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A NOTE ON EXPONENTIALLY BOUNDED STOPPING TIMES OF CERTAIN ONE-SAMPLE SEQUENTIAL RANK ORDER TESTS

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ABSTRACT. A simple proof of the exponential boundedness of the stopping time of the one-sample sequential probability ratio tests (SPRT's) is obtained.

KEY WORDS AND PHASES. Nonparametric; finite sure termination; test for symmetry.
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1. INTRODUCTION AND SUMMARY.

Stein [1] gave a very mild condition under which the stopping time of a SPRT based on a sequence of i.i.d. random variables is exponentially bounded. It is well known that exponential boundedness of the stopping time implies finite sure termination and the finiteness of the moment-generating function of the stopping time. Weed, Bradley and Govindarajulu [2] propose one-sample sequential probability ratio tests (SPRTs) based on Lehmann alternatives and also study their finite sure termination. Extending the work of Sethuraman [3], Govindarajulu [4] has obtained a very mild Stein-type of condition (namely, that a certain random variable U(Z) is not identially zero) for the exponential boundedness of the stopping time of the one-sample SPRT's. Here, using a general result of Wijsman [5] a much simpler proof of the same assertion is obtained.

2. NOTATION

Let $Z_1, Z_2, \cdots$ be a sequence of i.i.d. random variables having a continuous distribution function (d.f.) F(x). Assume that we observe the Z's sequentially one at a time. We wish to test

$$H_0\text{: } F(z) + F(-z) = 1 \quad \text{for all } z. \tag{2.1}$$

Let

$$G(z) = P(|Z| \le z | Z < 0) = \{F(0) - F(-z)\}/F(0), \text{ and}$$

$$H(z) = P(Z \le z | Z \ge 0) = \{F(z) - F(0)\}/\{1-F(0)\}, \quad \text{for all } z \ge 0,$$

and G and H are zero for $z < 0$. Alternatively, we write

$$H_0\text{: } G(z) = H(z) \quad \text{for all } z \text{ and } F(0) = 1/2$$

and take the alternative hypothesis to be

$$H_a\text{: } G(z) \ne H(z) \quad \text{for some } z.$$

Consider a subalternative hypothesis given by

$$H_A:\ H(z) = G^A(z) \quad \text{for all } z,\ A>0,\ A\neq 1,$$

$$A \text{ is specified, } F(0) = \lambda_0,\ \lambda_0 \text{ specified.}$$

Let t denote the number of stages to which the experiment has proceeded. (That is $Z_1, Z_2, \cdots, Z_t$ will be observed in t stages.) Let $X_1, \cdots, X_m$ $[Y_1, \cdots, Y_n]$ denote the absolute values [values] of those Z's among $Z_1, \cdots, Z_t$ that are negative [positive]. Also note that m is binomially distributed with parameters t and $\lambda = F(0)$, $0 < \lambda < 1$. Let $V_i = |Z_i|$, $i = 1, \cdots, t$. Let $G_m[H_n]$ denote the empirical d.f. based on $X_1, \cdots, X_m$ $[Y_1, \cdots, Y_n]$. Further, let $\Delta = (\Delta_1, \cdots, \Delta_t)$ where $\Delta_i = 1$ or 0 according as V_{it} corresponds to a negative or positive Z respectively where $V_{1t} \leq \cdots \leq V_{tt}$ denote the ordered $V_1, \cdots, V_t$. Also, let

$$L_t(A,\delta) = P_t(\Delta=\delta|A)/P_t(\Delta=\delta|A=1) = 2^t P_t(\Delta=\delta|A). \tag{2.2}$$

From Weed, Bradley and Govindarajulu [2] or Govindarajulu [4, (2.6)] we have

$$L_t(A,\delta) = 2^t \frac{t!}{t^t} \lambda_0^m (1-\lambda_0)^n \prod_{i=1}^{t} \{\lambda_t G_m(V_i) + A(1-\lambda_t)H_n(V_i)\}^{-1} \tag{2.3}$$

where

$$\lambda_t = m/t,\ G_k(z) = \sum_{1}^{k} I(X_i;z)/k \quad \text{and} \quad H_k(z) = \sum_{j=1}^{k} I(Y_j;z)/k, \tag{2.4}$$

where $I(c,d) = 1$ if $c \leq d$ and zero elsewhere. Let

$$W_0(Z;z) = [I(Z;0) + AI(0;Z)]I(|Z|;z) = I(X;z) + AI(Y;z)$$

and

$$W_t(z) = \lambda_t G_m(z) + A(1-\lambda_t)H_n(z).$$

(Note that $EW_0(Z;z) = W(z) = \lambda G(z) + A(1-\lambda)H(z)$.)

With the above notation, one can write

$$L_t(A,\delta) = 2^t t! t^{-t} \lambda_0^m (1-\lambda_0)^n A^n / \prod_{i=1}^{t} W_t(V_i). \tag{2.5}$$

Hence

$$\ell_t(A,\delta) = \ell n L_t(A,\delta) = t\ \ell n\ 2 + \ell n\ t! - t\ \ell n\ t + m\ \ell n\ \lambda_0$$
$$+ n\ \ell n\ (1-\lambda_0) + n\ \ell n\ A - \sum_{i=1}^{t} \ell n\ W_t(V_i).$$

Then the rank order SPRT (ROSPRT) for testing H_0 against H_A is given by:

Accept H_0 [reject H_0] if $\ell_t \leq b$ $[\ell_t \geq a]$ and take one more observation of $b < \ell_t < a$ where $b < 0 < a$ are some fixed constants. The stopping time is:

$$T = r \text{ if } b < \ell_t < a \text{ for } t = 1, \cdots, r-1 \text{ and } \ell(r) \leq b \text{ or } \ell_r \geq a\ (r=1,2,\cdots). \tag{2.6}$$

We will be interested in the termination properties of T. Also define

$$T^* = r \text{ if } |\ell_t| < d = \max(-b,a) \text{ for } t = 1, \cdots, r-1 \text{ and } |\ell_r| \geq d. \tag{2.7}$$

Then one can easily see that $T > r \Rightarrow T^* > r$. Hence $P(T > r) \leq P(T^* > r)$. So it suffices to show the exponential boundedness of T^*, which will be done in the sequel.

3. CERTAIN RESULTS PERTAINING TO EXPONENTIALLY BOUNDED STOPPING TIMES.

Wijsman [5] has obtained some general theorems which assert that the stopping time of a sequential procedure is exponentially bounded unless a certain function of the underlying random variable is zero with probability one. In the following we will state only the results that are relevant to our needs here.

LEMMA 3.1. (Wijsman [5, (Corollary 2.2)]). Let $Z, Z_1, Z_2, \cdots$ be i.i.d. random variables taking values in L and having common distribution F. Let $L^*_t = L^*_t(\underline{Z}^t), \underline{Z}^t = (Z_1, \cdots, Z_t)$ be a sequence of real valued statistics and let T* be defined by (2.7). Assume, further, that there is a measurable function $f: L \to R$, a measurable subset S of L with $P(Z \in S) > 0$ and $f(z) \neq 0$ whenever $z \in S$, and a set $S_0 \supset S$ such that for every positive integer r the function

$$K_n = \sup\{|L_{t+r}(Z^t, y^r) - L_t - \sum_1^r f(y_j)|: \ y_1, \cdots, y_r \in S_0\}$$

is measurable. If $K_n \to_{\exp} 0$, then T* is exponentially bounded.

PROOF. (See Wijsman [5, p.298]). In the following we will state a useful lemma which is more germane to nonparametric needs.

LEMMA 3.2. (Wijsman[5]). Let $Z, Z_1, Z_2, \cdots$ be a sequence of i.i.d. random variables taking values in a measurable space (L, A) and having the common d.f. F(z). Further, let I be a real interval (perhaps infinite) and let f be a bounded, real valued function on $I \times L$, such that $f(x, \cdot)$ is A-measurable for every $x \in I$ and $f(\cdot, z)$ is non-decreasing for every $z \in L$ or non-increasing for every $z \in L$. Also let

$$J_n(x) = \sum_1^n f(x, Z_i)/n, \quad J(x) = \int f(x,z)F(dz)$$

and

$$R_n = \sup\{|J_n(x) - J(x)|: \ x \in I\}.$$

Then $R_n \to_{\exp} 0$.

PROOF. (See Wijsman [5, p. 308]).

4. CERTAIN USEFUL LEMMAS.

In this section we give some results as lemmas which will be used in the sequel.

LEMMA 4.1. (Sethuraman [3, p. 1326]). Let $-1 < \beta < 0 < \alpha$.

(i) For $\beta \leq x \leq \alpha$ we have

$$\ell n(1+x) \geq x/(1+\beta) - \alpha(\alpha-\beta)/[(1+\alpha)(1+\beta)] \tag{4.1}$$

and for $0 \leq |x| < 1$

$$\ell n(1+x) \leq x. \tag{4.2}$$

DEFINITION 4.1. The stopping time T of a sequential procedure is said to be exponentially bounded if for any t there is $c < \infty$ and $0 \leq \rho < 1$ such that

$$P(T > t) \leq c\rho^t, \ t = 1, 2, \cdots . \tag{4.3}$$

Notice that T* is exponentially bounded implies that T is exponentially bounded.

DEFINITION 4.2. A sequential procedure is said to have property Q if it terminates finitely and the stopping time possesses a finite moment-generating function (m • g • f •) under P; that is $P(T < \infty) = 1$ and $E(\exp(\eta T)) < \infty$ for some $\eta > 0$ (here E denotes expectation under P).

LEMMA 4.2. A sequential procedure has property Q if T is exponentially bounded.

PROOF. Easy to construct.

LEMMA 4.3. For every $\varepsilon > 0$, and sufficiently large t, and $0 < \lambda < 1$, there exists $0 \leq \rho(\varepsilon,\lambda) < 1$ such that

$$P[\lambda_t \ell n\, \lambda_t - \lambda \ell n\, \lambda \geq \varepsilon] \leq \rho^t(\varepsilon,\lambda)$$

$$P[\ell n\{(1-\lambda_t)^{1-\lambda_t}/(1-\lambda)^{1-\lambda}\} \geq \varepsilon] \leq \rho^t(\varepsilon,\lambda).$$

PROOF. See Lemma 4.3 in Govindarajulu [4].

5. MAIN RESULTS.

In this section we prove the main result pertaining to the exponential boundedness of T*.

THEOREM 5.1. Define

$$\begin{aligned} U(Z) &= \ell n 2 + \ell n\, \{A(1-\lambda_0)\} - \ell n\{A(1-\lambda_0)/\lambda_0\} I(Z;0) \\ &\quad - \ell n\, W(|Z|) - \int_0^\infty \{W_0(Z;v)/W(v)\}\bar{F}(dv) \end{aligned} \tag{5.1}$$

where

$$\bar{F}(v) = \lambda G(v) + (1-\lambda)H(v). \tag{5.2}$$

Then T is exponentially bounded under F unless

$$P_F(U(Z) = 0) = 1.$$

PROOF. Because of Lemma 5.1 it suffices to show that

$$\ell_{t+r} - \ell_t - \sum_{j=1}^{r} U(Z_j) \to_{\exp} 0 \text{ as } t \to \infty.$$

Towards this we can write

$$\ell_{t+r} - \ell_t = r\, \ell n\, 2 + B_1 + B_2 + B_3 \tag{5.3}$$

where

$$B_1 = \ell n(t+r)! - \ell n t! - (t+r)\ell n(t+r) + t\, \ell n\, t$$

$$B_2 = r\, \ell n\{A(1-\lambda_0)\} - \ell n\{A(1-\lambda_0)/\lambda_0\} \sum_{i=t+1}^{t+r} I(Z_i;0)$$

and

$$\begin{aligned} B_3 &= -\sum_{j=t+1}^{t+r} \ell n\, W_{t+r}(V_j) - \sum_{i=1}^{t} \ell n\{W_{t+r}(V_i)/W_t(V_i)\} \\ &= B_4 + B_5 \text{ (say).} \end{aligned}$$

After using Stirlings approximation to factorials, we have

$$B_1 \to -r \quad \text{as } t \to \infty. \tag{5.4}$$

Denoting Z_{t+i} by y_i, we have

$$B_2 = r\, \ell n\{A(1-\lambda_0)\} - \ell n\{A(1-\lambda_0)/\lambda_0\} \sum_{j=1}^{r} I(y_i;0) \tag{5.5}$$

$$B_4 = -\sum_{j=1}^{r} \ell n\, W_{t+r}(|y_j|) \to_{\exp} -\sum_{j=1}^{r} \ell n\, W(|y_j|) \tag{5.6}$$

since $W_t(\cdot) \to_{\exp} W(\cdot)$ uniformly in $(\cdot)$. Next consider B_5. Since, for fixed v

$$W_t(v) = t^{-1} \sum_{i=1}^{t} W_0(Z_i;v)$$

we can write

$$\ell n\{W_{t+r}(v)/W_t(v)\} = -\ell n\{(t+r)/t\} + \ell n\left[\frac{\sum_{i=1}^{t+r} W_0(Z_i;v)}{\sum_{i=1}^{t} W_0(Z_i;v)}\right]$$
$$= -\ell n\{(t+r)/t\} + \ell n\left[\frac{\sum_{i=1}^{r} W_0(y_i;v)}{tW_t(v)}\right] \tag{5.7}$$

Now,

$$\sum_{i=1}^{t} - \ell n\{(t+r)/t\} \to -r \quad \text{as } t \to \infty. \tag{5.8}$$

Consider

$$\sum_{i=1}^{t} - \ell n\{1 + \sum_{1}^{r} \frac{W_0(y_j;V_i)}{tW_t(V_i)}\}.$$

Define

$$A(t,\varepsilon) = \{\omega: \sup_v |W_t(v) - W(v)| \le \varepsilon\},$$

where ω denotes an outcome. Note that $\bar{A}(t,\varepsilon)$ is exponentially bounded.

On $A(t,\varepsilon)$

$$\sum_{i=1}^{t} \ell n\{1 + \sum_{1}^{r} \frac{W_0(y_j,V_i)}{tW_t(V_i)}\} \le \sum_{i=1}^{t} \ell n\{1 + \sum_{1}^{r} \frac{W_0(y_j;V_i)}{t(W(V_i)-\varepsilon)}\}$$
$$= \sum_{i=1}^{t}\left[\sum_{1}^{r} \frac{W_0(y_j;V_i)}{t(W(V_i)-\varepsilon)} - \frac{\frac{1}{2}\left(\sum_{1}^{r} \frac{W_0(y_j;V_i)}{t(W(V_i)-\varepsilon)}\right)^2}{1+\theta_i\sum_{1}^{r} \frac{W_0(v)}{t(W(V_i)-\varepsilon)}}\right], \quad 0 < \theta_i < 1. \tag{5.9}$$

Now

$$\text{L.H.S.} - \sum_{i=1}^{t}\sum_{1}^{r} \frac{W_0(y_j;V_i)}{tW(V_i)} \le \frac{1}{t}\sum_{i=1}^{t}\sum_{1}^{r} W_0(y_j;V_i)\left[\frac{1}{W(V_i)-\varepsilon} - \frac{1}{W(V_i)}\right]$$
$$+ \frac{1}{2}\frac{1}{t}\sum_{i=1}^{t} \frac{r(1+A)}{t(W(V_i)-\varepsilon)^2}$$
$$= \frac{\varepsilon\, r(1+A)}{t}\sum_{i=1}^{t} \frac{1}{W(V_i)(W(V_i)-\varepsilon)}$$
$$+ \frac{1}{2t}\sum_{i=1}^{t} \frac{r(1+A)}{t(W(V_i)-\varepsilon)^2} \tag{5.10}$$
$$\approx \varepsilon\, r(1+A)\int_0^\infty \frac{\bar{F}(dv)}{W(v)[W(v)-\varepsilon)]}$$
$$+ \frac{1}{2t}\int_0^\infty \frac{r(1+A)\bar{F}(d(v))}{(W(v)-\varepsilon)^2}.$$

Since ε is arbitrary, R.H.S. tends to zero as $t \to \infty$.

By taking $W_t(v) = W(v) + \varepsilon$, one can bound the difference from the other side. Also note that

$$\sum_{i=1}^{t}\sum_{j=1}^{r} \frac{W_0(y_j;V_i)}{t\,W(V_i)} \approx \sum_{j=1}^{r}\int_0^\infty \frac{W_0(y_j;v)\bar{F}(dv)}{W(v)}.$$

Q.E.D.

6. GENERALIZED RANK ORDER SPRTs.

Suppose we estimate λ_0 by $\lambda_t = m/t$ and obtain $L_t^*(A,\delta)$. Considering $\ell^*(t) = \ell n\, L^*_t(A,\delta)$ and using Lemma 4.3, we can establish that for large t

$$\ell^*_{t+r} - \ell^*_t - \sum_{j=1}^{r} U^*(Z_j) \to_{exp} 0,$$

where $U^*(Z)$ is obtained by replacing λ_0 by λ in $U(Z)$ given by (5.1). Then, one can readily establish:

THEOREM 6.1. If $P(U^*(Z) = 0) \neq 1$, then the generalized rank order SPRT based on ℓ^*_t has property Q.

7. CONCLUDING REMARKS.

(i) Inversion of model H_A via the transformation $\bar{Z} = -1/Z$ produces the other Lehmann's class of alternatives given by $1 - \bar{H}(z) = \{1 - \bar{G}(z)^{\bar{A}}\}$ where $\bar{G}$, $\bar{H}$ are d.f.s associated with the random variable $\bar{Z}$, $\bar{A} = 1/A$ and $\bar{\lambda}_0 = \bar{F}(0) = \bar{A}/(1+A)$ if $\lambda_0 = 1/(1+A)$.

(ii) Although for simplicity, continuity of the distribution of Z is assumed, it is not necessary for the validity of the results. If F has a denumerable number of jumps, then F can be made continuous by the continuization process described in Govindarajulu [6, Remark 3.14.7]. Alternatively we can take $F(z) = \frac{1}{2}[P(X \leq z) + P(X < z)]$ as suggested by Wijsman [5, Equation (4.2)] and all the results will be valid.

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