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ON SOME SPACES OF SUMMABLE SEQUENCES AND THEIR DUALS

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ABSTRACT. Suppose that S is the space of all summable sequences α with $\|\alpha\|_S = \sup_{n\geq 0} \left|\sum_{j=n}^{\infty} \alpha_j\right|$ and J the space of all sequences β of bounded variation with $\|\beta\|_J = |\beta_0| + \sum_{j=1}^{\infty} |\beta_j - \beta_{j-1}|$. Then for α in S and β in J $\left|\sum_{j=0}^{\infty} \alpha_j\beta_j\right| \leq \|\alpha\|_S\|\beta\|_J$; this inequality leads to the description of the dual space of S as J. It, related inequalities, and their consequences are the content of this paper. In particular, the inequality cited above leads directly to the Stolz form of Abel's theorem and provides a very simple argument. Also, some other sequence spaces are discussed.

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1. INTRODUCTION.

Suppose S is the space of all summable sequences $s = (s_n)_{n=0}^{\infty}$, with norm given by $\|s\|_S = \operatorname{Sup}_{n\geq 0} \left|\sum_{j=n}^{\infty} s_j\right|$, the remainder in the sum of s after n-terms. An example of such an s, summable but not absolutely summable, is given by $s_n = \frac{(-1)^n}{n+1}$, n = 0, 1, 2, 3, ..., so that $\sum_{j=0}^{\infty} \frac{(-1)^n}{n+1}$ is an alternating convergent series. Here, as is readily verified, $\|s\|_S = \operatorname{Sup}_{n>0} \left|\sum_{j=n}^{\infty} \frac{(-1)^j}{j+1}\right| = \ln 2$, the the natural log of 2. This paper is based on the observation that S, with this norm, is naturally isomorphic to c_0, the space of all sequences having limit zero endowed with the supremum norm. Moreover, if we consider S on its own, we get several interesting results. For example, since S is isomorphic to c_0 and c_0*, the dual of c_0, is isomorphic to ℓ_1, the space of all absolutely summable sequences, then it follows that S*, the dual of S, is isomorphic to ℓ_1.

However, if we compute the dual of S without using the isomorphism with c_0, we find that S^* is isomorphic to J, the space of all sequences β of bounded variation with

$\|\beta\|_J = |\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|$; an example of such a β being given

by $\beta_n = \frac{1}{n+1}$ $n = 0, 1, 2, 3, \ldots$ so that $\|\beta\|_J = 1 + \sum_{n=1}^{\infty} [\frac{1}{n} - \frac{1}{n+1}] = 2$.

Of course, J is isomorphic to ℓ_1. Another example is that we get the following new inequality: If $\beta \in J$ and $s \in S$ $\left|\sum_{j=0}^{\infty} s_j\beta_j\right| < \|s\|_S \cdot \|\beta\|_J$, which can be used to give a proof of Abel's Continuity Theorem.

Similarly, we consider the space H of all sequences $\lambda = (\lambda_n)_{n=0}^{\infty}$ such that $\|\lambda\|_H = \sup_{n>0} \left|\sum_{j=0}^{n} \lambda_j\right|$ is finite; for example $\lambda_n = (-1)^n$, $n = 0, 1, 2, 3, \ldots$, so that $\|\lambda\|_H = 1$. We get the following inequality: If $\lambda \in H$, $\beta \in J$, and $\rho = \lim \beta$

then $\left|\lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n(\rho - \beta_{n-1})\right| \leq \|\lambda\|_H \cdot \|\beta\|_J$. Consequently we have that $\phi \in J^*$ if

and only if $\phi(\beta) = \lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n(\rho - \beta_{n-1})$ for $\beta \in J$.

Also, we give characterizations of the bounded linear operators on S easily obtained from a description of operators on c_0. These results, however, may be obtained using the uniform boundedness principle.

To make the presentation reasonably self-contained, we shall include a resume of pertinent results and definitions.

2. PRELIMINARIES

DEFINITION 2.1. Let $(X, \|\ \|_X)$ and $(Y, \|\ \|_Y)$ be two sequence spaces, which are Banach spaces with the respective norms. For a fixed $y = (y_n) \in Y$ define the mapping $\phi_y : X \to R$ by $\phi_y(x) = \sum_{n=0}^{\infty} x_n y_n$ for those $x = (x_n)$ in X for which the series converges.

Assume that $|\phi_y(x)| \leq M\|x\|_X$ with M some absolute constant, that is, ϕ_y is a bounded linear functional on X for any fixed $y \in Y$. Let X^* be the space of all bounded linear functionals ϕ on X, endowed with norm

$\|\phi\|_{X^*} = \sup_{\|x\|_X=1} |\phi(x)|$ and ψ the mapping from Y to X^* given by

$\psi(y) = \phi_y$. If ψ is onto and $\|y\|_Y = \|\phi_y\|_{X^*}$, then we say that Y is the dual of X, in this sense we write $Y = X^*$.

DEFINITION 2.2. We define the sequence spaces S, ℓ_1, ℓ_∞, c_0, and H respectively by

$$S = \{s = (s_n)_{n=0}^{\infty}, \text{ summable sequences such that } \|s\|_S = \sup_{p\geq 0} \left|\sum_{j=p}^{\infty} s_j\right|\},$$

$$\ell_1 = \{\delta = (\delta_n)_{n=0}^{\infty}, \|\delta\|_{\ell_1} = \textstyle\sum_{j=0}^{\infty} |\delta_j| < \infty\},$$

$$\ell_\infty = \{\mu = (\mu_n)_{n=0}^{\infty}, \|\mu\|_{\ell_\infty} = \sup_{j\geq 0} |\mu_j| < \infty\},$$

$$c_0 = \{\gamma = (\gamma_n)_{n=0}^{\infty}, \lim_{n\to\infty} \gamma_n = 0, \|\gamma\|_{c_0} = \sup_{n\geq 0} |\gamma_n|\}$$

$$H = \{\lambda = (\lambda_n)_{n=0}^{\infty},\ \|\lambda\|_H = \sup_{n\geq 0} \left|\sum_{j=0}^{n} \lambda_j\right| < \infty\}.$$

We believe that this is the first time that S is taken as a space with this norm.

THEOREM 2.3. The space S is a Banach space, isometrically isomorphic to c_0. The isometry $T:S \to c_0$ being given by $T(s) = (r_n(s))_{n=0}^{\infty}$ and $T^{-1}:c_0 \to S$ by $T^{-1}(\gamma) = (\gamma_n - \gamma_{n+1})_{n=0}^{\infty}$, where $r_n(s) = \sum_{j=n}^{\infty} \alpha_j$.

DEFINITION 2.4. Two Banach spaces $(X, \|\ \|_X)$ and $(Y, \|\ \|_Y)$ are isomorphic equivalents if there exist a one-to-one and onto linear mapping $T:X \to Y$ such that $N\|x\|_Y \leq \|Tx\|_Y \leq M\|x\|_X$ with N and M absolute constants. If there exists such a such a T with $\|Tx\|_Y = \|x\|_X$, then we say that X and Y are isometrically isomorphic.

Notice in Theorem 2.3 T and T^{-1} can be represented as infinite matrices,

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \ldots\ldots \\ 0 & 1 & 1 & 1 & 1 & \ldots\ldots \\ 0 & 0 & 1 & 1 & 1 & \ldots\ldots \\ 0 & 0 & 0 & 1 & 1 & \ldots\ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & \ldots\ldots \\ 0 & 1 & -1 & 0 & 0 & \ldots\ldots \\ 0 & 0 & 1 & -1 & 0 & \ldots\ldots \\ 0 & 0 & 0 & 1 & -1 & 0\ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}.$$

THEOREM 2.5. M is a bounded linear operator from c_0 into c_0 if and only if M is represented by an infinite matrix with columns in c_0 and rows in ℓ_1, uniformly bounded in ℓ_1. That is if $M = (m_{nk})$ then

i) $\lim_{n\to\infty} m_{nk} = 0$ for any fixed k,

ii) $\sup_{n\geq 0} \sum_{k=0}^{\infty} |m_{nk}| < \infty$.

THEOREM 2.6. ϕ is in $c_0{*}$ if and only if there is a $\delta = (\delta_n) \in \ell_1$ such that for any $\gamma = (\gamma_n) \in c_0$ $\phi(\gamma) = \sum_{n=0}^{\infty} \delta_n\gamma_n$, in which case there is only one such $\delta = (\delta_n) \in \ell_1$ and $\|\phi\|_{c_0{*}} = \|\delta\|_{\ell_1}$.

THEOREM 2.7. The space J is a Banach space, isometrically isomorphic to ℓ_1. The isometry $L:J \to \ell_1$ is given by $L(\beta) = (\beta_n - \beta_{n-1})_{n=0}^{\infty}$, $\beta_{-1} = 0$, and $L^{-1}:\ell_1 \to J$ by $L^{-1}(\delta) = (\sum_{j=0}^{n} \delta_j)_{n=0}^{\infty}$ with $\delta = (\delta_n) \in \ell_1$ and $\beta = (\beta_n) \in J$.

THEOREM 2.8. The space H is a Banach space isometrically isomorphic to ℓ_∞. The isometry $\psi:H \to \ell_\infty$ is given by $\psi(\lambda) = (\sum_{j=0}^{n} \lambda_j)_{n=0}^{\infty}$ and $\psi^{-1}:\ell_\infty \to H$ by $\psi^{-1}(\mu) = (\mu_n - \mu_{n-1})_{n=0}^{\infty}$ with $\mu_{-1} = 0$, $\mu = (\mu_n) \in \ell_\infty$, $\lambda = (\lambda_n) \in H$.

3. THE DUAL OF S

THEOREM 3.1 (Holder's type inequality). If $s = (s_n) \in S$ and $\beta = (\beta_n) \in J$ then $\left|\sum_{n=0}^{\infty} s_n\beta_n\right| \leq \|\beta\|_J \cdot \|s\|_S$.

Proof: For $N \geq 1$, we notice that

$$\left|\sum_{n=0}^{N-1} s_n\beta_n - r_0(s)\beta_0 - \sum_{n=1}^{N} r_n(s)[\beta_n - \beta_{n-1}]\right| = \left|r_N(s)\beta_N\right| = \left|r_N(s)\right|\cdot\left|\beta_N\right|,$$

where $r_n(s) = \sum_{j=n}^{\infty} s_j$. Now since s is summable and β is bounded, it follows that $\left|r_N(s)\right| \left|\beta_N\right| \to 0$ as $N \to \infty$. Therefore we have proved that

$$\sum_{n=0}^{\infty} s_n\beta_n = r_0(s)\beta_0 + \sum_{n=1}^{\infty} r_n(s)[\beta_n - \beta_{n-1}]; \text{ consequently}$$

$$\left|\sum_{n=0}^{\infty} s_n\beta_n\right| \leq \sup_{n\geq 0}\left|r_n(s)\right|\left(\left|\beta_0\right| + \sum_{n=1}^{\infty} \left|\beta_n - \beta_{n-1}\right|\right),$$

so that the theorem is proved.

Consider the mapping $\phi_\beta:S \to R$ defined by $\phi_\beta(s) = \sum_{n=0}^{\infty} s_n\beta_n$ where $\beta = (\beta_n)$ is a fixed element in J. In view of Theorem 3.1, we have ϕ_β is a bounded linear functional on S. At this point, a natural question is: are these all the bounded linear functionals on S? The next result tells us that the answer is yes.

THEOREM 3.2. (Duality Theorem). If $\phi \in S$ then there is a unique $\beta = (\beta_n) \in J$ such that $\phi = \phi_\beta$, that is, $\phi(s) = \sum_{n=0}^{\infty} s_n\beta_n$ for any $s = (s_n) \in S$. Moreover $\|\phi\|_{S*} = \|\beta\|_J$. Conversely if $\phi(s) = \sum_{n=0}^{\infty} s_n\beta_n$ then $\phi \in S*$ so that the mapping $\psi:J \to S*$ defined by $\psi(\beta) = \phi_\beta$ is an isometric isomorphism from J to $S*$.

Proof: If $\phi(s) = \sum_{n=0}^{\infty} s_n\beta_n$, then we already have seen by Theorem 3.1 that ϕ is a bounded linear functional on S, that is $\phi \in S*$. So it remains to prove the first part. Let $\phi \in S*$; then using Theorem 2.3, we have

$$\phi(s) = \phi(T^{-1} \circ T(s)) = (\phi \circ T^{-1})(T(s)) = (\phi \circ T^{-1})((\textstyle\sum_{j=n}^{\infty} s_j)_{n=0}^{\infty}).$$

Now notice that $\phi \circ T^{-1} \in c_0^*$, so that there is a $y = (y_n) \in \ell_1$ (see Theorem 2.6) such that

$$\phi(s) = (\phi \circ T^{-1})((\sum_{j=n}^{\infty} s_j)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} y_n(\sum_{j=n}^{\infty} s_j) \text{ and } \|\phi \circ T^{-1}\|_{c_0^*} = \|y\|_{\ell_1}. \quad (3.3)$$

Observe that (3.3) can be written $\phi(s) = \sum_{n=0}^{\infty} s_n \sum_{j=0}^{n} y_j$; therefore we may let $\beta = (\beta_n)$ where $\beta_n = \sum_{j=0}^{n} y_j$.

Since $\beta = L^{-1}(y)$, β is in J. Consequently given $\phi \in S^*$ there is a $\beta \in J$ such that $\phi = \phi_\beta$. On the other hand Theorem 2.6 tells us that $\|\phi \circ T^{-1}\|_{c_0^*} = \sum_{n=0}^{\infty} |y_n|$; therefore $\|\beta\|_J = \sum_{n=0}^{\infty} |y_n| = \|\phi \circ T^{-1}\|_{c_0^*} \leq \|T^{-1}\| \cdot \|\phi\|_{S^*}$. Now as $\|T^{-1}\| \leq 1$, it follows that $\|\beta\|_J \leq \|\phi\|_{S^*}$. Also notice that Theorem 3.1 implies $\|\phi\|_{S^*} \leq \|\beta\|_J$ (since $\phi = \phi_\beta$). Therefore putting together these last two inequalities we have $\|\phi\|_{S^*} = \|\beta\|_J$; consequently the mapping $\psi: J \to S^*$ defined by $\psi(\beta) = \phi_\beta$ is an isometry, so that the theorem is proved.

We have used Theorem 3.1 to characterize all bounded linear functionals on S. Now we are going to use this theorem again to give a proof of Abel's continuity theorem which is as follows.

THEOREM 3.4 (Abel's continuity theorem). If $\alpha = (\alpha_n)$ is summable and $f(z) = \sum_{n=0}^{\infty} \alpha_n z^n$, then $\lim_{z \to 1} f(z) = f(1)$, where z is restricted to approach the point 1 in such a way that $|z| < 1$ and $\frac{|1-z|}{1-|z|}$ remains bounded.

Proof: First of all, let C be a positive absolute constant such that $\frac{|1-z|}{1-|z|} \leq C$ and $|z| < 1$. Notice that $|\sum_{p=N}^{\infty} \alpha_p z^p| \leq \|(z^p)_{p=N}^{\infty}\|_J \cdot \|(\alpha_p)_{p=N}^{\infty}\|_S$ for $N \geq 1$ by the above inequality applied to the sequences (α_p) and (z^p), since $(z^p)_{p=N}^{\infty}$ is in J; in fact, since $|z| < 1$

$$\|(z^p)_{p=N}^{\infty}\|_J = |z|^N + |z^N - z^{N+1}| + |z^{N+1} - z^{N+2}| + \ldots\ldots =$$

$$= |z|^N + |1-z|(|z|^N + |z|^{N+1} + \ldots\ldots) = |z|^N(1 + \frac{|1-z|}{1-|z|}).$$

Now using the hypothesis we have $\|(z^p)_{p=N}^{\infty}\|_J \leq 1 + C$. Consequently, $|\sum_{p=N}^{\infty} \alpha_p z^p| \leq (1+C)\|(\alpha_p)_{p=N}^{\infty}\|_S \to 0$ as $N \to \infty$ since $(\alpha_n) \in S$. Then

$$|\sum_{p=0}^{\infty} \alpha_p z^p - \sum_{p=0}^{\infty} \alpha_p| \leq |\sum_{p=0}^{N-1} \alpha_p z^p - \sum_{p=0}^{N-1} \alpha_p| + |\sum_{p=N}^{\infty} \alpha_p z^p - \sum_{p=N}^{\infty} \alpha_p| = A+B.$$

For $\varepsilon > 0$, fix N so that $B \leq (2+C)\|(\alpha_p)_{p=N}^{\infty}\|_S \leq \varepsilon/2$. For $|z-1|$ sufficiently small, $A < \varepsilon/2$. Hence $\lim_{z \to 1} \sum_{p=0}^{\infty} \alpha_p z^p = \sum_{p=0}^{\infty} \alpha_p$. Therefore the theorem is proved.

4. THE DUAL OF J

Of course, for fixed $s = (s_n)$ in S the formula $\phi_s(\beta) = \sum_{j=0}^{\infty} s_j\beta_j$ defines a bounded linear functional on J. However, these are not all the members of J^*. As a key to our description of J^*, we note that if $\beta = (\beta_n)$ is in J with limit ρ then $\phi_s(\beta) = (\sum_{j=0}^{\infty} s_j)\rho + \sum_{j=0}^{\infty} s_j(\beta_j - \rho) = \lambda_0\rho + \sum_{j=1}^{\infty} \lambda_j(\rho - \beta_{j-1})$, where $\lambda_0 = \sum_{j=0}^{\infty} s_j$ and for $j \geq 1$ $\lambda_j = -s_{j-1}$. Here $\sum_{j=0}^{\infty}\lambda_j = 0$; so that λ is in S.

THEOREM 4.1 (Holder's type inequality). If $\lambda = (\lambda_n) \in H$, $\beta = (\beta_n) \in J$, and $\rho = \lim \beta$ then $\left|\lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n(\rho - \beta_{n-1})\right| \leq \|\lambda\|_H \cdot \|\beta\|_J$.

Proof: This inequality follows easily from the observation that

$$\lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n(\rho - \beta_{n-1}) = \lambda_0\beta_0 + \sum_{n=1}^{\infty} \Big(\sum_{j=0}^{n} \lambda_j\Big)[\beta_n - \beta_{n-1}]; \text{ so that}$$

$$\left|\lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n(\rho - \beta_{n-1})\right| \leq \sup_{n\geq 0} \left|\sum_{j=0}^{n} \lambda_j\right| \Big(|\beta_0| + \sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}|\Big).$$

As we did for S, we consider the mapping $\phi_\lambda: J \to R$ defined by $\phi_\lambda(\beta) = \lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n(\rho - \beta_{n-1})$ where $\lambda = (\lambda_n)$ is a fixed element in H. Theorem 4.1 assures us that ϕ_λ is a bounded linear functional on J. The claim of the next theorem is that these are all the bounded linear functionals on J. Here, our representation is in a slightly different sense than that of the Definition 2.1, the latter appearing more natural to us.

THEOREM (4.2) (Duality Theorem). If $\phi \in J^*$ then there is a unique $\lambda = (\lambda_n) \in H$ such that $\phi = \phi_\lambda$, that is, $\phi(\beta) = \lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n(\rho - \beta_{n-1})$ for any $\beta = (\beta_n) \in J$. Moreover $\|\phi\|_{J^*} = \|\lambda\|_H$. Conversely if $\phi = \phi_\lambda$ then $\phi \in J^*$. So that the mapping $\psi: H \to J^*$ defined by $\psi(\lambda) = \phi_\lambda$ is an isometric isomorphism. Here, $\rho = \lim \beta$.

Proof: By the observation preceding the theorem we see that it remains to prove the first part. Let $\phi \in J^*$; then in fact using Theorem 2.7 $\phi(\beta) = \phi(L^{-1} \circ L(\beta)) = (\phi \circ L^{-1})(L(\beta)) = (\phi \circ L^{-1})((\beta_n - \beta_{n-1})_{n=0}^{\infty})$, $\beta_{-1} = 0$. Now notice that $\phi \circ L^{-1} \in \ell_1^*$, so that there is a $\mu = (\mu_n) \in \ell_\infty$ such that

$$(4.3) \quad \phi(\beta) = (\phi \circ L^{-1})((\beta_n - \beta_{n-1})_{n=0}^{\infty}) = \sum_{n=0}^{\infty} \mu_n(\beta_n - \beta_{n-1}).$$

By Theorem 2.8 $\mu_n = \sum_{j=0}^{n} \lambda_j$ for $n \geq 0$ and some $\lambda = (\lambda_j) \in H$;

consequently (4.3) becomes $\phi(\beta) = \sum_{n=0}^{\infty} \Big(\sum_{j=0}^{n} \lambda_j\Big)(\beta_n - \beta_{n-1})$.

Observe now that

$$\sum_{n=0}^{N} \Big(\sum_{j=0}^{n} \lambda_j\Big)(\beta_n - \beta_{n-1}) = \lambda_0\beta_0 + \sum_{n=1}^{N} \Big(\sum_{j=0}^{n} \lambda_j\Big)(\beta_n - \beta_{n-1}) =$$

$$= \lambda_0\beta_N + \sum_{n=1}^{N}\lambda_n[\beta_N - \beta_{n-1}] = \lambda_0\beta_N + \sum_{n=1}^{N}\lambda_n[\beta_N - \rho + \rho - \beta_{n-1}] =$$

$= \lambda_0\rho + [\beta_N - \rho] \sum_{n=0}^{N} \lambda_n + \sum_{n=1}^{N} \lambda_n[\rho-\beta_{n-1}]$. Taking the limit as $N \to \infty$ we get

$$\sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} \lambda_j\right)(\beta_n - \beta_{n-1}) = \lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n[\rho - \beta_{n-1}]$$

since $[\beta_N - \rho] \sum_{n=0}^{N} \lambda_n \to 0$ as $N \to \infty$ ($\rho = \lim \beta$). Therefore we have

$\phi(\beta) = \lambda_0\rho + \sum_{n=1}^{\infty} \lambda_n[\rho - \beta_{n-1}]$.

Again by Theorem 2.8 $\|\phi \circ L^{-1}\|_{\ell_1^*} = \|\mu\|_{\ell_\infty} = \sup_{n\geq 0} |\mu_n| = \sup\left|\sum_{j=0}^{n} \lambda_j\right| = \|\lambda\|_H$ or $\|\lambda\|_H = \|\phi \circ L^{-1}\|_{\ell_1^*} \leq \|\phi\|_{J^*}\cdot\|L^{-1}\| \leq \|\phi\|_{J^*}$ since $\|L^{-1}\| \leq 1$. On the other hand Theorem 4.1 gives us that $|\phi(\beta)| \leq \|\lambda\|_H\|\beta\|_J$(since $\phi = \phi_\lambda$), which implies $\|\phi\|_{J^*} \leq \|\lambda\|_H$. Putting together these two inequalities we have $\|\phi\|_{J^*} = \|\lambda\|_H$, so that the theorem is proved.

5. CHARACTERIZATION OF OPERATORS ON S

In this section we characterize all bounded linear operators on S, even though an alternate proof can be given using the uniform boundedness principle. We rather use the characterization of operators on c_0. In fact we have the following result.

THEOREM 5.1. The linear mapping $A:S \to c_0$ is bounded if and only if there is an infinite matrix (a_{nk}) such that for any $s = (s_n) \in S$

$$As = \begin{bmatrix} a_{00} & a_{01} & a_{02} & \cdots & a_{0n} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1n} & \cdots \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ a_{n0} & a_{n1} & a_{n2} & \cdots & a_{nn} & \cdots \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \\ \cdot \\ \cdot \end{bmatrix}$$

satisfying

i) $\lim_{n\to\infty} a_{n_k} = 0$, for any fixed $k \geq 0$ i.e. the columns of the matrix are in c_0,

ii) $\sup_{n\geq 0} \sum_{k=0}^{\infty} \left|a_{n(k-1)} - a_{nk}\right| < \infty$ i.e. the rows of the matrix are uniformly bounded in J.

Here $a_{n(-1)} = 0$ for $n = 0, 1, 2, 3, \ldots$.

Proof: Note that the operator $P = AT^{-1}$ where T^{-1} is as in Theorem 2.3 ($c_0 \xrightarrow{T^{-1}} S \xrightarrow{A} c_0$) maps c_0 into c_0, that is $P:c_0 \to S \to c_0$, so that applying Theorem 2.5 we get the desired result.

THEOREM 5.2. The linear mapping $B:S \to S$ is bounded if and only if B is represented by an infinite matrix such that for some $A:S \to c_0$ we have

$B = A - \tilde{A}$, where $\tilde{A}$ is defined by $\tilde{A} = (\tilde{a}_{nk})$ with $\tilde{a}_{nk} = a_{(n+1)k}$.
($\tilde{A}$ is a shifting of A up by one row) Note that $a_{nk} = \sum_{j=n}^{\infty} b_{jk}$.

Proof: We define $A: S \overset{B}{\to} S \overset{T}{\to} c_0$, where T is as in Theorem 2.3. That is, $A = TB$ so that $B = T^{-1}A$.

Now using the matrix representation for T^{-1} we get

$$B = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \dots \\ 0 & 1 & -1 & 0 & 0 \dots \\ 0 & 0 & 1 & -1 & 0 \dots \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} A = [I - \tilde{I}]A = IA - \tilde{I}A = A - \tilde{A} \quad \text{where}$$

I is the identity matrix and $\tilde{I}$ is formed by the shifting of I up by one row.

Notice Theorem 5.2 is equivalent to saying that $B: S \to S$ is a bounded operator if B is an infinite matrix such that each column of B is summable, and if we consider the matrix $r(B)$ with $r(B) = (y_{nk})$, where $y_{nk} = \sum_{j=n}^{\infty} b_{jk}$, then then the rows of $r(B)$ are uniformly bounded in J.

THEOREM 5.3. The linear mapping $C: c_0 \to S$ is bounded if and only if C is represented by an infinite matrix such that for some $M: c_0 \to c_0$ we have $C = M - \tilde{M}$, $\tilde{M}$ is a shifting of M up by one row.

Proof: Notice $c_0 \overset{C}{\to} S \overset{T}{\to} c_0$, T as in Theorem 2.3. So define $M = TC$; therefore $C = T^{-1}M = M - \tilde{M}$.

Note here that the M's are special A's as are the B's. The C's are special B's. The reader might wish to compare these last three theorems with Exercises 45 and 46, pg. 77, of [1].

We would like to point out that the spaces H and J may also be found in [1] (pg. 240). Another space there, denoted by cs, is the space of all summable sequences $s = (s_j)$ normed with norm on H, $\max_{n \geq 0} \left| \sum_{j=0}^{n} s_j \right|$. It may be shown, as pointed out there, that cs* looks like J. We hasten here to note that J is not the dual of cs in the sense of Definition 2.1. In particular, the inequality $\left| \sum_{j=0}^{\infty} s_j \beta_j \right| < \|s\|_{cs} \cdot \|\beta\|_J$ does not hold. Consider the example:

Let $s = (s_n)$ be given by

$s_0 = 1$, $s_1 = s_2 = s_3 = s_4 = 0$, $s_5 = -2$, $s_6 = s_7 = \dots = 0$; also, $\beta = (\beta_n)$:

$\beta_0 = 0$, $\beta_1 = -1$, $\beta_2 = -2$, $\beta_3 = -3$, $\beta_4 = -4$, $\beta_5 = \beta_6 = \dots = -4$.

Notice that $\sum_{j=0}^{\infty} s_j \beta_j = 8$, $\|\beta\|_J = 4$ and $\|s\|_{cs} = \max_{n \geq 0} \left| \sum_{j=0}^{n} s_j \right| = 1$. Therefore

$\left| \sum_{j=0}^{\infty} s_j \beta_j \right| \leq [\max_{n \geq 0} \left| \sum_{j=0}^{n} s_j \right|] \cdot \|\beta\|_J$ does not hold.

To illustrate further the difference between cs and S, we note that cs and S are equivalent Banach spaces: if s is in S then $\|s\|_{cs} \leq 2\|s\|_S$ and $\|s\|_S \leq 2\|s\|_{cs}$. So that ϕ is in S^* if and only if ϕ is in cs*.

For $j \geq 0$ let e_j be the sequence τ such that $\tau_j = 1$ and if $k \neq j$ $\tau_k = 0$.

Now for ϕ in $S^*, \phi(s) = \sum_{j=0}' s_j\phi(e_j)$; the sequence $\{\phi(e_j)\}_{j=0}^{\infty}$ is in J and $\|\{\phi(e_j)\}\|_J = \|\phi\|_{S^*}$. But if β is the sequence of the example and ϕ is defined on S by $\phi(s) = \sum_{j=0}^{\infty}\beta_j s_j$ (so that $\beta_j = \phi(e_j)$) then $\|\beta\|_J = 4$. So the natural association of ϕ in cs* with the sequence $\{\phi(e_i)\}$ in J is not an isometry. There is an isometry from cs* onto J, obtained by noting that cs is isometrically isomorphic to the space c of all convergent sequences normed with norm in ℓ_∞; the dual c* being isometrically isomorphic to ℓ_1. This representation of c* as ℓ_1, and also the representation of cs* as J, is complicated by the fact that the sequence $\{e_j\}$ does not have dense span in c. However, the latter representation (of cs* as J) is made somewhat simpler by applying Theorems 2.7 and 3.5. For s in S let $S_N(s) = \sum_{j=0}^{N} s_j$, the Nth partial sum of s, and $\sigma(s) = \sum_{j=0}^{\infty} s_j$. Let ϕ be in cs* with $\phi(e_j) = \beta_j$. Now, with L as in Theorem 2.7, we have $\phi(s) = \sum_{j=0}^{\infty} s_j\phi(e_j) = \sum_{j=0}^{\infty}\beta_j s_j =$ $\beta_0 r_0(s) + \sum_{n=1}^{\infty} r_n(s)[\beta_n - \beta_{n-1}] = \beta_0\sigma(s) + \sum_{n=1}^{\infty}[\sigma(s) - S_{n-1}(s)][\beta_n - \beta_{n-1}] =$ $\sigma(s)\nu_0 + \sum_{n=1}^{\infty} S_{n-1}(s)\nu_n$ ($\nu_0 = \lim\beta$, $\nu_n = \beta_{n-1} - \beta_n$); so $\nu = L(\mu)$, with $\mu_0 = \lim\beta$, $\mu_n = \lim\beta + \beta_0 - \beta_n$. These computations suggest that if one uses (as in [1]) the space cs rather than S the norm to be used in J should be given by $\|\beta\| = |\lim\beta| + \sum_{n=1}^{\infty}|\beta_n - \beta_{n-1}|$. However, the norm used there is the same as ours (see 1, pg. 239). With this norm, $\|\beta\|$, on J, the Holder's type inequality we get is $\left|\sum_{j=0}^{\infty}\alpha_j\beta_j\right| \le \|\alpha\|_{cs}\|\beta\|$, α in S and β in J.

The reader might wish to compare the remarks of this last paragraph with some of D. J. H. Garling in pages 999-1000 of [2] and page 964 of [3].

These ideas were suggested by a norm on certain power series spaces introduced by the latter author in an investigation [4] of Fourier sine series and Lidstone series.

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