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LAPLACE TRANSFORM PAIRS OF N-DIMENSIONS

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ABSTRACT. In this paper I prove a theorem to obtain new n-dimensional Laplace transform pairs.

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1. INTRODUCTION.

The generalization of the well-known Laplace transform

$$L\{f(t)\ ;\ s\} = \int_0^\infty \exp(-st)f(t)dt \qquad (1.1)$$

to n-dimensional Laplace transform is represented as follows:

$$L_n\{f(t_1,t_2,\ldots t_n;s_1,s_2,\ldots,s_n\} = L_n\{f\}$$

$$= \int_0^\infty \int_0^\infty \cdots \int_0^\infty \exp(-\sum_{k=1}^{n} s_k t_k)\ f\ dt_1 dt_2 \ldots dt_n. \qquad (1.2)$$

In this paper I consider a method of computing Laplace transform pairs of n-dimensions from known one-dimensional Laplace transforms. The multi-dimensional Laplace transform pairs are useful in the solution of partial differential equations (see [1], [3] and [4]).

2. THEOREM. Let

(i) $L_1\{f(t);s\} = \phi(s)$

(ii) $L_1\{\sqrt{t}\ \phi(\frac{1}{t});s\} = F(s)$

(iii) $L_1\{t^3 f(t^4);s\} = G(s)$

(iv) $L_1\{t^4 f(t^4);s\} = H(s)$

and let $f(t)$, $\sqrt{t}\,\phi(\frac{1}{t})$, $t^3 f(t^4)$, $t^4 f(t^4)$ be continuous and absolutely integrable in $(0,\infty)$. Then

$$L_n\left\{\frac{(\frac{1}{t_1}+\ldots+\frac{1}{t_n})^3}{(t_1\ldots t_n)^{1/2}}F[\frac{1}{64}(\frac{1}{t_1}+\ldots+\frac{1}{t_n})^2];s_1,\ldots,s_n\right\}$$

$$= 2^{10}\pi^{\frac{n+1}{2}}\frac{G(\sqrt{s_1}+\ldots+\sqrt{s_n})}{(s_1\ldots s_n)^{1/2}} + 2^9\pi^{\frac{n+1}{2}}\frac{\sqrt{s_1}+\ldots+\sqrt{s_n}}{(s_1\ldots s_n)^{1/2}}H(\sqrt{s_1}+\ldots+\sqrt{s_n}),$$

$$n = 2,3,4\ldots \tag{2.1}$$

provided the integral on the left exists as an absolutely convergent in each of the variables.

PROOF: From (i), we have

$$\phi(\frac{1}{s}) = \int_0^\infty e^{-t/s}f(t)dt = \int_0^\infty e^{-u/s}f(u)du,$$

$$\sqrt{t}\,\phi(\frac{1}{t}) = \int_0^\infty \sqrt{t}\,e^{-u/t}f(u)du. \tag{2.2}$$

Let us multiply both sides of (2.2) by e^{-st}, $Re(s) > 0$, and integrate between the limits $(0,\infty)$. Then on changing the order of integrations on the resulting right hand integral (permissible by Fubini's theorem, on account of absolute convergence), we obtain

$$\int_0^\infty e^{-st}\sqrt{t}\,\phi(\frac{1}{t})dt = \int_0^\infty f(u)\,[\int_0^\infty \sqrt{t}\,e^{-st-u/t}dt]du.$$

We then evaluate the inner integral on the right (see [5], page 22) and use (ii) on the left to get the following result:

$$F(s) = \frac{\sqrt{\pi}}{2}\int_0^\infty (1+2\sqrt{us})s^{-3/2}e^{-2\sqrt{us}}f(u)du,$$

$$s^{3/2}F(s) = \frac{\sqrt{\pi}}{2}\int_0^\infty (1+2\sqrt{us})e^{-2\sqrt{us}}f(u)du. \tag{2.3}$$

Next let us write (2.3) in the form

$$(\frac{1}{t_1}+\ldots+\frac{1}{t_n})^3F[\frac{1}{64}(\frac{1}{t_1}+\ldots+\frac{1}{t_n})^2]$$

$$= 256\sqrt{\pi}\int_0^\infty e^{-\frac{\sqrt{u}}{4}\sum\frac{1}{t_i}}f(u)du + 64\sqrt{\pi}\int_0^\infty(\frac{1}{t_1}+\ldots+\frac{1}{t_n})e^{-\frac{\sqrt{u}}{4}\sum\frac{1}{t_i}}\cdot\sqrt{u}f(u)du$$

$$= 1024\sqrt{\pi}\int_0^\infty e^{-\frac{u^2}{4}\sum\frac{1}{t_i}}u^3f(u^4)du + 256\sqrt{\pi}\int_0^\infty(\frac{1}{t_1}+\ldots+\frac{1}{t_n})e^{-\frac{u^2}{4}\sum\frac{1}{t_i}}u^5f(u^4)du$$

We multiply both sides by $(t_1\dots t_n)^{-1/2}\exp(-\sum s_i t_i)$, integrate with respect to t_i between the limits $(0,\infty)$ and then change the order of integrations in the resulting integral on the right, permissible by Fubini's theorem, on account of absolute convergence.

This gives

$$\int_0^\infty \dots \int_0^\infty \exp(-\sum s_i t_i) \frac{(\frac{1}{t_1}+\dots+\frac{1}{t_n})^3}{(t_1\dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1}+\dots+\frac{1}{t_n})^2]dt_1\dots dt_n$$

$$= 1024\sqrt{\pi}\int_0^\infty u^3 f(u^4)\left[\int_0^\infty \frac{1}{\sqrt{t_1}}\exp(-s_1t_1-\frac{u^2}{4t_1})dt_1\dots\int_0^\infty \frac{1}{\sqrt{t_n}}\exp(-s_nt_n-\frac{u^2}{4t_n})dt_n\right]du$$

$$+ 256\sqrt{\pi}\int_0^\infty u^5 f(u^4)\left[\int_0^\infty\dots\int_0^\infty \left(\frac{1}{t_1^{3/2}\sqrt{t_2\dots t_n}}+\frac{1}{\sqrt{t_1}\,t_2^{3/2}\sqrt{t_3\dots t_n}}+\dots+\frac{1}{\sqrt{t_1\dots t_{n-1}}\,t_n^{3/2}}\right)\right.$$

$$\left.\cdot\exp(-s_it_i-\frac{u^2}{4}\sum\frac{1}{t_i})dt_1\dots dt_n\right]du. \qquad (2.4)$$

Evaluating the inner integrals on the right by (see [5], page 22, results 6 and 7)

$$\int_0^\infty \frac{1}{\sqrt{t}}\exp(-st-\frac{u^2}{4t})dt = \sqrt{\frac{\pi}{s}}\,e^{-u\sqrt{s}},\quad \int_0^\infty t^{-3/2}\exp(-st-\frac{u^2}{4})dt = \frac{2\sqrt{\pi}}{u}\,e^{-u\sqrt{s}}$$

we get

$$L_n\left\{\frac{(\frac{1}{t_1}+\dots+\frac{1}{t_n})^3}{(t_1\dots t_n)^{1/2}} F[\frac{1}{64}(\frac{1}{t_1}+\dots+\frac{1}{t_n})^2];s_1,\dots,s_n\right\} \qquad (2.5)$$

$$= 1024\,\pi^{\frac{n+1}{2}}(s_1\dots s_n)^{-1/2}\int_0^\infty \exp(-u\sum\sqrt{s_i})u^3 f(u^4)du$$

$$+ 512\,\pi^{\frac{n+1}{2}}\frac{(\sqrt{s_1}+\dots+\sqrt{s_n})}{(s_1\dots s_n)^{1/2}}\int_0^\infty \exp(-u\sum\sqrt{s_i})u^4 f(u^4)du.$$

The proof is complete if we use (iii) and ((iv) on the right hand side of (2.5).

3. APPLICATIONS: n-dimensional Laplace transform pairs.

Let $f(t) = t^v$; so that $L_1\{t^v;s\} = \frac{\Gamma(v+1)}{s^{v+1}} = \phi(s)$. Then

$$L_1\{\sqrt{t}\,\phi\,((\tfrac{1}{t}),s\} = L_1\{\Gamma(v+1)t^{v+3/2};s\} = \frac{\Gamma(v+1)\Gamma(v+5/2)}{x^{v+5/2}} = F(s),$$

$$L_1\{t^3 f(t^4);s\} = L_1\{t^{4v+3};s\} = \frac{\Gamma(4v+4)}{s^{4v+4}} = G(s),$$

$$L_1\{t^4 f(t^4);s\} = L_1\{t^{4v+4};s\} = \frac{\Gamma(4v+t)}{s^{4v+5}} = H(s).$$ Hence from (2.1), we get

$$L_n\{(t_1\cdots t_n)^{-1/2}(\frac{1}{t_1}+\cdots+\frac{1}{t_n})^{-2v-2};s_1,\ldots,s_n\}$$

$$= \frac{\pi^{\frac{n+1}{2}}\Gamma(4v+4)}{8^{2v-7}\Gamma(v+1)\Gamma(v+5/2)}(s_1\cdots s_n)^{-1/2}(\sqrt{s_1}+\cdots+\sqrt{s_n})^{-4v-4}$$

$$+ \frac{\pi^{\frac{n+1}{2}}\Gamma(4v+5)}{8^{2v-6}\Gamma(v+1)\Gamma(v+5/2)}(s_1\cdots s_n)^{-1/2}(\sqrt{s_1}+\cdots+\sqrt{s_n})^{-4v-4}. \quad (3.1)$$

Similarly if we take f to be the following

$$f(t) = \begin{cases} t^{c-1}\ {}_0F_3(a,b,c;kt) \\ t^{v}\exp(-\sqrt{t}) \\ J_v^2(\sqrt{2t} \\ t^{\alpha}\ {}_pF_q\left[\begin{matrix}(a);\\(b);\end{matrix}t\right] \end{cases}$$

in the theorem, then we obtain the following n-dimensional Laplace transform pairs:

$$L_n\left\{\frac{(t_1\cdots t_n)^{-1/2}}{(\frac{1}{t_1}+\cdots+\frac{1}{t_n})^{2c}}\ {}_1F_2\left[\begin{matrix}c+3/2;\\a\ ,\ b\ ;\end{matrix}\ \frac{64k}{(\frac{1}{t_1}+\cdots+\frac{1}{t_n})^2}\right];\ s_1,\ldots,s_n\right\}$$

$$= \frac{2\pi^{\frac{n+1}{2}}\Gamma(4c)\ \ (s_1\cdots s_n)^{-1/2}}{8^{2c}\Gamma(c)\Gamma(c+3/2)(\sqrt{s_1}+\cdots+\sqrt{s_n})^{4c}}\ {}_4F_3\left[\begin{matrix}2c,2c+1/2,2c+1,2c+2;\\a,b,c;\end{matrix}\ \frac{256k}{(\sqrt{s_1}+\cdots+\sqrt{s_n})^4}\right]$$

$$+ \frac{\pi^{\frac{n+1}{2}}\Gamma(4c+1)(s_1\cdots s_n)^{-1/2}}{8^{2c}\Gamma(c)\Gamma(c+3/2)(\sqrt{s_1}+\cdots+\sqrt{s_n})^{4c}}\ {}_4F_3\left[\begin{matrix}2c+1/2,2c+1,2c+3/2,2c+2;\\a,b,c;\end{matrix}\ \frac{256k}{(\sqrt{s_1}+\cdots+\sqrt{s_n})^4}\right],$$

$$\mathrm{Re}(c) > 0. \quad (3.2)$$

$$L_n\left\{\frac{\frac{1}{t_1}+\ldots+\frac{1}{t_n}}{(t_1\ldots t_n)^{1/2}}\left(\frac{1}{64}\left(\frac{1}{t_1}+\ldots+\frac{1}{t_n}\right)^2-\frac{1}{4}\right)^{-v-3/2}P_1^{-2v-3}\left[\frac{4}{\frac{1}{t_1}+\ldots+\frac{1}{t_n}}\right];s_1,\ldots,s_n\right\}$$

$$=\frac{\pi^{n/2}\Gamma(4v+4)}{2^{v-4}\Gamma(2v-5)}(s_1\ldots s_n)^{-1/2}\exp\left(\frac{1}{8}(\sqrt{s_1}+\ldots+\sqrt{s_n})^2\right)D_{-4v-4}\left(\frac{1}{\sqrt{2}}(\sqrt{s_1}+\ldots+\sqrt{s_n})\right)$$

$$+\frac{\pi^{n/2}\Gamma(4v+5)\ (\sqrt{s_1}+\ldots+\sqrt{s_n})}{2^{v-5/2}\Gamma(2v+5)(s_1\ldots s_n)^{1/2}}\exp\left(\frac{1}{8}(\sqrt{s_1}+\ldots+\sqrt{s_n})^2\right)D_{-4v-5}\left(\frac{1}{\sqrt{2}}(\sqrt{s_1}+\ldots+\sqrt{s_n})\right),$$

$$\mathrm{Re}(v) > -1. \qquad (3.3)$$

$$L_n\left\{\frac{(t_1\ldots t_n)^{-1/2}\left(\frac{1}{t_1}+\ldots+\frac{1}{t_n}\right)^3}{\left(\frac{1}{64}\left(\frac{1}{t_1}+\ldots+\frac{1}{t_n}\right)^2+1\right)-1}\ Q^2_{v-1/2}\left(\frac{1}{64}\left(\frac{1}{t_1}+\ldots+\frac{1}{t_n}\right)^2+1\right);s_1,\ldots,s_n\right\}$$

$$=\frac{\pi^{\frac{n}{2}+1}\sin(v+3/2)\pi(\sqrt{s_1}+\ldots+\sqrt{s_n})^{v/2}}{2^{\frac{v}{8}-9}\sin(v-1/2)\pi(s_1\ldots s_n)^{1/2}}$$

$$\cdot\ G^{03}_{50}\left[\frac{8\sqrt{2}}{(\sqrt{s_1}+\ldots+\sqrt{s_n})^2}\middle|\,2-\frac{v}{3},-\frac{v}{4},-\frac{1}{2}-\frac{v}{4},\frac{3}{2}+\frac{v}{4},2+\frac{v}{4}\right]$$

$$+\frac{\pi^{\frac{n}{2}+1}\sin(v+3/2)\pi(\sqrt{s_1}+\ldots+\sqrt{s_n})^{\frac{v+3}{2}}}{s^{v-73/8}\sin(v-1/2)\pi(s_1\ldots s_n)^{1/2}}\ \cdot$$

$$G^{03}_{50}\left[\frac{8\sqrt{2}}{(\sqrt{s_1}+\ldots+\sqrt{s_n})^2}\middle|\,\frac{9}{4}-\frac{v}{4},-\frac{1}{4}-\frac{v}{4},-\frac{3}{4}-\frac{v}{4},\frac{7}{4}+\frac{v}{4},\frac{9}{4}+\frac{4}{4}\right] \qquad (3.4)$$

$$L_n\left\{\frac{(t_1\ldots t_n)^{-1/2}}{\left(\frac{1}{t_1}+\ldots+\frac{1}{t_n}\right)^{2\alpha+2}}\ {}_{p+2}F_q\left[\begin{matrix}(a),\alpha+1,\alpha+5/2;\\(b);\end{matrix}\ \frac{64}{\left(\frac{1}{t_1}+\ldots+\frac{1}{t_n}\right)^2}\right];s_1,\ldots,s_n\right\}$$

$$=\frac{\pi^{\frac{n+1}{2}}\Gamma(4\alpha+4)(s_1\ldots s_n)^{-1/2}}{4(8)^{2\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+5/2)(\sqrt{s_1}+\ldots+\sqrt{s_n})^{4\alpha+4}}$$

$$\cdot\ {}_{p+4}F_q\left[\begin{matrix}(a),2\alpha+2,2\alpha+5/2,2\alpha+3,2\alpha+7/2;\\(b);\end{matrix}\ \frac{256}{(\sqrt{s_1}+\ldots+\sqrt{s_n})^4}\right]$$

$$+\frac{\pi^{\frac{n+1}{2}}\Gamma(4\alpha+5)(s_1\ldots s_n)^{-1/2}}{8^{2\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+5/2)(\sqrt{s_1}+\ldots+\sqrt{s_n})^{4\alpha+4}}$$

$$\cdot\ {}_{p+4}F_q\left[\begin{matrix}(a),2\alpha+3,2\alpha+7/2,2\alpha+4,2\alpha+9/2;\\(b);\end{matrix}\ \frac{256}{(\sqrt{s_1}+\ldots+\sqrt{s_n})^4}\right],$$

$$\mathrm{Re}(\alpha) > 1. \qquad (3.5)$$

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