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ON AN ASYMPTOTIC FORMULA FOR THE NIVEN NUMBERS

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ABSTRACT. A Niven number is a positive integer which is divisible by its digital sum. A discussion of the possibility of an asymptotic formula for $N(x)$ is given. Here, $N(x)$ denotes the number of Niven numbers less than x. A partial result will be presented. This result will be an asymptotic formula for $N_k(x)$ which denotes the number of Niven numbers less than x with digital sum k.

KEY WORDS AND PHRASES. Digital sums, asymptotic formula, Niven number.
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1. INTRODUCTION.

In Kennedy et al [1], the concept of a Niven number was introduced as any positive integer n which is divisible by its digital sum $s(n)$. One of the first questions about the set, N, of Niven numbers which was investigated was the status of

$$\lim_{x \to \infty} \frac{N(x)}{x} \tag{1.1}$$

where $N(x)$ denotes the number of Niven numbers less than x. (In what follows, we will use the convention that if A is a set of integers, then $A(x)$ will be the number of members of A less than x.) This limit, if it exists, is called the "natural density" of the set N.

Even though this was answered in Kennedy and Cooper [2], (the natural density of N is zero), other questions demanded attention. In particular, "Can an asymptotic formula for $N(x)$ be determined?" That is, does there exist a function $f(x)$ such that

$$\lim_{x \to \infty} \frac{N(x)}{f(x)} = 1 \ ? \tag{1.2}$$

If such an $f(x)$ exists, then the usual notation to indicate this is,

$$N(x) \sim f(x) . \tag{1.3}$$

The following notation will be used to arrive at a partial answer to this question. Let k be a positive integer. Then k may be written in the form

$$k = 2^a 5^b t \tag{1.4}$$

where $(t, 10) = 1$. We define:

$$S_k = \{x : s(x) = k\}, \tag{1.5}$$

$$N_k = S_k \cap N , \tag{1.6}$$

$$\bar{e}(k) = \max\{a, b\} , \tag{1.7}$$

and

$$e(k) = \text{order of } 10 \text{ modulo } t . \tag{1.8}$$

In what follows, we will develop an asymptotic formula for $N_k(x)$.

2. AN ASYMPTOTIC FORMULA WHEN $k = 2^a5^b3^c$.

Such a formula for $N_k(x)$ can easily be found for k of the form $2^a5^b3^c$ when $c = 0$, 1 , or 2 . This is given in Theorem 2.1 with the help of the following lemma.

LEMMA 1. Let k , n be integers. Then

$$\binom{\left[\frac{n - \bar{e}(k) - 1}{e(k)}\right] + 1}{k} \leq N_k(10^n) . \tag{2.1}$$

PROOF. Here, the square brackets denote the greatest integer function and the parentheses denote a binomial coefficient. Note that an integer of the form

$$\sum_{i=0}^{f} c_i 10^{\bar{e}(k) + ie(k)} \tag{2.2}$$

where $c_i \in \{0, 1\}$, $c_0 + c_1 + c_2 + \ldots + c_f = k$, and

$$f = \left[\frac{n - \bar{e}(k) - 1}{e(k)}\right] \tag{2.3}$$

is a Niven number with digital sum k . But the sequence

$$\{c_i\}_{i=0}^{f} \tag{2.4}$$

can be rearranged exactly

$$\binom{f+1}{k} \tag{2.5}$$

ways, and each of these will determine a Niven number with digital sum k . Therefore, we have that (2.1) holds.

THEOREM 2. Let $k = 2^a5^b3^c$ where $c = 0, 1$, or 2 . Then

$$N_k(x) \sim (\log x)^k/k! . \tag{2.7}$$

PROOF. Let n be the positive integer such that

$$10^n \leq x < 10^{n+1} . \tag{2.8}$$

For k of the given form, $e(k) = 1$, and it follows from Tang [3] that

$$S_k(10^n) \sim \binom{n-m}{k} \sim n^k/k! \tag{2.9}$$

for m not dependent on n. By Lemma 1, we have

$$\binom{n-\bar{e}(k)}{k} \leq N_k(10^n) \leq S_k(10^n) \tag{2.10}$$

and so,

$$N_k(10^n) \sim n^k/k! \quad , \tag{2.11}$$

since each side of (2.10) is asymptotic to $n^k/k!$. From

$$N_k(10^n) \leq N_k(x) \leq N_k(10^{n+1}) \tag{2.12}$$

and

$$N_k(10^n) \sim n^k/k! \sim (n+1)^k/k! \sim N_k(10^{n+1}) \quad , \tag{2.13}$$

it readily follows that

$$N_k(x) \sim (\log x)^k/k! \tag{2.14}$$

because $n = [\log x] \sim \log x$.

3. A LOWER BOUND FOR $N(x)$.

It should be noted here that Lemma 1 can be used to determine a lower bound for $N(x)$. In fact, the search for such a lower bound led to the method that will give us an asymptotic formula for $N_k(x)$. To determine this lower bound, let $k = 2^m$ for some positive integer m. Then $e(k) = 1$ and $\bar{e}(k) = m$, and we have by Lemma 1 that

$$\binom{n-m}{2^m} \leq N_{2^m}(10^n) \quad . \tag{3.1}$$

Thus,

$$N(10^n) \geq N_{2^m}(10^n) \geq \binom{n-m}{2^m} \geq n^{2^m-1} \tag{3.2}$$

eventually. That is, there exists an integer K such that

$$N(10^n) \geq n^{2^m-1} \quad , \tag{3.3}$$

for all $n \geq K$. Note that the rightmost inequality of (3.2) follows since

$$\binom{n-m}{2^m} \tag{3.4}$$

is a polynomial in n of degree 2^m. As in the proof of Theorem 2, we can then establish that eventually

$$N(x) \geq (\log x)^{2^m-1} \quad , \tag{3.5}$$

for any integer m. Therefore, for any integer t,

$$N(x) \geq (\log x)^t \tag{3.6}$$

eventually.

4. AN ASYMPTOTIC FORMULA FOR $N_k(x)$.

In what follows, for positive integers k and n, we will denote the decimal representation of $x \in [0, 10^n)$, where $[0, 10^n)$ is the set of non-negative integers less than 10^n, as

$$\sum_{i=0}^{n-1} x_i 10^i \quad . \tag{4.1}$$

Note that initial zeros will be allowed so that x will have n digits. For each $j = 0, 1, 2, \ldots, e(k)-1$ we also define the finite sequences $B(x,j)$ and $T(x)$ by

$$T(x) = \left\{x_i\right\}_{i=0}^{\bar{e}(k)-1} \tag{4.2}$$

and

$$B(x,j) = \left\{x_{\bar{e}(k)+j+ie(k)}\right\}_{i=0}^{f} \tag{4.3}$$

where

$$f = \left[\frac{n - \bar{e}(k) - 1}{e(k)}\right] \quad . \tag{4.4}$$

Using (4.2) and (4.3), we now define the relation $\cong$ on $[0, 10^n)$ by: $x \cong y$ if and only if $T(x) = T(y)$ and $B(x,j)$ is a rearrangement of the terms of $B(y,j)$ for each j.

It is clear that $\cong$ is an equivalence relation on $[0, 10^n)$. For x a member of $[0, 10^n)$, let $\langle x \rangle$ denote the equivalence class containing x. The following lemma will be used to help count the number of Niven numbers with digital sum k.

LEMMA 3. Let $x, y \in [0, 10^n)$. Then $x \cong y$ implies that $s(x) = s(y)$ and $x \equiv y \pmod k$.

PROOF. Since $x \cong y$, we have that $T(x) = T(y)$ and $B(x,j)$ is a rearrangement of the terms of $B(y,j)$ for each $j = 0, 1, 2, \ldots, e(k) - 1$. Thus

$$\sum_{i=0}^{\bar{e}(k)-1} x_i = \sum_{i=0}^{\bar{e}(k)-1} y_i \tag{4.5}$$

and

$$\sum_{j=0}^{e(k)-1} \sum_{i=0}^{f(j)} x_{\bar{e}(k)+j+ie(k)} = \sum_{j=0}^{e(k)-1} \sum_{i=0}^{f(j)} y_{\bar{e}(k)+j+ie(k)} \tag{4.6}$$

where for each j,

$$f(j) = \left[\frac{n - \bar{e}(k) - 1 - j}{e(k)}\right] . \qquad (4.7)$$

Hence, by adding corresponding sides of (4.5) and (4.6), we have that $s(x) = s(y)$.

Note also that for each j,

$$10^{\bar{e}(k) + j + re(k)} \equiv 10^{\bar{e}(k) + j + te(k)} \pmod k \qquad (4.8)$$

for any pair of non-negative integers r and t. Thus, $x \equiv y \pmod k$.

LEMMA 4. Let $x \in [0, 10^n)$. Then $x \in N_k$ if and only if $\langle x\rangle \subseteq N_k$.

PROOF. Since $x \in \langle x\rangle$, it is immediate that $\langle x\rangle \subseteq N_k$ implies that $x \in N_k$. Conversely, suppose that $y \in \langle x\rangle$. Then $x \cong y$ and by Lemma 3, $k = s(x) = s(y)$ and $y \equiv x \equiv 0 \pmod k$. Therefore, $y \in N_k$ and we have that $\langle x\rangle \subseteq N_k$.

Note that Lemma 4 states either an equivalence class contains only Niven numbers, or it contains only non-Niven numbers.

For a finite sequence

$$\left\{a_i\right\}_{i=0}^{m} \qquad (4.9)$$

of digits, let

$$d_t = \#\left\{i : a_i = t\right\} , \qquad (4.10)$$

for $t = 0, 1, 2, \ldots, 9$. Here, the $\#$ symbol denotes the cardinality of the set. For example, if $t = 3$, then d_3 is the number of terms of the sequence equal to 3. Therefore, the number of finite sequences which can be formed by rearranging the terms of (4.9) is given by the multinomial coefficient

$$\binom{m+1}{d_0, d_1, \ldots, d_9} . \qquad (4.11)$$

We will use this fact to develop an asymptotic formula for $N_k(x)$ for any integer k.

LEMMA 5. Let $x \in N_k \cap [0, 10^n)$. Then $\#\langle x\rangle$ is a polynomial in f of degree less than or equal to k where

$$f = \left[\frac{n - \bar{e}(k) - 1}{e(k)}\right] . \qquad (4.12)$$

PROOF. Note that each $y \in \langle x\rangle$ may be found by rearranging the terms of $B(x,j)$ for various j's. Let

$$d_t(j) = \#\left\{0 \leqslant i \leqslant f : x_{\bar{e}(k) + j + ie(k)} = t\right\} . \qquad (4.13)$$

By the previous discussion, the number of such y's which can be formed by these rearrangements is given by

$$\prod_{j=0}^{e(k)-1} \binom{f+1}{d_0(j), d_1(j), \ldots, d_9(j)} . \qquad (4.14)$$

But each factor of this product is a polynomial in f of degree

$$\sum_{i=1}^{9} d_i(j) \quad , \tag{4.15}$$

and so, $\#\langle x\rangle$ is a polynomial in f of degree

$$\sum_{j=0}^{e(k)-1} \sum_{i=1}^{9} d_i(j) \tag{4.16}$$

which is less than or equal to k .

THEOREM 6. Let n, k be positive integers. Then

$$N_k(10^n) \sim cn^k \tag{4.17}$$

for a constant c which depends upon k .

PROOF. Since

$$N_k(10^n) = \sum \#\langle x\rangle \quad , \tag{4.18}$$

where the sum is taken over the collection of equivalence classes induced by $\cong$ on $N_k \cap [0, 10^n)$, we have that $N_k(10^n)$ is a polynomial in f of degree not exceeding k by Lemma 5 . Thus, all we need to do in order to show that $N_k(10^n)$ has degree k is to construct a Niven number, x, with digital sum k such that $\langle x\rangle$ is a polynomial in f of degree k . Such an x is

$$\sum_{i=0}^{k-1} 10^{\bar{e}(k) + ie(k)} \quad . \tag{4.19}$$

Here,

$$\#\langle x\rangle = \binom{f}{k} \quad , \tag{4.20}$$

which is a polynomial in f of degree exactly k . Thus,

$$N_k(10^n) \sim c_1 f^k \tag{4.21}$$

for some c_1 which is dependent on k . Since

$$f = \left[\frac{n - \bar{e}(k) - 1}{e(k)}\right] \quad , \tag{4.22}$$

we have that $f \sim n/e(k)$ and therefore, $N_k(10^n) \sim cn^k$ where $c = c_1/(e(k))^k$.

Finally, using an argument similiar to that in the proof of Theorem 2 , we have the following corollary.

COROLLARY 7. Let k, x be positive integers. Then

$$N_k(x) \sim c(\log x)^k \tag{4.23}$$

where c depends on k .

5. CONCLUSION.

Thus, a partial answer concerning an asymptotic formula for $N(x)$ has been presented. As was shown by Theorem 2, exact values of the constant c can be found for certain integers k. In fact, given a particular k, it is indeed possible to determine the exact form that c will be. This would involve an investigation of the partitions of k with summands less than or equal to 9, and the number of solutions to certain diophantine congruences. We feel that this is a subject for future study. The determination of an asymptotic formula for $N(x)$, however, will be left an an open problem.

REFERENCES

1. KENNEDY, R.E., GOODMAN, T.A., BEST, C.H. Mathematical Discovery and Niven Numbers, The MATYC Journal 14 (1980), 21-25.

2. KENNEDY, R.E., COOPER, C.N. On the Natural Density of the Niven Numbers, College Mathematics Journal 15 (1984), 309-312.

3. TANG, S.C. An Improvement and Generalization of Bellman-Shapiro's Theorem on a Problem in Additive Number Theory, Proc. Amer. Math. Soc. 14 (1963), 199-204.