that if $3 \nmid h_3$ then O_6 has a relative integral basis over O_3, but here $h_3 = 21$ so $3|h_3$ and also a relative integral basis exists.

3. EXISTENCE OF A RELATNIVE INTEGRAL BASIS: BY SOME CONDITIONS ON n FOR $O_6(\sqrt[3]{n},\sqrt{-3}) / O_3(\sqrt[3]{n})$.

Now here we will show that for some $n \in z$ this extension has relative integral basis.

THEOREM 3.1. If $3 \nmid h_3$, then O_{K_6} has relative integral basis over O_3 for Type 1.

PROOF. By Theorem in [7, p. 222], O_6 has a relative integral basis over $O_3 \iff D_{6/3} / \sqrt{-3} = 3_{11} / \sqrt{-3} = 1/3_1$ is a P.I. in O_6 generated by an element of k_3. But $3_1 = (-3,\sqrt[3]{n}+1)$ when $3 \nmid ab$ and $3_1 = (-3, \sqrt[3]{n})$ when $3|ab$ in Type I and $(3) = (3_1\cdot 3^2_2)$ for Type 2.

Now, $3 \nmid h_3$ so $(3_1)^3 = (-3,\sqrt[3]{n} \pm 1)^3 = (3)$ or $(3_1)^3 = (-3,\sqrt[3]{n})^3 = (3)$ for P.I., so: $3_1 = (-3,\sqrt[3]{n} \pm 1)$ or $(-3,\sqrt[3]{n}) = 3_1$ is P.I. Then $1/3_1$ also generates a P.I. in O_6. In Type II, $(3) = (3_1\cdot 3^2_2)$, it is dependent on ideals 3_1 and 3_2; therefore in this case, a relative basis may exist or may not exist.

But it may be that $3|h_3$ and again O_6 has relative basis over O_3. For example in $k_3 = Q(\sqrt[3]{213})$, $h_3 = 21$ and $3_1 = (\sqrt[3]{123} - 6)$, so $3 | 21$ and a relative integral basis exists. Therefore the inverse of Theorem 3.1 is not true in general.

Bondi has shown in [8] for $k_3 = Q(\sqrt[3]{n})$:

THEOREM 3.2. One of the following statements holds:

$$3 \nmid h_3 \iff \begin{cases} 1) & n = 3 \\ 2) & n = p\,,\quad p = \text{prime}\,,\quad p \equiv -1 \pmod 3 \\ 3) & n = 3p \text{ or } 9p,\quad p = \text{prime} \equiv 2 \text{ or } 5 \pmod 9 \\ 4) & n = p\cdot q \ (p,q \text{ are primes}),\quad p \equiv 2 \text{ and } q \equiv 5 \pmod 9 \\ 5) & n = p^2\cdot q \ (p,q \text{ are distinct primes}),\ p \equiv q \equiv 2 \text{ or } 5 \pmod 9. \end{cases}$$

DEFINITION 3.3. A number n is called a Honda number if it is a number in the table for Theorem 3.2.

By Theorems 3.1 and 3.2 we have:

THEOREM 3.4. If n is a Honda number in type I, $O_6(\sqrt[3]{n}, \sqrt{-3})$ necessarily has a relative integral basis over $O_3(\sqrt[3]{n})$.

4. RELATIVE INTEGRAL BASIS OF $O_{K_6}(\sqrt[3]{n},\sqrt{-3}) \,/\, O_{K_3}(\sqrt[3]{n})$.

We proved in Theorem 3.1 that if $3 \nmid h_3$, then 3_1 is P.I. only for Type I. Therefore a relative integral basis for O_6/O_3 exists, since by the theorem in [3, p. 201], $\operatorname{disc}\left(1, \frac{\frac{3+\sqrt{-3}}{2}}{3_1}\right) = d_{K_6/k_3} = 3_1$. Therefore $\left[1, \frac{\frac{3+\sqrt{-3}}{2}}{3_1}\right]$ is a relative integral basis for O_6 over O_3, so:

$$O_{K_6} = \left[1, \frac{\frac{3+\sqrt{-3}}{2}}{3_1}\right]\cdot O_{k_3}\ .$$

5. CONSTRUCTION OF RELATIVE INTEGRAL BASIS FOR $O_6(\sqrt[3]{n},\sqrt{-3})/O_2(\sqrt{-3})$.

Since $O_2(\sqrt{-3})$ is P.I.D., then by (2.3) $O_6(\sqrt[3]{n},\sqrt{-3})/O_2(\sqrt{-3})$ has a relative integral basis.

THEOREM 5.1. Let $\lambda = \frac{3_1}{\frac{1}{2}(3+\sqrt{-3})}$. For Type I and 3_1 is a P.I., if $3|a$ then

$$O_6 = \left[1, \frac{\sqrt[3]{ab^2}}{\lambda}, \frac{\sqrt[3]{a^2b}}{\lambda^2}\right]\cdot O_2\ ,$$

and if $3|b$ then

$$O_6 = \left[1, \frac{\sqrt[3]{ab^2}}{\lambda^2}, \frac{\sqrt[3]{a^2b}}{\lambda}\right]\cdot O_2\ .$$

PROOF. Since $N_{6/2}(\sqrt[3]{ab^2}/\lambda)$ and $N_{6/2}(\sqrt[3]{a^2b}/\lambda^2)$ are in O_2, then are integers.

If $d_{6/2} = \operatorname{disc}(1, \sqrt[3]{ab^2}/\lambda, \sqrt[3]{a^2b}/\lambda^2)\cdot O_2$, then by the theorem in [3, p. 201] $x = [1, \sqrt[3]{ab^2}/\lambda, \sqrt[3]{a^2b}/\lambda^2]$ is a relative integral basis of O_6/O_2, so we are going to compute disc(x).

$$\operatorname{disc} x = \begin{vmatrix} 1, & \sqrt[3]{ab^2}/\lambda, & \sqrt[3]{a^2b}/\lambda^2 \\ 1, & \rho\sqrt[3]{ab^2}/\lambda', & \rho^2\sqrt[3]{a^2b}/\lambda'^2 \\ 1, & \rho^2\sqrt[3]{ab^2}/\lambda'', & \rho\sqrt[3]{a^2b}/\lambda''^2 \end{vmatrix}^2$$

$$= (ab)^2\cdot\frac{(3+\sqrt{-3})^6}{3^4\cdot 2^6}\begin{vmatrix} 3_1^2 & 3_1 & 1 \\ 3_1'^2 & 3_1'\rho & \rho^2 \\ 3_1''^2 & 3_1''\rho^2 & \rho \end{vmatrix}^2$$

$$= \frac{(ab)^2\cdot(3+\sqrt{-3})^6}{3^4\cdot 2^6}\left\{3_1^2(3_1'\rho^2-3''\rho)+3_1'^2(3_1''\rho^2-3_1\rho)+3_1''^2(3_1\rho^2-3_1'\rho)\right\}^2$$

$$= 3^2\cdot a^2b^2 .$$

For "Type I" we have $d_{6/3} = f_0^2 = \mathrm{disc}\ x$, so $x = [1, \sqrt[3]{ab^2}/\lambda, \sqrt[3]{a^2b}/\lambda^2]$ is a relative integral basis of O_6/O_3. See Cohn et al. [7,9,10].

ILLUSTRATION 5.2. For $K_3 = Q(\sqrt[3]{213})$, the ideal $3_1 = (\sqrt[3]{213}-6)$ is P.I. and $3|ab^2$ (Type I, $3|a$), so

$$O_{K_6} = \left[1, \frac{\sqrt[3]{213}}{\lambda}, \frac{\sqrt[3]{213^2}}{\lambda^2}\right]\cdot O_2 , \quad \text{where} \quad \lambda = \frac{\sqrt[3]{213}-6}{\frac{1}{2}(3+\sqrt{-3})} .$$

We have to mention that if 3_1 is not a P.I., this is still an open question.

THEOREM 5.3. Assume $k_3 = Q(\sqrt[3]{ab^2})$, $(3) = 3_1^3 = (\sqrt[3]{ab^2} \pm n)^3$, for $3 \nmid ab$ and "Type I", then:

$$O_{K_6} = \left[1, \frac{\sqrt[3]{ab^2}+n-\sqrt{-3}}{3_1}, \frac{\sqrt[3]{ab^2}+\sqrt[3]{a^2b}+t_1}{3_1}\right] O_{k_2} , \tag{5.4}$$

where $t_1 = 0$ for $a = 3k+1$ and $b = 3k+2$ or conversely and $t_1 = 1$ for $a = b = 3k+1$ and $t_1 = -1$ for $a = b = 3k+2$.

PROOF. Now

$$\frac{\sqrt[3]{ab^2}\pm n-\sqrt{-3}}{3_1} = \alpha_1 , \qquad \frac{\sqrt[3]{ab^2}+\sqrt[3]{a^2b}+t_1}{3_1} = \alpha_2$$

are integrals because $N_{K_6/K_2}(\alpha_1)$ and $N_{K_6/K_2}(\alpha_2)$ are integers. We take $x = [1,\alpha_1,\alpha_2]$ and $t_2 = + n - \sqrt{-3}$:

$$\mathrm{disc}(x) = \begin{vmatrix} 1 & \frac{\sqrt[3]{ab^2}+t_2}{3_1} & \frac{\sqrt[3]{ab^2}+\sqrt[3]{a^2b}+t_1}{3_1} \\ 1 & \frac{\rho\sqrt[3]{ab^2}+t_2}{3_1'} & \frac{\rho\sqrt[3]{ab^2}+\rho^2\sqrt[3]{a^2b}+t_1}{3_1'} \\ 1 & \frac{\rho^2\sqrt[3]{ab^2}+t_2}{3_1''} & \frac{\rho^2\sqrt[3]{ab^2}+\rho\sqrt[3]{a^2b}+t_1}{3_1''} \end{vmatrix}^2$$

$$= \frac{1}{3^2}\begin{vmatrix} 3_1 & \sqrt[3]{ab^2}+t_2 & \sqrt[3]{a^2b}+t_3 \\ 3_1' & \rho\sqrt[3]{ab^2}+t_2 & \rho^2\sqrt[3]{a^2b}+t_3 \\ 3_1'' & \rho^2\sqrt[3]{ab^2}+t_2 & \rho\sqrt[3]{a^2b}+t_3 \end{vmatrix}^2$$

for $t_3 = t_1 - t_2$.

$$\mathrm{disc}(x) = \Big[3_1\cdot(\rho^2ab+\rho t_3\sqrt[3]{ab^2}+t_2\rho\sqrt[3]{a^2b}+t_2\cdot t_3-\rho\, ab-\rho^2t_2\cdot\sqrt[3]{a^2b}-t_3\rho^2\sqrt[3]{ab^2}-t_2\cdot t_3)$$
$$+3_1'\cdot(\rho^2ab+t_2\sqrt[3]{a^2b}+t_3\rho^2\sqrt[3]{ab^2}+t_2\cdot t_3-\rho\, ab-t_3\sqrt[3]{ab^2}-t_2\cdot\sqrt[3]{a^2b}-t_2\cdot t_3)$$
$$+3_1''\cdot(\rho^2ab+t_3\sqrt[3]{ab^2}+t_2\rho^2\sqrt[3]{a^2b}+t_2\cdot t_3-\rho\, ab-t_2\sqrt[3]{a^2b}-\rho t_3\sqrt[3]{ab^2}+t_2\cdot t_3)\Big]\cdot\frac{1}{3^2}$$

$$\mathrm{disc}(x) = \left[\pm 3nab\rho^2 \mp 3nab\rho + 3ab\rho t_2 - 3ab\rho^2t_2\right]\cdot\frac{1}{3^2}$$

For $t_2 = + n - \sqrt{-3}$ we have

$$\mathrm{disc}(x) = \left[3nab(\rho^2-\rho)-3abt_2(\rho^2-\rho) = 3ab(\rho^2-\rho)(n-t_2) = 3ab(\rho^2-\rho)(n-n+\sqrt{-3})\right]\cdot\frac{1}{3^2}$$

$$\mathrm{disc}(x) = 1/3^2\cdot 3^2\cdot a^2b^2\cdot 3^2$$

$$\mathrm{disc}(x) = 3^2\cdot a^2b^2\ ,\quad \text{since}\quad d_{K_6/k_2} = f_0^2 = (3ab)^2 = \mathrm{disc}(x)\ .$$

Thus (5.4) is a relative integral basis for O_{K_6} over O_{k_2}.

Also we have the same result for the case $t_2 = -n-\sqrt{-3}$.

ILLUSTRATION 5.5. (1) If $k_3 = Q(\sqrt[3]{2})$, $3_1 = (\sqrt[3]{2}+1)$ then $n = 1$ and $a = 3k+2$, $b = 3k+1$, so $t_1 = 0$. Therefore:

(1) $$O_{K_6} = \left[1,\ \frac{\sqrt[3]{2}+1-\sqrt{-3}}{3_1},\ \frac{\sqrt[3]{2}+\sqrt[3]{4}+0}{3_1}\right]\cdot O_{k_2}\ .$$

(2) For $K_3 = Q(\sqrt[3]{5})$, $3_1 = (\sqrt[3]{5}-2)$, $n = -2$ and $t_1 = 0$. We have

$$O_{K_6} = \left[1,\ \frac{\sqrt[3]{3}-2-\sqrt{-3}}{3_1},\ \frac{\sqrt[3]{5}+\sqrt[3]{25}+0}{3_1}\right]\cdot O_{k_2}\ .$$

For all Honda numbers 3_1 is necessarily P.I. so for such n we can construct a relative integral basis as in (5.4) for O_{K_6/K_2}.

THEOREM 5.6. The relative integral basis in "Type II" of O_{K_6} over O_{K_2} is:

$$O_{K_6} = \left[1,\ \frac{\sqrt[3]{ab^2}-1}{\sqrt{-3}},\ \frac{1+\sqrt[3]{ab^2}+\sqrt[3]{a^2b}}{3}\right]\cdot O_{k_2}\ .$$

PROOF. For "Type II" (i.e. $a \equiv \pm b \bmod 9$), $\theta_0 = \frac{1+\sqrt[3]{ab^2}+\sqrt[3]{a^2b}}{3}$ satisfies in equation $\theta_0^3 - \theta_0^2 + \theta_0\cdot\frac{1-ab}{9} - \frac{1-a+a^2b+ab^2}{27} = 0$, so then it is an integral and also $(\sqrt[3]{ab^2}-1)/\sqrt{-3}$ is an integral, because: From $(\sqrt[3]{ab^2}-1)/\sqrt{-3} = t$, we have $(\sqrt[3]{ab^2})^3 = (\sqrt{-3}\,t+1)^3$, so $-3\sqrt{-3}\,(t^3-t) = ab^2-1+9t^2$ and at last we have the equation: $t^6+t^4+\frac{(ab^2-1)^2}{27}+t^2\cdot\frac{(1+2ab^2)}{3} = 0$ which shows t is an integral. We take $x = [1,t,\theta_0]$, then

$$\mathrm{disc}(x) = \begin{vmatrix} 1 & \frac{\sqrt[3]{ab^2}-1}{\sqrt{-3}} & \frac{1+\sqrt[3]{ab^2}+\sqrt[3]{a^2b}}{3} \\ 1 & \frac{\rho\sqrt[3]{ab^2}-1}{\sqrt{-3}} & \frac{1+\rho\sqrt[3]{ab^2}+\rho^2\sqrt[3]{a^2b}}{3} \\ 1 & \frac{\rho^2\sqrt[3]{ab^2}-1}{\sqrt{-3}} & \frac{1+\rho^2\sqrt[3]{ab^2}+\rho\sqrt[3]{a^2b}}{3} \end{vmatrix}^2$$

$$= \left[\frac{\theta_0}{3\sqrt{-3}}(\rho^2\sqrt[3]{ab^2}-1-\rho\sqrt[3]{ab^2}+1)+\frac{\theta_0'}{3\sqrt{-3}}(\sqrt[3]{ab^2}-1-\rho^2\sqrt[3]{ab^2}+1)+\frac{\theta_0''}{3\sqrt{-3}}(\rho\sqrt[3]{ab^2}-1-\sqrt[3]{ab^2}+1)\right]^2$$

$\mathrm{disc}(x) = \left[\frac{3ab(\rho^2-\rho)}{3\sqrt{-3}}\right]^2 = a^2b^2$. Since $d_{K_6/K_3} = f_0^2 = (ab)^2 = \mathrm{disc}(x)$, then

$x = \left[1,\ \frac{\sqrt[3]{ab^2}-1}{\sqrt{-3}},\ \frac{\sqrt[3]{ab^2}+\sqrt[3]{a^2b}+1}{3}\right]$ is a relative integral basis of O_{K_6}/O_{k_2}.

ILLUSTRATION 5.7. For $k_3 = Q(\sqrt[3]{10})$, $a \equiv \pm b \bmod 9$, so

$$O_{K_6} = \left[1,\ \frac{\sqrt[3]{10}-1}{\sqrt{-3}},\ \frac{\sqrt[3]{10}+\sqrt[3]{10^2}+1}{3}\right]\cdot O_{k_2}\ .$$

Here we will give another theorem for computing a relative integral basis of O_{K_6} over O_{K_2} for $\pm n = 3t+1$ no matter whether 3_1 is a P.I. in O_{k_3} or not.

THEOREM 5.8. Let $n = 3t+1$, $m = -n$ be square-free in $k_3 = Q(\sqrt[3]{n})$ for Type I, then

$$O_{K_6} = \left[\frac{1+\sqrt[3]{n}+\sqrt[3]{n^2}}{\sqrt{-3}}, \sqrt[3]{n}, \sqrt[3]{n^2}\right]\cdot O_{k_2},$$

or

$$O_{K_6} = \left[\frac{1-\sqrt[3]{m}+\sqrt[3]{m^2}}{\sqrt{-3}}, \sqrt[3]{m}, \sqrt[3]{m^2}\right]\cdot O_{k_2}.$$

PROOF. At first we will show that $t = (1+\sqrt[3]{n}+\sqrt[3]{n^2})/\sqrt{-3}$ is an integral. We take

$$t = \frac{(1-\sqrt[3]{n})(1+\sqrt[3]{n}+\sqrt[3]{n^2})}{(1+\sqrt[3]{n})\cdot -\sqrt{-3}}, \qquad \frac{(1-n)-t\sqrt{-3}}{-\sqrt{-3}\,t} = \sqrt[3]{n}, \qquad \frac{((1-n)-t\sqrt{-3})^3}{3t^3\cdot\sqrt{-3}} = n,$$

$$(1-n)^3 + 3t\sqrt[3]{-3} - 3 + \sqrt{-3}\,(1-n)^2 - 9t^2(1-n) - nt^3\cdot 3\sqrt{-3} = 0,$$

$$\left[\sqrt{-3}\,(3t^3 - 3t(1-n)^2 - 3nt^3)\right]^2 = \left[-(1-n)^3 + 9t^2(1-n)\right]^2,$$

or briefly:

$$-27(1-n)^2t^6 - 27t^4(1-n)^2(2n+1) - 9t^2(1-n)^4 - (1-n)^6 = 0,$$

$$t^6 + (2n+1)t^4 + \frac{(1-n)^2}{3}\cdot t^2 + \frac{(1-n)^4}{27} = 0,$$

which shows that t is an integral. Now we take

$$x = \left[\frac{1+\sqrt[3]{n}+\sqrt[3]{n^2}}{\sqrt{-3}}, \sqrt[3]{n}, \sqrt[2]{n^2}\right],$$

$$\mathrm{disc}(x) = \begin{vmatrix} \frac{1+\sqrt[3]{n}+\sqrt[3]{n^2}}{\sqrt{-3}} & \sqrt[3]{n} & \sqrt[3]{n^2} \\ \frac{1+\rho\sqrt[3]{n}+\rho^2\sqrt[3]{n^2}}{\sqrt{-3}} & \rho\sqrt[3]{n} & \rho^2\sqrt[3]{n^2} \\ \frac{1+\rho^2\sqrt[3]{n}+\rho\sqrt[3]{n^2}}{\sqrt{-3}} & \rho^2\sqrt[3]{n} & \rho\sqrt[3]{n^2} \end{vmatrix}^2$$

$$= n^2\left[\frac{1+\sqrt[3]{n}+\sqrt[3]{n^2}}{\sqrt{-3}}\cdot(\rho^2-\rho) + \frac{1+\rho\sqrt[3]{n}+\rho^2\sqrt[3]{n^2}}{\sqrt{-3}}\cdot(\rho^2-\rho) + \frac{1+\rho^2\sqrt[3]{n}+\rho\sqrt[3]{n}}{\sqrt{-3}}\cdot(\rho^2-\rho)\right]^2$$

$$= n^2\left[\frac{\rho^2-\rho}{\sqrt{-3}}(3+0+0)\right]^2 = 3^2\cdot n^2$$

Since $d_{6/2} = f_0^2 = (3n)^2 = \mathrm{disc}(x)$, then

$$O_{K_6} = \left[\frac{1+\sqrt[3]{n}+\sqrt[3]{n^2}}{\sqrt{-3}}, \sqrt[3]{n}, \sqrt[3]{n^2}\right]\cdot O_{k_2}.$$

We can apply the same proof for $m = -n$.

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