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DIAGONALIZATION OF A SELF-ADJOINT OPERATOR ACTING ON A HILBERT MODULE

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ABSTRACT. For each bounded self-adjoint operator T on a Hilbert module H over an H*-algebra A there exists a locally compact space $\mathfrak{M}$ and a certain A-valued measure μ such that H is isomorphic to $L^2(\mu)\otimes A$ and T corresponds to a multiplication with a continuous function. There is a similar result for a commuting family of normal operators. A consequence for this result is a representation theorem for generalized stationary processes.

KEY WORDS AND PHRASES. H*-algebra, Hilbert module, A-linear operator. 1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. PRIMARY: 46H25. SECONDARY: 46K15, 47B10, 47A67, 46G10, 60G10.

1. INTRODUCTION.

The diagonalization theorem states that for each bounded self-adjoint linear operator T acting on a Hilbert space H there exists a measure space (S, μ) and a real valued measurable function h(s) such that H is isomorphic to $L^2(S, \mu)$ and T corresponds to the multiplication with h(s). Furthermore, the space (S, μ) could be selected in such a way that there is a Hausdorff topology on S with respect to which h(s) is continuous, S is locally compact and which makes μ a regular Borel measure. In this note we shall give a suitable generalization of this fact.

The situation is somewhat more complex in our case. The space $L^2(S, \mu)$ needs to be replaced by the tensor product $L^2(\mu)\otimes A$, which is less manageable. This space is properly defined below.

2. PRELIMINARIES.

Let A be a proper H*-algebra (Ambrose [1]) and let $\tau A = \left\{xy \mid x, y \in A\right\}$ be its trace-class (Saworotnow and Friedell [2]); let X be a locally compact Hausdorff space and let μ be a positive τA-valued Borel measure on X. The last statement means that μ is defined on the class β of all Borel subsets Δ of X having the property that $\Delta \subset Q$ for some compact set Q, and μ is such that $(\mu(\Delta)x, x) \geq 0$ for all $\Delta \in \beta$ and each $x \in A$. Members of β will be called bounded Borel sets (a bounded Borel set is a Borel set included in a compact set). Note that the scalar-valued function $m\Delta = \operatorname{tr}\mu\Delta, \Delta \in \beta$, is an ordinary Borel measure on X; it coincides with the total variation $|\mu|$ (Definition in III.1.4 of Dunford and Schwartz [3]) of μ.

Let S(X) and S(X,A) be respectively the classes of all complex-valued and A-valued simple functions of X. One can define the integrals for members $\psi(x) = \Sigma_i \lambda_i \phi_{\Delta i}(x)$ and $\xi(x) = \Sigma_i a_i \phi_{\Delta i}(x)$ ($\Delta_i \epsilon \beta$, $a_i \epsilon A$ and λ_i's are complex numbers) of S(X) and S(X,A) in the usual way by setting

$$\int \psi d\mu = \Sigma \lambda_i \mu\Delta_i \text{ and } \int \xi d\mu = \Sigma a_i \mu\Delta_i \tag{2.1}$$

and then extending it to larger classes using the norms

$$||\psi|| = \int |\psi| dm = \Sigma |\lambda_i| m\Delta_i \tag{2.2}$$

and

$$||\xi|| = \Sigma ||a_i|| m\Delta_i. \tag{2.3}$$

Let L(X) and B(X,A) denote respectively the classes of those functions to which the integrals are extendable in this fashion. (Note that S(X) is dense in L(X) and S(X,A) is dense in B(X,A)).

Then it is easy to see that

$$\tau(\int \psi d\mu) \leq ||\psi|| \text{ and } \tau(\int \xi d\mu) \leq ||\xi|| \tag{2.4}$$

hold for all $\psi \epsilon L(X)$ and $\xi \epsilon B(X,A)$. (For a discussion of integrals of this type we refer the reader to Bogdanowicz [4]).

LEMMA 1. If $a \epsilon A$ and either $\psi \epsilon L(X)$ or $\psi \epsilon B(X,A)$, then $a\psi \epsilon B(X,A)$ and $\int a\psi d\mu = a\int \psi d\mu$. If $\psi \epsilon S(X,A)$ and $\psi \geq 0$ m-almost everywhere then $tr\int \psi d\mu \geq 0$.

PROOF. The first assertion is easy to verify. Let ψ be a simple function such that "$\psi(x) \geq 0$" holds outside of some set $\Delta \epsilon \beta$ with $m\Delta = tr\mu\Delta = 0$. Then ψ can be represented in the form $\psi = \Sigma_{i=1}^n a_i \phi_{\Delta i}$ with $\Delta_1, \Delta_2, \ldots, \Delta_n$ disjoint ($\Delta_i \epsilon \beta$) and $a_i \geq 0$ for each i for which "$m\Delta_i = \tau(\mu\Delta_i) = tr(\mu\Delta_i) > 0$" holds. Then $tr\int \psi d\mu = tr\,\Sigma_i a_i \mu\Delta_i = \Sigma_i tr(a_i \mu\Delta_i) = \Sigma_i tr\sqrt{\mu\Delta_i}\, a_i \sqrt{\mu\Delta_i} \geq 0$.

Let $L^2(\mu) = \{f : X \longrightarrow C \mid f \text{ is m-measurable and } \int |f|^2 dm < \infty\}$ ($m = tr\mu$) be the set of all square m-measurable complex-valued functions. Then there is a τA-valued inner product

$$[\psi_1, \psi_2] = \int \bar{\psi}_1 \psi_2 d\mu \tag{2.5}$$

defined on $L^2(\mu)$ such that $(\psi_1, \psi_2) = tr[\psi_2, \psi_1] = \int \bar{\psi}_2 \psi_1 dm$ is an ordinary scalar product on $L^2(\mu)$ making $L^2(\mu)$ a Hilbert space.

LEMMA 2. Let $\psi_1, \psi_2, \ldots \psi_n \epsilon L^2(\mu)$ and let $a_1, a_2, \ldots, a_n \epsilon A$. Then

$$tr\,\Sigma_{i,j} a_i^* \int \bar{\psi}_i \psi_j d\mu a_j \geq 0 \tag{2.6}$$

PROOF. Let $n(\psi)$ denote the norm on $L^2(\mu)$: $n(\psi)^2 = (\psi,\psi) = \int |\psi|^2 dm$. Let $\epsilon > 0$ be arbitrary; let $\eta_1, \eta_2, \ldots \eta_n \epsilon S(X)$ be such that $n(\psi_i - \eta_i) < \epsilon$ for $i = 1,2,\ldots n$. Then

$$\left| tr\,\Sigma a_i^* \int \bar{\psi}_i \psi_j d\mu a_j - tr\,\Sigma a_i^* \int \bar{\psi}_i \psi_j d\mu a_j \right| =$$

$$\left| \Sigma tr(a_j a_i^* \int (\bar{\psi}_i \psi_j - \bar{\eta}_i \eta_j) d\mu) \right| \leq \Sigma \tau(a_j a_i^*) \tau(\int (\bar{\psi}_i \psi_j - \bar{\eta}_i \eta_j) d\mu) \leq$$

$$\Sigma ||a_j|| \cdot ||a_i^*|| \int |\bar{\psi}_i \psi_j - \bar{\eta}_i \eta_j| dm \leq \Sigma ||a_j|| \cdot ||a_i^*|| (\int |\bar{\psi}_i| |\psi_i - \eta_j| dm + \int |\bar{\psi}_i - \bar{\eta}_i| |\eta_j| dm \leq$$

$$\Sigma ||a_i|| \cdot ||a_j^*|| (n(\psi_i).n(\psi_j - \eta_j) + n(\psi_i - \eta_i).n(\eta_j)) \leq \Sigma_{i=1} \epsilon (2n(\psi_i) + \epsilon) ||a_i|| \cdot ||a_j^*||$$

and the last sum can be made arbitrarily small by selecting ϵ small enough. On the other hand one can see that

$$tr(\Sigma_{i,j} a_i^* \int \bar{\eta}_i \eta_j d\mu a_j) = tr\int (\Sigma_j a_j \eta_j)(\Sigma_i a_i \eta_i)^* d\mu \geq 0 \tag{2.7}$$

since $(\Sigma_j a_j \eta_j)(\Sigma_i a_i \eta_i)^*$ is positive and simple. Hence $\mathrm{tr}\,\Sigma a_i^* \int \bar{\psi}_i \psi_j d\mu a_j \geq 0$.

COROLLARY. The expression $z = \Sigma_{i,j}(a_i^* \int \bar{\psi}_i \psi_j d\mu a_j)$ is a positive member of τA.

PROOF. Note that the expression $(za,a) = \mathrm{tr}(a^*za)$ is of the same form as $\mathrm{tr}z$. Hence $(za,a) \geq 0$ for each $a \in A$.

Now consider the space K of all tensors $f = \Sigma_{i=1}^n \psi_i \otimes a_i$ with $\psi_1, \psi_2, \ldots, \psi_n \in L^2(\mu)$ and $a_1, a_2, \ldots, a_n \in A$. Define the positive form $[f,g]$ on K by setting

$$[f,g] = \Sigma_{i,j} a_i^* (\int \bar{\psi}_i \eta_j d\mu) b_j \tag{2.8}$$

(here $g = \Sigma_j \eta_j \otimes b_j$). Let $\mathcal{N} = \{f \in K : [f,f] = 0\}$, $K' = K|\mathcal{N}$; we define $L^2(\mu) \otimes A$ to be the completion of K' with respect to the norm $||f|| = \sqrt{\tau[f,f]}$ (modulo the set $\mathcal{N}$). It is not difficult to see that $L^2(\mu) \otimes A$ is a Hilbert module.

Let h be a bounded continuous real valued function on X. Define the operator T_h on $L^2(\mu) \otimes A$ by setting

$$T_h(f) = T_h(\Sigma \psi_i \otimes a_i) = \Sigma(\psi_i h) \otimes a_i \tag{2.9}$$

Then T_h is a bounded self-adjoint (in the sense that $[T_h(f),g] = [f,T_h(g)]$ holds). Also T_h is A-linear (additive and A-homogeneous in the sense that $T_h(fa) = T_h(f)a$ for all $f \in L^2(\mu) \otimes A$, $a \in A$).

The fact that T_h is bounded (in the sense that "$||T_h(f)|| \leq M||f||$" holds for some M) can be verified directly, using §10 of Naimark [5]. Let $f = \Sigma_i \psi_i \otimes a_i$ be a fixed member of K. Consider the positive linear functional

$$p(y) = \mathrm{tr}[f, Ty(f)] = \mathrm{tr}\,\Sigma a_i^* \int \bar{\psi}_i y \psi_j d\mu a_j \tag{2.10}$$

on the space BC(X) of all bounded continuous (complex) functions on X. It follows from the proposition I in subsection 4 of §10 in Naimark [5] that $p(h^*h) \leq ||h^*h||_\infty p(e) = ||h||_\infty^2 p(e)$. Thus:

$$||T_h(f)||^2 = \mathrm{tr}[T_h(f),T_h(f)] = \mathrm{tr}[f,T_{h^*h}(f)] = p(h^*h) \leq ||h||_\infty^2 p(e) =$$

$$||h||_\infty^2 \mathrm{tr}[f,f] = ||h||_\infty^2 ||f||^2. \tag{2.11}$$

We also see that $||T_h|| \leq ||h||_\infty$. It turns out that each bounded self-adjoint A-linear operator is of the form T_h described above.

3. MAIN RESULTS.

Definition. An A-linear operator T on a Hilbert module H is said to be cyclic if there exists $f_o \in H$ such that the set $\{\Sigma_{k=0}^n \lambda_k T^k(f_o)a_k : a_k \in A, \lambda_k \text{ complex}\}$ is dense in H (we assume that $T^o(f_o) = If_o = f_o$).

THEOREM 1. For each bounded A-linear self-adjoint operator T on a Hilbert module H there exists a locally compact Hausdorff space X, a τA-valued positive regular measure μ defined on the class β of bounded (dominated by compact sets) Borel subsets of X and a bounded continuous real valued function h on X such that H is isometrically isomorphic to $L^2(\mu) \otimes A$ and T corresponds to the operator T_h (described above) acting on $L^2(\mu) \otimes A$. If T is cyclic, then X is homeomorphic to the compact subset of the real line.

PROOF. Let B be the commutative B*-algebra generated by T and the identity operator I (note that each member of B is A-linear). Let $\mathcal{M}$ be the set of maximal ideals of B, let τ be the standard Gelfand topology on $\mathcal{M}$ and let $S \longrightarrow S(M)$ be the Gelfand map of B into the continuous complex functions on $\mathcal{M}$. Note that $\mathcal{M}$ is homeomorphic to the spectrum of T, which is a compact subset of the real line. We consider 2 cases.

CASE I. First assume that there exists $f_o \in H$ such that the set

$$H^1 = \left\{ \Sigma_{i=1}^n S_i(f_o)a_i : S_i \in B, a_i \in A \right\} \tag{3.1}$$

is dense in H (this is equivalent to the statement that T is cyclic).

Let β be the class of all Borel subsets of $\mathfrak{M}$ (each $\Delta \in \beta$ is bounded since $\mathfrak{M}$ is compact) and let $\Delta \longrightarrow P_\Delta$ be a spectral measure on β (§17, Proposition II in subsection 4 of Naimark [5]) such that $S = \int_{\mathfrak{M}} S(M)dP_M$. Note that each P_Δ is A-linear since it commutes with linear maps $f \longrightarrow fa (a \in A)$ (which commute with all $S \in B$). Then map

$$\Delta \longrightarrow \mu_\Delta = [f_o, P_\Delta f_o] \tag{3.2}$$

is a τA-valued positive measure on β, and for each $S \in B$ we have

$$\int_{\mathfrak{M}} S(M)d\mu(M) = \int S(M)d[f_o, P_M f_o] = [f_o, \int S(M)dP_M] = [f_o, Sf_o] \tag{3.3}$$

(here, as above, [,] denotes the generalized inner product on H). In this case we can take $X = \mathfrak{M}$. The correspondence

$$Sf_o \longleftrightarrow S(M) \tag{3.4}$$

is a (linear) isomorphism between the linear subspace $K = \left\{ S_{fo} \middle| S \in B \right\}$ of H and C(X) = $C(\mathfrak{M})$. This correspondence can be extended in the obvious way to the isomorphism between the closure of K and the Hilbert space $L^2(\mu)$. The τA-valued inner product is also preserved by this correspondence: if $S_1, S_2 \in B$ then

$$[S_1 f_o, S_2 f_o] = [f_o, S_1^* S_2 f_o] = \int \bar{S}_1(M)S_2(M)d\mu(M) \tag{3.5}$$

We extend this isomorphism to a correspondence between H^1 and a dense subset of $L^2(\mu)\otimes A$ by setting

$$\Sigma_k S_k(f_o)a_k \longleftrightarrow \Sigma\, S_k(M)\otimes a_k \tag{3.6}$$

This correspondence also preserves the (vector) inner product: if $f = \Sigma\, S_k(f_o)a_k$ and $g = \Sigma Q_i(f_o)b_i$, then

$$[f,g] = \Sigma_{k,i} a_k^* [S_k(f_o), Q_i(f_o)]b_i = \Sigma_{k.i} a_k^* \int \bar{S}_k(M)Q_i(M)d\mu b_i \tag{3.7}$$

We extend it to an isomorphism between H and $L^2(\mu)\otimes A$. It is easy to check that T correponds to the operator T_h of multiplication with function h(M) = T(M):

$$T(\Sigma_k S_k(f_o)a_k = \Sigma_k TS_k(f_o)a_k \longleftrightarrow \Sigma_k T(M)S_k(M)\otimes a_k \tag{3.8}$$

The function h is real valued since $T^* = T$, and $||h||_\infty \leq ||T||$.

Note also that in this case $\mathfrak{M}$ is homeomorphic to the spectrum of T, which is a compact subset of the real line. This implies the last assertion of the theorem.

CASE II. Now let us consider the general case. For any $f \in H$ let H(f) be the closure of the set $\left\{ \Sigma_{i=1}^n S_i(f)a_i : S_i \in B, a_i \in A \right\}$. Then it follows from Lemma 2 in Saworotnow [6] that $f \in H(f)$. Also both H(f) and its orthogonal complement $H(f)^\perp$ (which coincides with the set $H(f)^p = \left\{ g \in H : [g,h] = 0 \text{ for all } h \in H(f) \right\}$ (Lemma 3 of Saworotnow [6])) are invariant under T.

It follows from this fact and Zorn's Principle that there exists a set $\left\{ f_\gamma : \gamma \in \Gamma \right\}$ of mutually orthogonal members of H such that $H = \Sigma_\gamma \otimes H(f_\gamma)$, $H(f_\gamma) \perp H(f_\beta)$ if $\gamma \neq \beta$, and each $H(f_\gamma)$ is invariant under T.

For each $\gamma \in \Gamma$ and $S \in B$ let S_γ be the restriction of S to $H(f_\gamma)$, and let $B_\gamma = \left\{ S_\gamma : S \in B \right\}$. It follows from part I (case I) of this proof that for each $\gamma \in \Gamma$ there exists a compact Hausdorff space $(\mathfrak{M}_\gamma, \tau_\gamma)$, a τA-valued positive Borel measure μ_γ and a continuous real valued function $h_\gamma(\ )$ on $\mathcal{M}_\gamma$ such that $H(f_\gamma)$ is isomorphic to $L^2(\mu_\gamma)\otimes A$ and action of the operator T_γ (the restriction of T) corresponds to the multiplication with h_γ on $L^2(\mu_\gamma)$. Note also that $h_\gamma(M) \leq ||T||$ for each $M\in\mathcal{M}_\gamma$.

Let $X = \dot{U}\ \mathcal{M}_\gamma$ and let τ be the topology on X defined by the requirement that a set $O\subset X$ is open $(O\in\tau)$ if and only if $O\cap\mathcal{M}_\gamma$ belongs to τ_γ for each $\gamma\in\Gamma$. Let β be the class of all bounded Borel subsets of X. For each $\Delta\in\beta$ there are indices (we use a simplified notation here) $1,2,\ldots,n\in\Gamma$ such that $\Delta\subset\bigcup_{i=1}^{n}\mathcal{M}_i$. We set

$$\mu(\Delta) = \Sigma_{i=1}^{n}\mu_i(\Delta\cap\mathcal{M}_i) \tag{3.9}$$

Then β is a ring and μ is a positive τA-valued measure on β. We define the function h on X by setting $h(M) = h_\gamma(M)$ where $\gamma\in\Gamma$ is such that $M\in\mathcal{M}_\gamma$. Then it is easy to see that h has the required properties.

To complete the proof it is now sufficient to show that $L^2(\mu)\otimes A = \Sigma_\gamma L^2(\mu_\gamma)\otimes A$. First note that each $L^2(\mu_\gamma)$ is included in $L^2(\mu)$ and that $L^2(\mu) = \Sigma_\gamma L^2(\mu_\gamma)$ (easy to verify). Now let $f\in L^2(\mu)\otimes A$. For each $\epsilon > 0$ one can find $g = \Sigma_{i=1}^{n}\psi_i\otimes a_i$ such that $||f-g|| < \epsilon$ with $\psi_i\in L^2(\mu)$. But each ψ_i can be approximated in $L^2(\mu)$ by expressions of the form $\Sigma_{j=1}^{n}\phi_j$ with $\phi_j\in L^2(\mu_{\gamma_j})$ for some $\gamma_1, \gamma_2,\ldots,\gamma_n\in\Gamma$. Thus f can be approximated (as close as we please) by members $\Sigma_{i=1}^{n}(\Sigma_j\,\phi_j)\otimes a_i$ of $\Sigma_\gamma L^2(\mu_\gamma)\otimes A$, i.e., g is a member of $\Sigma_\gamma L^2(\mu_\gamma)\otimes A$.

Conversely, let $f\in\Sigma_\gamma L^2(\mu_\gamma)\otimes A$; then f can be approximated by finite sums of expressions of the type $\Sigma_{i=1}^{n}\psi_i\otimes a_i$ with $a_i\in A$ and $\psi_1,\psi_2,\ldots\psi_n$ belonging to some $L^2(\mu\beta)$ with $\beta\in\Gamma$. We may conclude that $f\in L^2(\mu)\otimes A$ since $L^2(\mu_\gamma)\subset L^2(\mu)$ for each γ. The reader should be able to give a precise argument here.

THEOREM 2. Let Z be a family of bounded A-linear operators on a Hilbert module H (over an H*-algebra A) such that each member of Z and its adjoint (with respect to the generalized inner product) commute with any other member of Z. In particular, Z could be a commutative *-algebra of A-linear operators on H. Then there exists a locally compact Hausdorff space X, a τA-valued positive Borel measure μ on X and a map $T \longrightarrow h_T$ of Z into complex valued functions on X such that H is isomorphic to $L^2(\mu)\otimes A$ and each T corresponds to multiplication with some function h_T. Moreoever $||h_T||_\infty \leq ||T||$ for each $T\in Z$.

PROOF. The proof is essentially the same as the proof of Theorem 1 above. We use the *-algebra of operators generated by Z (and the identity operator I) instead of the algebra generated by the operator T (and I).

COROLLARY 1. Each *-representation of a commutative *-algebra by bounded A-linear operators is of the form $x \longrightarrow T_h$, where T_h is an operator of multiplication with a complex valued function $h = h_x$ described before Theorem 1.

This corollary could be considered as a generalization of Theorem 65 in Mackey [7] if we disregard the fact that Mackey considers more general (self-adjoint) algebras and we do not specify the space X on which the functions $h = h_x$ act (also our Hilbert module does not have to be separable (as a Hilbert space)).

COROLLARY 2. Let G be a commutative locally compact group with composition + and let $t \longrightarrow U_t$ be a *-representation of G by A-linear unitary operators acting on a Hilbert module H. Assume that there exists a vector $f_0\in H$ such that the submodule H_0, generated by the vectors of the form $U_t(f_0)$, is dense in H. Then there exists a compact Hausdorff space $\mathcal{M}$, a positive τA-valued Borel measure μ on $\mathcal{M}$ and a map

$t \longrightarrow g_t$ of G into the continuous functions on $\mathfrak{M}$ such that H is (isometrically) isomorphic to $L^2(\mu)\otimes A$ and each U_t corresponds to multiplication members of $L^2(\mu)$ with g_t.

The map $t \longrightarrow g_t$ has the following properties (for each $t\in G$ and all $M\in\mathfrak{M}$):

$$g_0(M) = 1 \text{ (here 0 is the identify of G)} \tag{3.10}$$

$$|g_t(M)| = 1 \tag{3.11}$$

$$g_{-t}(M) = \bar{g}_t(M) \tag{3.12}$$

$$g_{t+s}(M) = g_t(M)g_s(M) \tag{3.13}$$

It is appropriate at this point to mention a certain application of the last corollary. Let G, A and H be as above, and let $\xi:G \longrightarrow H$ be a generalized stationary process (Saworotnow [8]), i.e., ξ is an H-valued function on G such that $(\xi(t+r),\xi(s+r)) = (\xi(t),\xi(s))$ for all $t,r,s\in G$. Let H_ξ be the submodule generated by the vectors of the form $\xi(t)$, $t\in G$ (H_ξ = closure of $\left\{\sum_{k=1}^n \xi(t_k)a_k : t_k\in G\right\}$).

For each $t\in G$ consider the operator U_t on H_ξ defined by

$$U_t(\textstyle\sum_{k=1}^n \xi(t_k)a_k) = \textstyle\sum_{k=1}^n \xi(t_k+t)a_k \text{ and let } f_0 = \xi(0). \tag{3.14}$$

Then the map $t \longrightarrow U_t$ is a representation of G by A-linear unitary operators and it is easy to see that the assumptions of Corollary 2 are fulfilled. Let $\mathfrak{M}$, μ and g_t be as in Corollary 2 and let f(M) be the member of $C(\mathfrak{M})$ corresponding to $f_0 = \xi(0)$. Then the space H_ξ is isomorphic to $L^2(\mu)\otimes A$ and each U_t corresponds to multiplication of members of $L^2(\mu)$ with g_t. For each $t\in G$ let $h_t(M) = g_t(M)f(M)$. In this fashion we arrived at a concrete representation of the abstract stationary process ξ by the complex valued continuous function h_t defined on $\mathfrak{M}$. Note that the scalar product $(\xi(t),\xi(s))$ corresponds to the expression

$$\int h_t(M)\overline{h_s(M)}d\mu(M) = \int g_t(M)\overline{g_s(M)}f(M)\overline{f(M)}d\mu(M) =$$
$$\int g_t(M)g_{-s}(M)|f(M)|^2 d\mu(M) = \int g_{t-s}(M)|f(M)|^2 d\mu(M) \tag{3.15}$$

and this expression depends on t-s only and is independent of a particular choice of t and s.

4. CONCLUDING REMARK.

To conclude the paper we make the following remark about the operator T_h discussed above. It is easy to see that we do not need at all to assume existence of a (locally compact) topology on the space X (discussed at the beginning of this paper). Let μ be a positive τA-valued measure defined on some σ-ring of subsets of X. If h is any $tr\mu$-measurable essentially bounded real valued function on X then the corresponding operator T_h on $L^2(\mu)\otimes A$,

$$T_h(\textstyle\sum_i \psi_i \otimes a_i) = \textstyle\sum_i (\psi_i h)\otimes a_i \tag{3.16}$$

is also self-adjoint, A-linear and bounded. The fact that T_h is bounded can be verified in the same way as above using the algebra B of all essentially bounded $tr\mu$-measurable complex-valued functions on X.

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