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INTERIOR AND EXTERIOR SOLUTIONS FOR BOUNDARY VALUE PROBLEMS IN COMPOSITE ELASTIC AND VISCOUS MEDIA

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ABSTRACT. We present the solutions for the boundary value problems of elasticity when a homogeneous and istropic solid of an arbitrary shape is embedded in an infinite homogeneous isotropic medium of different properties. The solutions are obtained inside both the guest and host media by an integral equation technique. The boundaries considered are an oblong, a triaxial ellipsoid and an elliptic cyclinder of a finite height and their limiting configurations in two and three dimensions. The exact interior and exterior solutions for an ellipsoidal inclusion and its limiting configurations are presented when the infinite host medium is subjected to a uniform strain. In the case of an oblong or an elliptic cylinder of finite height the solutions are approximate. Next, we present the formula for the energy stored in the infinite host medium due to the presence of an arbitrary symmetrical void in it. This formula is evaluated for the special case of a spherical void. Finally, we analyse the change of shape of a viscous incompressible ellipsoidal region embedded in a slowly deforming fluid of a different viscosity. Two interesting limiting cases are discussed in detail.

KEY WORDS AND PHRASES. Isotropic solid, composite media, strain energy, viscous inhomogeneity, triaxial ellipsoid.
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1. INTRODUCTION.

Composite media problems arise in various fields of mechanics and geophysics. In this paper we first present the solutions for boundary value problems of elastostatics when a homogeneous and isotropic solid of an arbitrary shape is embedded in an infinite homogeneous isotropic medium of different properties. The solutions are obtained inside both the guest and the host media. The boundaries considered are an oblong, an ellipsoid with three unequal axes, and elliptic cylinder of finite height and their limiting configurations in two and three dimensions. The exact interior and exterior solutions for an ellipsoidal inclusion and its limiting configurations are presented when the infinite host media is subjected to a uniform strain. For other configurations the solution presented are approximate ones. Next we present the formula for the energy stored in the infinite host medium due to the presence of an arbitrary symmetrical void in it. This formula is evaluated for the special case of a spherical void. Finally, we analyse the change of shape of a viscous incompressible ellipsoidal region embedded in a slowly deforming fluid of a different viscosity. Two interesting limiting cases are discussed in detail.

The analysis is based on a computational scheme in which we first convert the boundary value problems to integral equations. Thereafter, we convert these integral equations to infinite set of algebraic equations. A judicial truncation scheme then helps us in achieving our results. Interesting feature of this computational technique is that the very first truncation of the algebraic system yields the exact solution for a triaxial ellipsoid and very good approximations for other configurations.

The main analysis of this article is devoted to three-dimensional problems of elasticity and viscous fluids. The limiting results for various two-dimensional problems can be deduced by taking appropriate limits.

2. MATHEMATICAL PRELIMINARIES

Let (x,y,z) be Cartesian coordinate system. A homogeneous three-dimensional solid of arbitrary shape of elastic constants λ_2 and μ_2 occupying region R_2 is embedded in an infinite homogeneous isotropic medium of R_1 of elastic constants λ_1 and μ_1. The elastic solid is assumed to be symmetrical with respect to the three coordinate axes and the origin $\underset{\sim}{0}$ of the coordinate system is situated at the centroid of R_2. Let S be the boundary of the region R_2 so that the entire region is $R = R_1 + S + R_2$. The stiffness tensors $C^{\alpha}_{ijk\ell}(\underset{\sim}{x})$, $\underset{\sim}{x} = (x,y,z) \in R_\alpha$, $\alpha = 1,2$ are constants and are defined as

$$C^{\alpha}_{ijk\ell} = \lambda_\alpha \delta_{ij}\delta_{k\ell} + \mu_\alpha(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \tag{2.1}$$

where δ's are Kronecker deltas. The latin indices have the range 1,2,3.

The integral equation which embodies this boundary value problem is derived in precisely the same fashion as the one in reference [1]. Indeed, the displacement field $\underset{\sim}{u}(\underset{\sim}{x})$ satisfies the integral equation

$$u_j(\underset{\sim}{x}) = u^0_j(\underset{\sim}{x}) + \Delta C_{i\ell km} \int_{R_2} G_{jm,k}(\underset{\sim}{x},\underset{\sim}{x}')u_{\ell,i'}(\underset{\sim}{x}')dR_2', \quad \underset{\sim}{x} \in R, \tag{2.2}$$

where subscript comma stands for differentiation, $\underset{\sim}{u}^0(\underset{\sim}{x})$ is the displacement field in the infinite host medium occupying the whole region R due to the prescribed stressed at infinity, $\Delta C_{i\ell km} = C^2_{i\ell km} - C^1_{i\ell km}$, while Green's function G_{km} satisfies the differential equation

$$C^1_{ijk\ell}G_{km,\ell j}\,(\underset{\sim}{x},\underset{\sim}{x}') = -\,\delta_{im}\,\delta(\underset{\sim}{x}-\underset{\sim}{x}'),\ \underset{\sim}{x},\underset{\sim}{x}' \in R, \tag{2.3}$$

and $\delta(\underset{\sim}{x}-x')$ is the Dirac delta function. Explicitly,

$$\begin{aligned} G_{ij}(\underset{\sim}{x},\underset{\sim}{x}') &= G_{ji}(\underset{\sim}{x},\underset{\sim}{x}') \\ &= \frac{1}{8\pi}\left\{\frac{1}{\mu_1}\nabla^2\delta_{ij} + \left(\frac{1}{\lambda_1+2\mu_1} - \frac{1}{\mu_1}\right)\frac{\partial^2}{\partial x_1 \partial x_j}\right\}|\underset{\sim}{x} - \underset{\sim}{x}'|. \end{aligned} \tag{2.4}$$

For the sake of completeness and for future reference we write down briefly the basic steps of the truncation scheme for solving the integral equation (2.2). To obtain the interior solution of the integral equation (2.2) when $\underset{\sim}{x} \in R_2$, we differentiate equation (2.2) n times to get

$$u_{j,p_1---p_n}(\underset{\sim}{x}) - u^0_{j,p_1---p_n} = (-1)^{n+1}\,\Delta C_{i\ell km}\int_{R_2} G_{j'm',\,k'p_1'---p_n'}(\underset{\sim}{x},\underset{\sim}{x}')u_{\ell,i'}(\underset{\sim}{x}')dR_2,$$
$$\underset{\sim}{x} \in R_2, \tag{2.5}$$

where p's have the values 1,2,3. Now we expand the quantities $u_{\ell,i'}(\underset{\sim}{x}')$ in Taylor series about the origin $\underset{\sim}{O}$ where $\underset{\sim}{x}' \in R_2$. Thus,

$$\begin{aligned} u_{\ell,i}(\underset{\sim}{x}') &= \sum_{s=0}^{\infty}\frac{1}{s!}\{u_{\ell,i'q_1'---q_s'}(\underset{\sim}{O})\}x'_{q_1}\ldots x'_{q_s} \\ &= \sum_{s=0}^{\infty}\frac{1}{s!}\{u_{\ell,i\,q_1---q_s}(\underset{\sim}{x})\}x'_{q_1}---x'_{q_s}, \end{aligned} \tag{2.6}$$

where q's have the values 1,2,3. Substituting these values in (2.5) and setting $\underset{\sim}{x} = \underset{\sim}{O}$, in both sides we obtain

$$u_{j,p_1---p_n}(\underset{\sim}{O}) - u^0_{j,p_1---p_n}(\underset{\sim}{O}) = (-1)^{n+1}\Delta C_{i\ell km}\sum_{s=0}^{\infty}\frac{1}{s!}T_{jm,kp_1---p_n,q_1---q_s}$$
$$\times\, u_{\ell,iq_1---q_s}(\underset{\sim}{O}), \tag{2.7}$$

where

$$T_{jm,kp_1---p_n,q_1---q_s} = \int_{R_2} G_{jm,kp_1---p_n}(\underset{\sim}{x},\underset{\sim}{O})x_{q_1}---x_{q_s}\,dR_2, \tag{2.8}$$

As in reference [1], taking $n = 0,1$, $s = 0$, in equation (2.5) we get $u_j(\underset{\sim}{O}) = u^0_j(\underset{\sim}{O})$, and

$$u_{j,p}(\underset{\sim}{O}) - u^0_{j,p}(\underset{\sim}{O}) = \Delta C_{i\ell km}T_{jm,kp}u_{\ell,i}(0), \tag{2.9}$$

respectively, where

$$\begin{aligned} T_{jm,kp} &= \int_{R_2} G_{jm,kp}(\underset{\sim}{x},\underset{\sim}{O})dR_2 \\ &= \{\mu_1^{-1}\delta_{jm}t_{kp\ell\ell} + (M_1^{-1}-\mu_1^{-1})t_{jmkp}\}, \end{aligned} \tag{2.10}$$

while $M_1 = \lambda_1 + 2\mu_1$ and t_{jmkp} are the shape factors

$$t_{jmkp} = \frac{1}{8\pi}\int_{R_2}\frac{\partial^4 r}{\partial x_j \partial x_m \partial x_k \partial x_p}\,dR_2,\ r = |\underset{\sim}{x}|. \tag{2.11}$$

Now we substitute the value $\Delta C_{jmkp} = C^2_{jmkp} - C^1_{jmkp}$ from (2.1) in (2.9) and get

$$u_{j,p}(\underset{\sim}{Q}) - u^0_{j,p}(\underset{\sim}{Q}) = \Delta\lambda T_{jk,kp}u_{\ell,\ell}(\underset{\sim}{Q})$$
$$+ \Delta\mu(T_{jm,kp}u_{m,k}(\underset{\sim}{Q}) + T_{jm,kp}u_{k,m}(\underset{\sim}{Q})). \tag{2.12}$$

When we decompose $u_{j,p}(\underset{\sim}{Q})$ into the symmetric and antisymmetric parts $u_{jp}(\underset{\sim}{Q})$ and $a_{jp}(\underset{\sim}{Q})$ respectively, as we did in reference [1] and define

$$T^{\pm}_{jm,kp} = \frac{1}{2}(T_{jm,kp} \pm T_{pm,kj}),$$

we find that relation (2.9) yields the following two relations

$$u_{jp}(\underset{\sim}{Q}) - u^0_{jp}(\underset{\sim}{Q}) = \Delta C_{i\ell km}T^+_{jm,kp}u_{\ell i}(\underset{\sim}{Q}), \tag{2.13}$$

$$a_{jp}(\underset{\sim}{Q}) - a^0_{jp}(\underset{\sim}{Q}) = \Delta C_{i\ell km}T^-_{jm,kp}u_{\ell i}(0). \tag{2.14}$$

Equation (2.13) gives rise to the relation

$$u_{kk}(\underset{\sim}{Q})[1 + \frac{\Delta\lambda}{M_1}] - \frac{2\Delta\mu}{M_1}[t_{11kk}u_{11}(\underset{\sim}{Q})+t_{22kk}u_{22}(\underset{\sim}{Q})+t_{33kk}u_{33}(\underset{\sim}{Q})] = u^0_{kk}(\underset{\sim}{Q}). \tag{2.15}$$

The values of $u_{11}(\underset{\sim}{Q})$, $u_{22}(\underset{\sim}{Q})$, $u_{33}(\underset{\sim}{Q})$ in terms of the known constants $u^0_{11}(\underset{\sim}{Q})$, $u^0_{22}(\underset{\sim}{Q})$, $u^0_{33}(\underset{\sim}{Q})$ are given by the matrix equation

$$B\bar{\underset{\sim}{u}} = \bar{\underset{\sim}{u}}^0, \tag{2.16}$$

where the column vectors $\bar{\underset{\sim}{u}}$ and $\bar{\underset{\sim}{u}}^0$ are

$$\bar{\underset{\sim}{u}} = \begin{bmatrix} u_{11}(\underset{\sim}{Q}) \\ u_{22}(\underset{\sim}{Q}) \\ u_{33}(\underset{\sim}{Q}) \end{bmatrix}, \quad \bar{\underset{\sim}{u}}^0 = \begin{bmatrix} u^0_{11}(\underset{\sim}{Q}) \\ u^0_{22}(\underset{\sim}{Q}) \\ u^0_{33}(\underset{\sim}{Q}) \end{bmatrix}, \tag{2.17}$$

while the elements b_{ij}, $i,j = 1,2,3$ of the matrix B are given as

$$b_{ij} = (1-2\frac{\Delta\mu}{\mu_1}t_{iikk})\delta_{ij} - \frac{\Delta\lambda}{M_1}t_{iikk} - 2\Delta\mu(\frac{1}{M_1} - \frac{1}{\mu_1})t_{iijj}, \tag{2.18}$$

and the suffices i and j are not summed. Furthermore, the values of $u_{ij}(\underset{\sim}{Q})$, $i \neq j$, $i,j = 1,2,3$ are given in terms of the known constants $u^0_{ij}(\underset{\sim}{Q})$, $i \neq j$ by the relation

$$u_{ij}(\underset{\sim}{Q}) = [1-\Delta\mu\{\mu_1^{-1}(t_{jjkk}+t_{iikk}) + 4(M_1^{-1} - \mu_1^{-1})t_{iijj}\}]^{-1}u^0_{ij}(\underset{\sim}{Q}). \tag{2.19}$$

Similarly, equation (2.14) yields the values of non-zero components of $a_{ij}(\underset{\sim}{Q})$, $i \neq j$ in the form

$$a_{ij}(\underset{\sim}{Q}) = a^0_{ij}(\underset{\sim}{Q}) + \frac{\Delta\mu}{\mu_1}(t_{jjkk}-t_{iikk})u_{ij}(\underset{\sim}{Q}), \quad i \neq j,\ i,j = 1,2,3, \tag{2.20}$$

where $u_{ij}(\underset{\sim}{Q})$ is defined by (2.19).

Finally, substituting the above values of $u_{ij}(\underset{\sim}{Q})$ and $a_{ij}(\underset{\sim}{Q})$ in the expansions

$$u_i(\underset{\sim}{x}) = u_i^0(\underset{\sim}{Q}) + (u_{ik}(\underset{\sim}{Q}) + a_{ik}(\underset{\sim}{Q}))x_k, \quad \underset{\sim}{x} \in R_2, \tag{2.21}$$

yield the required approximate inner solution where $x_1=x$, $x_2=y$, $x_3=z$. Relation (2.21) gives the exact solution for an ellipsoidal inclusion and its limiting configurations when the infinite host medium is subjected to a uniform prescribed stress.

In the case of elastic inclusions which are symmetrical with respect to the three coordinate axes and have only one characteristic length as in the case of a sphere, a cube etc., there are only two distinct non-zero shape factors, namely, t_{1111}, t_{1122}. Indeed, since relation (2.9) yields

$$t_{kkmm} = \frac{1}{8\pi}\int_{R_2} \nabla^2(\nabla^2 r)dR_2 = \frac{1}{8\pi}\int_{R_2} [-8\pi\delta(\underset{\sim}{x})]dR_2 = -1, \tag{2.22}$$

it follows that in this case

$$t_{1111} = t_{2222} = t_{3333};\ t_{1122} = t_{2233} = t_{3311};\ t_{11kk} = t_{22kk} = t_{33kk} = -\frac{1}{3}. \tag{2.23}$$

When we substitute these relations in (2.15) and (2.16), we get the simplified results,

$$u_{kk}(\underset{\sim}{Q}) = [1 + \frac{\Delta K}{M_1}]^{-1}\, u_{kk}^0(\underset{\sim}{Q});\ K = \lambda + \frac{2}{3}\mu,\ \Delta K = \Delta\lambda + \frac{2}{3}\Delta\mu, \tag{2.24}$$

$$\bar{\underset{\sim}{u}} = \frac{1}{3}\begin{bmatrix} c & d & d \\ d & c & d \\ d & d & c \end{bmatrix} \underset{\sim}{u}^0, \tag{2.25}$$

where

$$c = 2A^{-1}+C^{-1}, \quad d = -A^{-1} + C^{-1},$$

$$A = 1+2\Delta\mu\{\frac{1}{3\mu_1} - (M_1^{-1}-\mu_1^{-1})(t_{1111}-t_{1122})\},$$

$$C = 1 + \frac{\Delta K}{M_1}\ .$$

Similarly, in this case, results (2.19) and (2.20) yield

$$u_{ij}(\underset{\sim}{Q}) = [1+2\Delta\mu\{\frac{1}{3\mu_1} - 2(M_1^{-1}-\mu_1^{-1})t_{iijj}\}]^{-1}\, u_{ij}^0(\underset{\sim}{Q}),\ i \neq j,\ i,j = 1,2,3, \tag{2.26}$$

and

$$a_{ij}(\underset{\sim}{Q}) = a_{ij}^0(\underset{\sim}{Q}). \tag{2.27}$$

The above values of $u_{ij}(0)$ and $a_{ij}(0)$ when substituted in the expansions (2.21) give rise to the required inner solution in this case. In order to complete the analysis of this section we need the values of the shape factors of various inclusion. They are presented in the next section.

3. VALUES OF THE SHAPE FACTORS FOR VARIOUS SOLIDS

(i) Oblong. Let the faces of the oblong be given by $x = \pm a$, $y = \pm b$, $z = \pm c$ so that the region R_2 is $|x| < a$, $|y| < b$, $|z| < c$. In this case,

$$\left.\begin{aligned} t_{1111} &= -\frac{2}{\pi}\tan^{-1}\frac{bc}{a\Delta} + \frac{1}{\pi}\frac{abc}{\Delta}\cdot\frac{a^2+\Delta^2}{a^2\Delta^2+b^2c^2}\,, \\ t_{1122} &= -\frac{1}{\pi}\cdot\frac{abc}{(a^2+b^2)\Delta}\,,\quad t_{1133} = -\frac{1}{\pi}\cdot\frac{abc}{(a^2+c^2)\Delta}\,,\quad t_{11kk} = -\frac{2}{\pi}\tan^{-1}\frac{bc}{a\Delta}\,, \end{aligned}\right\} \tag{3.1}$$

where $\Delta = (a^2+b^2+c^2)^{1/2}$ and k is summed. All the other shape factors can be obtained by permutations.

For a cube of edge $2a$, the above values reduce to

$$\left.\begin{aligned} t_{1111} &= t_{2222} = t_{3333} = -\frac{1}{3} + \frac{1}{\pi\sqrt{3}}\,, \\ t_{1122} &= t_{2233} = t_{3311} = -\frac{1}{2\pi\sqrt{3}}\,, \\ t_{11kk} &= t_{22kk} = t_{33kk} = -\frac{1}{3}\,. \end{aligned}\right\} \tag{3.2}$$

When we take the limit $c \to \infty$ in relations (3.1), we obtain the following values of the non-zero shape factors for an infinite rectangular cylinder occupying the region R_2: $|x| < a$, $|y| < b$, $-\infty < z < \infty$.

$$\left.\begin{aligned} t_{1111} &= -\frac{2}{\pi}\tan^{-1}\frac{b}{a} + \frac{ab}{\pi(a^2+b^2)}\,,\quad t_{1122} = t_{2211} = -\frac{ab}{\pi(a^2+b^2)}\,, \\ t_{2222} &= -\frac{2}{\pi}\tan^{1}\frac{a}{b} + \frac{ab}{\pi(a^2+b^2)}\,. \end{aligned}\right\} \tag{3.3}$$

Setting $b = a$ in the above formulas we obtain the values of the corresponding shape factors for an infinite square cylinder occupying the region R_2 : $|x| < a$, $|y| < a$, $-\infty < z < \infty$. These values are

$$t_{1111} = -\frac{1}{2} + \frac{1}{2\pi}\,,\quad t_{1122} = t_{2211} = -\frac{1}{2\pi}\,,\quad t_{2222} = -\frac{1}{2} + \frac{1}{2\pi}\,. \tag{3.4}$$

The limiting results (3.3) and (3.4) agree with the ones obtained in reference [1].

(ii) Triaxial Ellipsoid. Let the equation of the surface of the ellipsoidal elastic solid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a \ge b \ge c > 0,$$

where a, b and c are the lengths of the semi-principal axes of the ellipsoid. In this case R_2 is the region $x^2/a^2 + y^2/b^2 + z^2/c^2 < 1$, and the values of the non-vanishing shape factors are

$$t_{iiii} = -\frac{3abc}{4}\int_0^\infty \frac{u\,du}{(u+a_i^2)^2 R_u}\,,\quad i = 1,2,3, \tag{3.5a}$$

$$t_{iijj} = -\frac{abc}{4}\int_0^\infty \frac{u\,du}{(u+a_i^2)(u+a_j^2)R_u}\,,\ i \neq j,\ i,j = 1,2,3, \tag{3.5b}$$

where $R_u = [(u+a^2)(u+b^2)(u+c^2)]^{1/2}$, $a_1 = a$, $a_2 = b$, $a_3 = c$ and the suffices i and j are not summed. For a prolate spheroid with the semi-principal axes a,b,b, $a \geq b$, the foregoing shape factors reduce to

$$t_{1111} = -\frac{3ab^2}{4}\int_0^\infty \frac{udu}{(u+a^2)^{5/2}(u+b^2)} = \frac{1}{2}(1-k^2)(3L_1-1), \tag{3.6a}$$

$$t_{2222} = t_{3333} = -\frac{3ab^2}{4}\int_0^\infty \frac{udu}{(u+b^2)^3(u+a^2)^{1/2}} = -\frac{3}{8}+\frac{1}{16}(1-k^2)+\frac{3}{16}(3+k^2)L_1, \tag{3.6b}$$

$$t_{1122} = t_{1133} = -\frac{ab^2}{4}\int_0^\infty \frac{udu}{(u+b^2)^2(u+a^2)^{3/2}} = \frac{1}{12}(1-k^2)-\frac{1}{4}(3-k^2)L_1, \tag{3.6c}$$

$$t_{2233} = -\frac{ab^2}{4}\int_0^\infty \frac{udu}{(u+a^2)^{1/2}(u+b^2)^3} = \frac{1}{3}t_{2222}, \tag{3.6d}$$

where

$$L_1 = \frac{(1-k^2)}{k4}\left[\frac{1}{2k}\log\left(\frac{1+k}{1-k}\right)-1-\frac{1}{3}k^2\right],\ k^2 = 1-\frac{b^2}{a^2}. \tag{3.7}$$

In the limit when $b \to a$, i.e. $k \to 0$, in relations (3.6) and (3.7) we find that $L_1 \to 1/5$ and the shape factors for a sphere of radius a are

$$t_{iiii} = -\frac{1}{5},\ t_{iijj} = -\frac{1}{15},\ i \neq j,\ t_{iikk} = -\frac{1}{3}, \tag{3.8}$$

$i,j,k = 1,2,3$ and the suffices i and j are not summed.

Similarly, the shape factors for the oblate spheroid with semi-principal axes a,a,b, $a \geq b$ derived from relations (3.5) are

$$t_{1111} = t_{2222} = -\frac{3a^2b}{4}\int_0^\infty \frac{udu}{(u+a^2)3(u+b^2)^{1/2}} = -\frac{3}{8}+\frac{1}{16}(1+\kappa^2)+\frac{3}{16}(3-\kappa^2)L_2, \tag{3.9a}$$

$$t_{3333} = \frac{3a^2b}{4}\int_0^\infty \frac{udu}{(u+b^2)^{5/2}(u+a^2)} = \frac{1}{2}(1+\kappa^2)-(3L_2-1) \tag{3.9b}$$

$$t_{1133} = t_{2233} = -\frac{a^2b}{4}\int_0^\infty \frac{udu}{(u+a^2)(u+b^2)^{3/2}} = \frac{1}{12}(1+\kappa^2)-\frac{1}{4}(3+\kappa^2)L_2, \tag{3.9c}$$

$$t_{1122} = -\frac{a^2b}{4}\int_0^\infty \frac{udu}{(u+a^2)^3(u+b^2)^{1/2}} = \frac{1}{3}t_{1111}, \tag{3.9d}$$

where

$$L_2 = \left(\frac{1+\kappa^1}{\kappa 4}\right)\left(\frac{1}{\kappa}\tan^{-1}\kappa-\frac{1}{3}+\frac{1}{3}\kappa^2\right),\ \kappa^2 = \left(\frac{a^2}{b^2}-1\right). \tag{3.10}$$

When $b \to a$, we again recover the corresponding results for a spherical inclusion.

In the limit $\kappa \to \infty$ in relations (3.9) and (3.10), $L_2 \to 1/3$, $\kappa^2(3L_2-1) \to 0$, we find that the values of the non-vanishing shape factors for an extremely thin oblate spheroid are

$$t_{1111} = t_{2222} = -1/8,\quad t_{1133} = t_{2233} = -1/6,\quad t_{1122} = -1/24.$$

Similarly, when $a \to \infty$ in results (3.5), we deduce the shape factors for the infinite elliptic cylinder occupying the region $R_2 : y^2/b^2 + z^2/c^2 < 1$, $-\infty < x < \infty$. Then the non-zero values of the shape factors are

$$t_{2222} = -\frac{3bc}{4}\int_0^\infty \frac{udu}{(u+b^2)^{5/2}(u+c^2)^{1/2}} = \frac{-c(b+2c)}{2(b+c)^2} = -\frac{1}{2}e^{\xi_0}\sinh\xi_0(2-e^{-\xi_0}\cosh\xi_0),$$

$$t_{3333} = -\frac{3abc}{4}\int_0^\infty \frac{u\,du}{(u+b^2)^{1/2}(u+c^2)^{5/2}} = \frac{-b(c+2b)}{2(b+c)^2} = -\frac{1}{2}e^{-\xi_0}\cosh\xi_0(2-e^{-\xi_0}\sinh\xi_0),$$

$$t_{2233} = t_{3322} = -\frac{bc}{4}\int_0^\infty \frac{u\,du}{(u+b^2)^{3/2}(u+c^2)^{3/2}} = \frac{-bc}{2(b+c)^2} = -\frac{1}{4}e^{-2\xi_0}\sinh 2\xi_0, \qquad (3.11)$$

and $b/c = \coth\xi_0$, which agree with the known results [1]. When $c \to b$, we recover the shape factors for a circular cylindrical inclusion while in the limit $\xi_0 \to 0$ we obtain the corresponding values for an infinite strip.

(iii) Elliptic Cylinder of Finite Height. Let the elliptic cylinder of height $2h$ occupy the region

$$R_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1, \ |z| < h,$$

where a and b are the lengths of its semi principal axes. In this case the values of shape factors are

$$t_{1111} = \frac{3hab}{\pi}\int_0^{\pi/2} [-\cos^2\eta + \frac{a^2\cos^4\eta}{(L)^2}\{1-\frac{1}{3}\frac{h^2}{(L^2+h^2)}\}]\frac{d\eta}{(L)^2\sqrt{L^2+h^2}}, \qquad (3.12a)$$

$$t_{2222} = \frac{3hab}{\pi}\int_0^{\pi/2} [-\sin^2\eta + \frac{b^2\sin^4\eta}{(L)^4}\{1-\frac{h^2}{3(L^2+h^2)}\}]\frac{d\eta}{(L)^2\sqrt{L^2+h^2}}, \qquad (3.12b)$$

$$t_{3333} = \frac{3hab}{\pi}\int_0^{\pi/2} [-\sin^2\eta + \frac{h^2\sin^2\eta}{(h^2+a^2\cos^2\eta)}\{1-\frac{b^2\sin^2\eta}{3(L^2+h^2)}\}]\frac{d\eta}{(h^2+a^2\cos^2\eta)\sqrt{L^2+h^2}}, \qquad (3.12c)$$

$$t_{1122} = \frac{hab}{\pi}\int_0^{\pi/2} [-\cos^2\eta + \frac{3b^2\sin^2\eta\cos^2\eta}{(L)^2}\{1-\frac{h^2}{3(L^2+h^2)}\}]\frac{d\eta}{(L)^2\sqrt{L^2+h^2}}, \qquad (3.12d)$$

$$t_{1133} = \frac{hab}{\pi}\int_0^{\pi/2} [-\sin^2\eta + \frac{3a^2\sin^2\eta\cos^2\eta}{(h^2+a^2\cos^2\eta)}\{1-\frac{b^2\sin^2\eta}{3(L^2+h^2)}\}]\frac{d\eta}{(h^2+a^2\cos^2\eta)\sqrt{L^2+h^2}}, \qquad (3.12e)$$

$$t_{2233} = \frac{hab}{\pi}\int_0^{\pi/2} [-\cos^2\eta + \frac{3b^2\sin^2\eta\cos^2\eta}{(h^2+b^2\sin^2\eta)}\{1-\frac{a^2\cos^2\eta}{3(L^2+h^2)}\}]\frac{d\eta}{(h^2+b^2\sin^2\eta)\sqrt{L^2+h^2}}, \qquad (3.12f)$$

where

$$L^2 = (a^2\cos^2\eta + b^2\sin^2\eta).$$

In order to get the corresponding shape factors of a circular cylinder of radius a and height $2h$, we let $b \to a$ in the above formulas and obtain

$$\left.\begin{aligned} t_{1111} &= t_{2222} = -\frac{3}{16}(\ell+\ell^3), t_{3333} = -1 + \frac{1}{2}(3\ell-\ell^3), \\ t_{1133} &= t_{2233} = -\frac{1}{4}(\ell-\ell^3), \ \ell = h/\ a^2+h^2. \end{aligned}\right\} \qquad (3.13)$$

These results, in turn, yield the shape factors of a circular disc of radius a and small thickness $2h$. They are

$$\left.\begin{aligned} t_{1111} &= t_{2222} = -\frac{3}{16}(\frac{h}{a} + \frac{h^3}{2a^3}), \ t_{3333} = -1 + \frac{1}{2}(\frac{3h}{a} - \frac{5}{2}\frac{h^3}{a^3}), \\ t_{1122} &= \frac{1}{3}t_{1111}, t_{1133} = t_{2233} = -\frac{1}{4}(\frac{h}{a} - \frac{3h^3}{2a^3}). \end{aligned}\right\} \qquad (3.14)$$

4. EXACT INTERIOR AND EXTERIOR SOLUTIONS FOR AN ELLIPSOIDAL ENCLOSURE OCCUPYING REGION $R_2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1.$

When the infinite host (homogeneous and isotropic) elastic medium occupying the whole region R is subjected to a prescribed uniform stress T along the direction of z-axis, the components of the vectors $u^0(\underset{\sim}{x})$ and $\underset{\sim}{\tau}^0(\underset{\sim}{x})$ are given as

$$u_1^0(\underset{\sim}{x}) = -\frac{T\sigma_1}{E_1}x,\ u_2^0(\underset{\sim}{x}) = -\frac{T\sigma_1}{E_1}y,\ u_3^0(\underset{\sim}{x}) = \frac{T}{E_1}z,\ \tau_{22}^0(\underset{\sim}{x}) = T,\quad \underset{\sim}{x} \in R, \tag{4.1}$$

where σ's have to be defined. Accordingly, in this case

$$\begin{aligned} &u_{11}^0(\underset{\sim}{0}) = u_{22}^0(\underset{\sim}{0}) = -\frac{T\sigma_1}{E_1},\ u_{33}^0(\underset{\sim}{0}) = \frac{T}{E_1}, \\ &a_{ij}^0(\underset{\sim}{0}) = 0,\ i \neq j,\ j = 1,2,3. \end{aligned} \tag{4.2}$$

When this infinite host medium has an isotropic elastic ellipsoidal inclusion of Lame's constants λ_2, μ_2 occupying the region R_2:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1,$$

then the exact inner solution is given by

$$u_1(\underset{\sim}{x}) = (u_{11}(0))x,\ u_2(\underset{\sim}{x}) = (u_{22}(\underset{\sim}{0}))y,\ u_3(\underset{\sim}{x}) = (u_{33}(0))z,\ \underset{\sim}{x} \in R_2, \tag{4.3}$$

where the constants $u_{11}(\underset{\sim}{0})$, $u_{22}(\underset{\sim}{0})$ and $u_{33}(\underset{\sim}{0})$ are given by the matrix equation (2.17). Thus

$$B\bar{\underset{\sim}{u}} = \bar{\underset{\sim}{u}}^0 = -\frac{T}{E_1}\begin{bmatrix}\sigma_1\\ \sigma_1\\ -1\end{bmatrix}, \tag{4.4}$$

or

$$B(\bar{\underset{\sim}{u}} = \begin{bmatrix}u_{11}(\underset{\sim}{0})\\ u_{22}(\underset{\sim}{0})\\ u_{33}(\underset{\sim}{0})\end{bmatrix}) = \bar{\underset{\sim}{u}}^0 = \begin{bmatrix}u_{11}^0(\underset{\sim}{0})\\ u_{22}^0(\underset{\sim}{0})\\ u_{33}^0(\underset{\sim}{0})\end{bmatrix} = -\frac{T}{E_1}\begin{bmatrix}\sigma_1\\ \sigma_1\\ -1\end{bmatrix},$$

where the components b_{ij} of the matrix B are defined in (2.18).

The solution of equation (4.4), after me simplifications, is given as

$$u_{11}(\underset{\sim}{0}) = \alpha(g_1h_2-g_2h_1),\ u_{22}(\underset{\sim}{0}) = \alpha(h_1f_2-h_2f_1),\ u_{33}(\underset{\sim}{0}) = \alpha(f_1g_2-f_2g_1), \tag{4.5}$$

where

$$f_1 = 1 - \frac{2\Delta\mu}{\mu_1}t_{11kk} - \frac{\Delta\lambda}{M_1}(t_{11kk}-t_{22kk}) = 2\Delta\mu(\frac{1}{M_1} - \frac{1}{\mu_1})(t_{1111} - t_{1122}), \tag{4.6a}$$

$$g_1 = -1 + 2\frac{\Delta\mu}{\mu_1}t_{22kk} - \frac{\Delta\lambda}{1}(t_{11kk}-t_{22kk}) - 2\Delta\mu(\frac{1}{M_1} - \frac{1}{\mu_1})(t_{1122}-t_{2222}) \tag{4.6b}$$

$$h_1 = -\frac{\Delta\lambda}{M_1}(t_{11kk}-t_{22kk}) - 2\Delta\lambda(\frac{1}{M_1} - \frac{1}{\mu_1})(t_{1133}-t_{2233}), \tag{4.6c}$$

$$f_2 = 1 - 2\frac{\Delta\mu}{\mu_1}t_{11kk} - \frac{\Delta\lambda}{M_1}(t_{11kk}+\sigma_1 t_{33kk}) - 2\Delta\mu(\frac{1}{M_1} - \frac{1}{\mu_1})(t_{1111}+\sigma_1 t_{1133}) \tag{4.6d}$$

$$g_2 = -\frac{\Delta\lambda}{M_1}(t_{11kk}+\sigma_1 t_{33kk}) - 2\Delta\mu(\frac{1}{M_1}-\frac{1}{\mu_1})(t_{1122}+t_{2233}), \tag{4.6e}$$

$$h_2 = \sigma_1(1-2\frac{\Delta\mu}{\mu_1}t_{33kk}) - \frac{\Delta\mu}{M_1}(t_{11kk}+t_{33kk}) - 2\Delta\mu(\frac{1}{M_1}-\frac{1}{\mu_1})(t_{1133}+\sigma_1 t_{3333}) \tag{4.6f}$$

$$\alpha = \frac{(1-2\sigma_1)T}{E_1\beta}, \tag{4.6g}$$

$$\beta = (g_1h_2-g_2h_1)(1+\frac{\Delta\lambda}{M_1}-\frac{2\Delta\mu}{M_1}t_{11kk}) + (h_1f_2-h_2f_1)(1+\frac{\Delta\lambda}{M_1}-\frac{2\Delta\mu}{M_1}t_{22kk})$$
$$+ (f_1g_2-f_2g_1)(1+\frac{\Delta\lambda}{M_1}-2\frac{\Delta\mu}{M_1}t_{33kk}), \tag{4.6h}$$

where t's are the shape factors of the ellipsoid as given by (3.5), when we set $a_1=a$, $a_2=b$, $a_3=c$. When we substitute the values of $u_{ii}(Q)$, $i = 1,2,3$, from (4.5) in (4.3), we get the exact inner solution.

Limiting Cases

To check these results we take the limits $b \to a$, $c \to a$, so that the ellipsoidal region reduces to the spherical region $x^2+y^2+z^2 < a^2$. Now for the sphere there are only two distinct non-zero shape factors, namely,

$$t_{1111} = t_{2222} = t_{3333} = -\frac{1}{5};\quad t_{1122} = t_{2233} = t_{3311} = -\frac{1}{15},$$

and consequently

$$t_{11kk} = t_{22kk} = t_{33kk} = -\frac{1}{3}.$$

Thus, for this limiting case we have

$$f_1 = -g_1 = 1 + 2\Delta\mu\{\frac{1}{3\mu_1}+\frac{2}{15}(\frac{1}{M_1}-\frac{1}{\mu_1})\}, \tag{4.7a}$$

$$h_1 = 0, \tag{4.7b}$$

$$f_2 = 1 + \frac{2\Delta\mu}{3\mu_1} + \frac{\Delta\lambda}{3M_1}(1+\sigma_1) + \frac{2\Delta\mu}{15}(\frac{1}{M_1}-\frac{1}{\mu_1})(3+\sigma_1), \tag{4.7c}$$

$$g_2 = \frac{\Delta\lambda}{3M_1}(1+\sigma_1) + \frac{2\Delta\mu}{15}(\frac{1}{M_1}-\frac{1}{\mu_1})(1+\sigma_1), \tag{4.7d}$$

$$h_2 = {}_1(1+\frac{2\Delta\lambda}{3\mu_1}) + \frac{\Delta\lambda}{3M_1}(1+\sigma_1) + \frac{2\Delta\mu}{15}(\frac{1}{M_1}-\frac{1}{\mu_1})(1+3\sigma_1) \tag{4.7e}$$

$$\alpha = -\frac{T}{[g_1(1+\frac{\Delta K}{M_1})\{1+2\Delta\mu(\frac{1}{3\mu_1}+\frac{2}{15}(\frac{1}{M_1}-\frac{1}{\mu_1}))\}]}, \tag{4.7f}$$

$$\beta = (1+\frac{\Delta K}{M_1})[g_1(2h_2-f_2-g_2)], \tag{4.7g}$$

where $K = \Delta\lambda + (2\Delta\mu)/3$. Also,

$$u_{11}(\underset{\sim}{Q}) = u_{22}(\underset{\sim}{Q}) = -\frac{T\sigma_1}{3E_1}(\frac{1}{A}+\frac{2}{C}) + \frac{T}{3E_1}(\frac{1}{C}-\frac{1}{A}), \tag{4.8a}$$

$$u_{33}(0) = -\frac{2T\sigma_1}{3E_1}(\frac{1}{C}-\frac{1}{A}) + \frac{T}{3E_1}(\frac{2}{A}+\frac{1}{C}), \tag{4.8b}$$

where

$$A = 1 + 2\Delta\mu\left\{\frac{1}{3\mu_1} + \frac{2}{15}\left(\frac{1}{M_1} - \frac{1}{\mu_1}\right)\right\} = \frac{\mu_1(7-5\sigma_1)+\mu_2(8-10\sigma_1)}{15\mu_1(1-\sigma_1)}\ , \tag{4.9}$$

$$C = \left(1+\frac{\Delta\kappa}{M_1}\right) = \frac{(1-2\sigma_1)[2\mu_1(1-2\sigma_2)+\mu_2(1+\sigma_2)]}{3\mu_1(1-\sigma_1)(1-2\sigma_2)}\ . \tag{4.10}$$

Substituting the values of $u_{ii}(\underset{\sim}{Q})$ from (4.8) to (4.10) in (4.3), we obtain the exact inner solution for the spherical inclusion. Expressing the components $u_i(\underset{\sim}{x})$ in spherical polar coordinates (r,θ,φ) we have for $r < a$,

$$u_r(\underset{\sim}{x}) = \frac{r}{2}\{[u_{11}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})] + [u_{33}(\underset{\sim}{Q}) - u_{11}(\underset{\sim}{Q})]\cos 2\theta\}, \tag{4.11a}$$

$$u_\theta(x) = -\frac{r}{2}[u_{33}(\underset{\sim}{Q}) - u_{11}(\underset{\sim}{Q})]\sin 2\theta. \tag{4.11b}$$

The corresponding non-vanishing stress components are

$$\tau_{rr}(\underset{\sim}{x}) = \lambda_2[2u_{11}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})] + \mu_2\{[u_{11}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})] + [u_{33}(\underset{\sim}{Q}) - u_{11}(\underset{\sim}{Q})]\cos 2\theta\}, \tag{4.12a}$$

$$\tau_{\theta\theta}(x) = \lambda_2[2u_{11}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})] + \mu_2\{[u_{11}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})] - u_{33}(\underset{\sim}{Q}) - u_{11}(\underset{\sim}{Q})]\cos 2\theta, \tag{4.12b}$$

$$\tau_{r\theta}(\underset{\sim}{x}) = -\mu_2[u_{33}(\underset{\sim}{Q}) - u_{11}(\underset{\sim}{Q})]\sin 2\theta, \tag{4.12c}$$

$$\tau_{\varphi\varphi}(\underset{\sim}{x}) = \lambda_2[2u_{11}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})] + 2\mu_2 u_{11}(\underset{\sim}{Q}). \tag{4.12d}$$

As far as the authors are aware [3,4] even these exact interior solutions for a sphere are new.

Interior solutions for a prolate-spheroidal enclosure of semi-principal axes a,b,b, $a \geq b$ are obtained by appealing to the corresponding shape factors. The values of the shape factors t_{iiii} and t_{iijj}, $i \neq j$, $i = j = 1,2,3$ (i and j are not summed), are given by relations (3.6) while,

$$t_{11kk} = \frac{1}{2}(1-k^2)(3L_1-1) + \frac{1}{2}(3-k^2)L_1 = -\frac{1}{3}(1-k^2) - k^2L_1,$$

$$t_{22kk} = t_{33kk} = -\frac{1}{3}\left(1+\frac{k^2}{2}\right) + \frac{1}{2}k^2 L_1.$$

The values of the shape factors and relations (4.3), (4.5) and (4.6) lead to the required exact solutions.

Similarly, using the shape factors of oblate spheroid as given by relations (3.9) we obtain from equations (4.3), (4.5) and (4.6) the exact solution for this limiting case. Formulas for various other configurations such as an elliptic disk can now also be derived.

In precisely the same manner we use the shape factors of the <u>oblong</u> as given by (3.1) and that of <u>elliptic cylinder of finite height</u> as given by (3.12) and derive the first approximation to the interior solutions of these cases from equations (4.3), (4.5) and (4.6). These results yield, in the limit, the corresponding formulas for the configurations such as a cube and a circular cylinder of finite height.

Let us now discuss the exact outer solution for an ellipsoidal enclosure occupying the region R_2: $x^2/a^2 + y^2/b^2 + z^2/c^2 < 1$. We have found that the exact inner solution in this case is given by relation (4.3) where the values of the constants $u_{ii}(\underset{\sim}{0})$, $i = 1,2,3$ are given explicitly by equations (4.5) and (4.6) in terms of the known shape factors of the ellipsoid. Substituting this inner solution in the governing integral equation (2.2) and setting $x_1=x$, $y_1=y$, $z_1=z$, $(x=\underset{\sim}{x},y,z)$, we have

$$u_j(\underset{\sim}{x}) = u_j^0(\underset{\sim}{x}) + \Delta\lambda u_{11}(\underset{\sim}{0}) + u_{22}(\underset{\sim}{0}) + u_{33}(\underset{\sim}{0})\} \int_{R_2} G_{jk,k}(\underset{\sim}{x},\underset{\sim}{x}')dR_2'$$

$$+ 2\,\Delta\mu u_{11}(\underset{\sim}{0}) \int_{R_2} G_{j1,1}(\underset{\sim}{x},\underset{\sim}{x}')dR_2' + u_{22}(\underset{\sim}{0}) \int_{R_2} G_{j2,2}(\underset{\sim}{x},\underset{\sim}{x}')dR_2'$$

$$+ u_{33}(\underset{\sim}{0}) \int_{R_2} G_{j3,3}(\underset{\sim}{x},\underset{\sim}{x}')dR_2', \quad \underset{\sim}{x} \in R_1, \tag{4.13}$$

where the components of Green's function $G_{ij}(\underset{\sim}{x},\underset{\sim}{x}')$ are given by (2.4). Various integrals in this relation can be evaluated in the following way.

$$\int_{R_2} G_{j1,1}(\underset{\sim}{x},\underset{\sim}{x}')dR_2' = \frac{1}{8\pi}\{\frac{2}{\mu_1}\delta_{ij}\frac{\partial}{\partial x_1}\int_{R_2}\frac{dR_2'}{|\underset{\sim}{x}-\underset{\sim}{x}'|}$$

$$+ (\frac{1}{M_1}-\frac{1}{\mu_1})\frac{\partial^2}{\partial x_1^2}(x_j\int_{R_2}\frac{dR_2'}{|\underset{\sim}{x}-\underset{\sim}{x}'|}) - (\frac{1}{M_1}-\frac{1}{\mu_1})\frac{\partial^2}{\partial x_1^2}\int_{R_2}\frac{x_j'dR_2'}{|x-x'|}$$

$$= \frac{1}{2}\{\frac{2}{\mu_1}\delta_{j1}\frac{\partial}{\partial x_1}(\phi(\underset{\sim}{x})) + (\frac{1}{M_1}-\frac{1}{\mu_1})\frac{\partial^2}{\partial x_1^2}(x_j\phi(\underset{\sim}{x}) - \phi_j(\underset{\sim}{x})), \tag{4.14}$$

where $\phi(\underset{\sim}{x})$ is the Newtonian potential due to the solid ellipsoid of unit density occupying region $R_2 : x^2/a^2 + y^2/b^2 + z^2/c^2 < 1$, at the point $\underset{\sim}{x} \in R_1$ and $\phi_j(x)$ is the Newtonian potential due to a solid ellipsoid of variable density x_j occupying the region R_2 at the point $\underset{\sim}{x} \in R_1$, that is

$$\phi(\underset{\sim}{x}) = \frac{1}{4\pi}\int_{R_2}\frac{dR_2'}{|x-x'|} = \frac{V}{4}\int_\xi^\infty \frac{du}{R_u}(1-\sum_{k=1}^{3}\frac{x_k^2}{(u+a_k^2)}), \quad \underset{\sim}{x} \in R_1, \tag{4.15}$$

$$\phi_j(\underset{\sim}{x}) = \frac{1}{4\pi}\int\frac{x_j'dR_2'}{|\underset{\sim}{x}-\underset{\sim}{x}'|} = \frac{V}{4}a_j^2 x_j\int_\xi^\infty\{1-\sum_{k=1}^{3}\frac{x_k^2}{u+a_k^2}\}\frac{du}{(u+a_j^2)R_u}, \quad \underset{\sim}{x} \in R_1,\ j = 1,2,3, \tag{4.16}$$

$$V = a_1a_2a_3 = abc, \quad R_u = \{(u+a_1^2)(u+a_2^2)(u+a_3^2)\}^{1/2},$$

and ξ is the positive root of

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1, \quad \underset{\sim}{x} \in R_1 (\xi>0).$$

Similarly, other integrals occuring in the right hand side of equation (4.13) can be evaluated. Substituting these values of the integrals in (4.13) we obtain

$$u_j(\underset{\sim}{x}) = u_j^0(\underset{\sim}{x}) + \frac{\Delta\lambda}{M_1}\{u_{11}(\underset{\sim}{Q}) + u_{22}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})\}\frac{\partial}{\partial x_j}(\phi(\underset{\sim}{x}))$$

$$+\Delta\mu\{\frac{2}{\mu_1}u_{jj}(\underset{\sim}{Q})\frac{\partial}{\partial x_j}(\phi(\underset{\sim}{x}))$$

$$+(\frac{1}{M_1}-\frac{1}{\mu_1})(u_{11}(\underset{\sim}{Q})\frac{\partial^2}{\partial x_1^2} + u_{22}(\underset{\sim}{Q})\frac{\partial^2}{\partial x_2^2} + u_{33}(\underset{\sim}{Q})\frac{\partial^2}{\partial x_3^2})(x_j\phi(\underset{\sim}{x})-\phi_j(\underset{\sim}{x}))\}, \tag{4.17}$$

where $j = 1,2,3$ and $\underset{\sim}{x} \in R_1$ and j is not summed. All the terms on the right side are known. Indeed, the first term is given by (4.1), the functions $\phi(\underset{\sim}{x})$ and $\phi_j(\underset{\sim}{x})$ are known from (4.15) and (4.16) while the quantities $u_{ii}(\underset{\sim}{Q})$, $i = 1,2,3$ are expressed in relations (4.5) and (4.6). Let us check this formula by considering the limiting case of a spherical inclusion.

Let $b \to a$, $c \to a$ in relations (4.14) to (4.17) so that

$$\phi(\underset{\sim}{x}) = \frac{a^3}{4}\int_{r^2-a^2}^{\infty}\frac{du}{(u+a^2)^{3/2}}(1-\frac{r^2}{u+a^2}) = \frac{a^3}{3r},\quad r = |x| > a, \tag{4.18}$$

$$\phi_j(\underset{\sim}{x}) = \frac{a^5}{4}x_j\int_{r^2-a^2}^{\infty}(1-\frac{r^2}{(u+a)^2})\frac{du}{(u+a^2)^{5/2}} = \frac{a^5x_j}{15r^3},\quad r = |\underset{\sim}{x}| > a, \tag{4.19}$$

where we have used the fact that $a^2 + \xi = r^2$. Substituting these values and the values of $u_{ii}(\underset{\sim}{Q})$ from (4.8) (for the sphere) in (4.17) we get the required exterior solution for the spherical enclosure, namely

$$u_j(\underset{\sim}{x}) = u_j^0(\underset{\sim}{x}) + \frac{\Delta\lambda}{M_1}\{2u_{11}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q})\}\frac{\partial}{\partial x_j}(\frac{a^3}{3r}) + \Delta\mu\{\frac{2}{\mu_1}u_{jj}(\underset{\sim}{Q})\frac{\partial}{\partial x_j}(\frac{a^3}{3r})$$

$$+(\frac{1}{M_1}-\frac{1}{\mu_1})[(u_{11}(\underset{\sim}{Q}))(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}) + u_{33}(\underset{\sim}{Q})\frac{\partial^2}{\partial x_3^2}][x_j(\frac{a^3}{3r}-\frac{a^5}{15r^3})]\},$$

$$|\underset{\sim}{x}| = r > a,\ j = 1,2,3. \tag{4.20}$$

Setting $u_j(\underset{\sim}{x}) = u_j^0(\underset{\sim}{x}) + u_j^s(\underset{\sim}{x})$, $|\underset{\sim}{x}| = r > a$, and writing $u_j^s(\underset{\sim}{x})$ in spherical polar coordinates we obtain

$$u_r^s(\underset{\sim}{x}) = -[\frac{\bar{A}}{r^2}+\frac{3\bar{B}}{r^4}] + [(\frac{5-4\sigma_1}{1-2\sigma_1})\frac{\bar{C}}{r^2}-\frac{9\bar{B}}{r^4}]\cos 2\theta, \tag{4.21a}$$

$$u_\theta^s(\underset{\sim}{x}) = -[2\frac{\bar{C}}{r^2}+\frac{6\bar{B}}{r^4}]\sin 2\theta, \tag{4.21b}$$

where

$$\frac{\bar{A}}{a^3} = -\frac{5T}{24\mu_1}\{\frac{(\mu_1-\mu_2)(5-4\sigma_1)}{(7-5\sigma_1)\mu_1+(8-10\sigma_1)\mu_2}\}$$

$$-\frac{T}{6\mu_1}\{\frac{\mu_1(1-2\sigma_2)-\mu_2\frac{(1+\sigma_2)}{(1+\sigma_1)}(1-2\sigma_1)}{2\mu_1(1-2\sigma_2)+\mu_2(1+\sigma_2)}\}, \tag{4.22a}$$

$$\frac{\bar{B}}{a^5} = \frac{T}{8\mu_1}\{\frac{(\mu_1-\mu_2)}{(7-5\sigma_1)\mu_1+(8-10\sigma_1)\mu_2}\}, \tag{4.22b}$$

$$\frac{\overline{C}}{a^3} = \frac{T}{8\mu_1}\left\{\frac{5(1-2\sigma_1)(\mu_1-\mu_2)}{(7-5\sigma_1)\mu_1+(8-10\sigma_1)\mu_2}\right\} . \tag{4.22c}$$

The corresponding stress components are

$$\tau^s_{rr}(\underset{\sim}{x}) = \lambda_1\Delta + 2\mu_1\left\{\left[\frac{2\overline{A}}{r^3}+\frac{12\overline{B}}{r^5}\right] + \left[-2\left(\frac{5-4\sigma_1}{1-2\sigma_1}\right)\frac{\overline{C}}{r^3}+\frac{36\overline{B}}{r^5}\right]\cos 2\theta\right\}, \tag{4.23a}$$

$$\tau^s_{r\theta}(\underset{\sim}{x}) = \mu_1\left\{-\frac{4(1+\sigma_1)\overline{C}}{(1-2\sigma_1)r^3}+\frac{48\overline{B}}{r^5}\right\}\sin 2\theta, \tag{4.23b}$$

$$\tau^s_{\theta\theta}(x) = \lambda_1\Delta + 2\mu_1\left\{-\left[\frac{\overline{A}}{r^3}+\frac{3\overline{B}}{r^5}\right] + \left[\left(\frac{1+4\sigma_1}{1-2\sigma_1}\right)\frac{\overline{C}}{r^3}-\frac{21\overline{B}}{r^5}\right]\cos 2\theta\right\}, \tag{4.23c}$$

$$\tau^s_{\varphi\varphi}(\underset{\sim}{x}) = \lambda_1\Delta + 2\mu_1\left\{-\left(\frac{\overline{A}}{r^3}+\frac{2\overline{C}}{r^3}+\frac{9\overline{B}}{r^5}\right] + 3\left[\frac{\overline{C}}{(1-2\sigma_1)r^3}-\frac{5\overline{B}}{r^5}\right]\cos 2\theta\right\}, \tag{4.23d}$$

where

$$\Delta = -\left(\frac{2\overline{C}}{r^3}\right)[1+3\cos 2\theta].$$

Spherical Void. When λ_2 and $\mu_2 \to 0$ in the above relations we obtain the corresponding interior and exterior solutions for a spherical void of radius a. For example, the components of the stress tensor at the outer surface of the void are

$$\tau^s_{rr}(a,\theta,\varphi) = -\frac{T}{2}(1+\cos 2\theta),\ \tau^s_{r\theta}(a,\theta,\varphi) = \frac{T}{2}\sin 2\theta,$$

$$\tau^s_{\theta\theta}(a,\theta,\varphi) = \frac{T}{2(7-5\sigma_1)}\{5(1-2\sigma_1) - (8+5\sigma_1)\cos 2\theta\},$$

$$\tau^s_{\varphi\varphi}(a,\theta,\varphi) = -\frac{3T}{2(7-5\sigma_1)}\{1+5\sigma_1\cos 2\theta\}.$$

Relations (4.23) agree with the known results and serve as a check on our formulas.

Finally, we present the outer solutions for the limiting case of the prolate spheroid where semi-axes are $a,b,b,\ a \ge b$, i.e.,

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2} < 1,\quad b^2 = a^2(1-e^2).$$

In this case relation (4.15) reduces to

$$\phi(\underset{\sim}{x}) = \frac{ab^2}{4}\int_\xi^\infty \frac{du}{\sqrt{u+a^2}\,(u+b^2)}\left\{1-\frac{x^2}{u+a^2}-\frac{y^2+z^2}{u+b^2}\right\} = \frac{ab^2}{4}\left[\frac{2}{ae}\tanh^{-1}\left(\sqrt{\frac{a^2-b^2}{\xi+a^2}}\right)\right.$$

$$-\frac{2x^2}{a^3e^3}\left\{\tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}}-\sqrt{\frac{a^2-b^2}{\xi+a^2}}\right\}$$

$$\left.-\frac{y^2+z^2}{a^3e^3}\left\{\frac{ae\sqrt{\xi+a^2}}{\xi+b^2}-\tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}}\right\}\right],$$

where $\underset{\sim}{x} \in R_1$, i.e., $\xi > 0$.

Similarly, relation (4.16) becomes

$$\phi_1(\underset{\sim}{x}) = \frac{a^3b^2}{4}\, x\Big[\frac{2}{a^3e^3}\Big\{\tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}} - \sqrt{\frac{a^2-b^2}{\xi+a^2}}\Big\}$$

$$-\frac{2x^2}{a^5e^5}\Big\{\tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}} - \sqrt{\frac{a^2-b^2}{\xi+a^2}} - \frac{1}{3}\left(\sqrt{\frac{a^2-b^2}{\xi+a^2}}\right)^3\Big\}$$

$$-\frac{2(y^2+z^2)}{a^5e^5}\Big\{\frac{1}{2}\,\frac{ae\sqrt{\xi+a^2}}{(\xi+b^2)} + \frac{\sqrt{a^2-b^2}}{5+a^2} - \frac{3}{2}\tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}}\Big\}\Big]$$

$$\phi_j(\underset{\sim}{x}) = \frac{ab^4}{4}\, x_j\Big[\frac{1}{a^3e^3}\Big\{\frac{ae\sqrt{\xi+a^2}}{(\xi+b^2)} - \tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}}\Big\}$$

$$-\frac{2x^2}{a^5e^5}\Big\{\frac{1}{2}\,\frac{ae\sqrt{\xi+a^2}}{\xi+b^2} + \sqrt{\frac{a^2-b^2}{\xi+a^2}} - \frac{3}{2}\tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}}\Big\}$$

$$-\frac{2(y^2+z^2)}{a^5e^5}\Big\{\frac{1}{4}\,\frac{ae(\sqrt{\xi+a^2})^3}{(\xi+b^2)^2} - \frac{5}{8}\,\frac{ae\sqrt{\xi+a^2}}{(\xi+b^2)} + \frac{3}{8}\tanh^{-1}\sqrt{\frac{a^2-b^2}{\xi+a^2}}\Big\}\Big],\quad \underset{\sim}{x} \in R_1(\xi>0),\ j=2,3.$$

Substituting these values of $\phi(\underset{\sim}{x})$ and $\phi_j(\underset{\sim}{x})$ $j = 1,2,3$ in equations (4.17) and using the limiting values of $u_{ii}(\underset{\sim}{0})$ from the inner solution for this limiting configuration, we readily derive the exact exterior solution for the prolate spheroid.

All the other limiting configurations can be handled in the same way.

5. ARBITRARY SYMMETRICAL CAVITY AND STRAIN ENERGY

By a symmetrical cavity we mean a cavity which is symmetrical with respect to three coordinate axes. Observe that this is also true for a symmetrical inclusion for which the method of finding the interior solution is given in Section 2.

Interior solutions in the case of an arbitrary symmetrical cavity embedded in an infinite elastic medium are obtained in terms of the shape factors of the inclusion by setting $\lambda_2 = 0$, $\mu_2 = 0$, in the analysis of Section 2. This interior solution yields the values of the displacement field at the outer surface of the inclusion. Indeed, due to the continuity of the displacement field across S we have

$$\underset{\sim}{u}^0(\underset{\sim}{x}_S) + \underset{\sim}{u}^s(\underset{\sim}{x}_S)|_+ = \underset{\sim}{u}(\underset{\sim}{x}_S)|_+ = \underset{\sim}{u}(\underset{\sim}{x}_S)|_-.$$

Thus

$$u_i^s(\underset{\sim}{x}_S)|_+ = u_i^s(\underset{\sim}{x})_S|_-, \tag{5.1}$$

where the superscript s implies the perturbed field. Since the inclusion is a cavity, the stress field vanishes inside S and due to the continuity of the tractions across S, we have

$$\tau_{ni}(\underset{\sim}{x}_S)|_+ = 0, \text{ or, } \tau^0_{ni}(\underset{\sim}{x}_S) + \tau^s_{ni}(\underset{\sim}{x}_S)|_+ = 0,$$

so that

$$\tau^s_{ni}(\underset{\sim}{x}_S)|_+ = -\tau^0_{ni}(\underset{\sim}{x}_S). \tag{5.2}$$

Thus from the interior solution derived by us, we can find the components of the displacement field $u_i^s(\underset{\sim}{x}_S)|_+$ by using formula (5.1). Formula (5.2) gives the values of the perturbation in the tractions across the outer surface S of the cavity in terms of the known values of $\tau_{ni}^0(\underset{\sim}{x}_S)$ due to the prescribed stresses to which the host medium is subjected.

The elastic energy E stored in the host medium due to the presence of the symmetrical cavity is given by the formula

$$E = \frac{1}{2} \int_S u_i^s(\underset{\sim}{x}_S)|_+ \, \tau_{ni}^s(\underset{\sim}{x}_S)|_+ dS. \tag{5.3}$$

Note that in the above formula we have dropped the second integral taken over the sphere of infinite radius because it vanishes when we appeal to the far-field behavior.

$$u_i^s(\underset{\sim}{x}) = 0(\frac{1}{r^2}), \quad \tau_{ij}^s(\underset{\sim}{x}) = 0(\frac{1}{r^3}) \quad \text{as} \quad r \to \infty,$$

of the displacement and the traction fields.

Let us illustrate fromula (5.3) for the spherical cavity embedded in the infinite host medium so that the region R_2 is $r < a$. For this purpose we assume that the prescribed stress field is such that we have the uniform tension T in the directions of x,y,z axes before the creation of the cavity. In this case the components of the displacement field are

$$u_i^0(\underset{\sim}{x}) = \frac{T}{2\mu_1} (\frac{1-2\sigma_1}{1+\sigma_1}) x_i, \quad \underset{\sim}{x} \in R, \tag{5.4a}$$

or

$$u_r^0(\underset{\sim}{x}) = \frac{T}{2\mu_i} (\frac{1-2\sigma_1}{1+\sigma_1}) r, \; u_\theta^0(\underset{\sim}{x}) = u_\varphi^0(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x} \in R. \tag{5.4b}$$

The corresponding non-vanishing components of the stress tensor $\tau_{ij}^0(\underset{\sim}{x})$ are

$$\tau_{11}^0(\underset{\sim}{x}) = \tau_{22}^0(\underset{\sim}{x}) = \tau_{33}^0(\underset{\sim}{x}) = T, \; \underset{\sim}{x} \in R, \tag{5.5a}$$

or

$$\tau_{rr}^0 = T, \quad \tau_{\theta\theta}^0 = \tau_{\varphi\varphi}^0 = \frac{3T\sigma_1}{1+\sigma_1}, \quad \underset{\sim}{x} \in R. \tag{5.5b}$$

Accordingly, in this case

$$u_{11}^0(\underset{\sim}{0}) = u_{22}^0(\underset{\sim}{0}) = u_{33}^0(\underset{\sim}{0}) = \frac{T}{2\mu_1} (\frac{1-2\sigma_1}{1+\sigma_1}),$$

$$u_{ij}^0(\underset{\sim}{0}) = a_{ij}^0 = 0, \quad i \neq j. \tag{5.6}$$

Substituting these values in relations (2.16), we get

$$u_{11}(\underset{\sim}{0}) = u_{22}(\underset{\sim}{0}) = u_{33}(\underset{\sim}{0}) = \frac{3T}{4\mu_1} (\frac{1-\sigma_1}{1+\sigma_1}), \; u_{ij}(\underset{\sim}{0}) = a_{ij}(\underset{\sim}{0}) = 0, \quad i \neq j \tag{5.7}$$

which yield the required exact interior solution

$$u_i(\underset{\sim}{x}) = \frac{3T}{4\mu_1}\left(\frac{1-\sigma_1}{1+\sigma_1}\right)x_i, \quad r < a, \tag{5.8a}$$

$$\tau_{ij}(\underset{\sim}{x}) = 0, \quad r < a, \tag{5.8b}$$

where we have used the fact that the region R_2 is void so that $\lambda_2 = \mu_2 = 0$. Hence, from relations (5.1), (5.2), (5.4a) and (5.5a) it follows that

$$\underset{\sim}{u}^S(\underset{\sim}{x}_S)|_+ = \frac{T}{4\mu_1}\underset{\sim}{x}_S, \quad |\underset{\sim}{x}_S| = a, \tag{5.9a}$$

$$\tau^S_{ni}(\underset{\sim}{x}_S)|_+ = -\tau^0_{ni}(\underset{\sim}{x}_S) = -\tau^0_{ij}(\underset{\sim}{x}_S)nj = -Tn_i(\underset{\sim}{x}_S), \quad |\underset{\sim}{x}_S| = a, \tag{5.9b}$$

where n_i are the components of the unit normal $\hat{\underset{\sim}{n}}(x_S)$ directed outwards at the point $\underset{\sim}{x}_S$ of S.

Finally, we substitute the above values in formula (5.3) and get the required value of the stored energy E as

$$E = \frac{T}{2}\int_{r=a}\underset{\sim}{u}^S(\underset{\sim}{x}_S)|_+ \cdot \hat{\underset{\sim}{n}}(\underset{\sim}{x}_S)dS = \frac{T}{2}\int_{r=a}\frac{Ta\underset{\sim}{n}(\underset{\sim}{x}_S)}{4\mu_1}\cdot\hat{\underset{\sim}{n}}(\underset{\sim}{x}_S)dS = \frac{\pi T^2a^3}{2\mu_1}.$$

6. ANALYSIS OF VISCOUS INHOMOGENEITY

The analysis of the displacement fields in elastic composite media can be applied to solve the problem of the slow deformation of an incompressible homogeneous viscous fluid ellipsoidal inhomogeneity embedded in an infinite homogeneous viscous fluid of different viscosity which is subjected to a devitorial constant pure strain rate whose principal axes are parallel to those of the ellipsoidal inclusion. This problem is of interest in the theory of the deformation of rocks and in the theory of mixing and homogenization of viscous fluids [5].

Let an infinite region R be filled with an incompressible homogeneous fluid of viscosity μ_1 and be subjected to devitorial uniform pure strain rate $\underset{\sim}{e}^0(\underset{\sim}{x})$, $\underset{\sim}{x} \in R$ with non-zero components:

$$e^0_1(\underset{\sim}{x}) = U, \quad e^0_{22}(\underset{\sim}{x}) = -\frac{U}{2}, \quad e^0_{33}(\underset{\sim}{x}) = -\frac{U}{2}, \quad \underset{\sim}{x} \in R, \tag{6.1a}$$

where U is positive constant so that the corresponding velocity components are

$$u^0_1(\underset{\sim}{x}) = Ux, \quad u^0_2(\underset{\sim}{x}) = -\frac{U}{2}y, \quad u^0_3(\underset{\sim}{x}) = -\frac{U}{2}z, \quad \text{div}\,\underset{\sim}{u}^{(0)}(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x} \in R. \tag{6.1b}$$

Then at time $t = 0$, let an ellipsoidal homogeneous viscous incompressible fluid of viscosity μ_2 which occupies the region R_0: $x^2/a_0^2 + y^2/b_0^2 + z^2/c_0^2 < 1$, $a_0 > b_0 > c_0$, be embedded in the infinite host medium which is subjected to the devitorial uniform pure strain rate $\underset{\sim}{e}^0(\underset{\sim}{x})$ as described in (6.1) so that the principal axis of $\underset{\sim}{e}^0(\underset{\sim}{x})$ are parallel to those of the ellipsoidal inclusion. Due to this uniform pure strain rate the ellipsoidal inclusion gets deformed to an ellipsoid at each subsequent instant. Let, at time t, the inclusion occupy the region R_2: $x^2/a^2 + y^2/b^2 + z^2/c^2 < 1$, $a > b > c$, where a,b,c are functions of time. Thus, $(4\pi/3)a_0b_0c_0 = (4\pi/3)abc$, i.e., $abc = a_0b_0c_0$.

The inner solution $\underset{\sim}{u}(\underset{\sim}{x})$, $\underset{\sim}{x} \in R_2$ at instant t is linear in x,y,z and is readily obtained from the analysis of the corresponding elastostatic problem of composite media by taking appropriate limits. The quantity $\underset{\sim}{u}(\underset{\sim}{x})$, which is displacement vector in the previous analysis, now represents velocity field in region R_1 and R_2. In both these regions we have to satisfy the equation of continuity

$$\operatorname{div} \underset{\sim}{u}(\underset{\sim}{x}) = 0, \quad \underset{\sim}{x} \in R_2 \quad \text{or} \quad R_1.$$

Secondly, while the tensor $e_{ij}(\underset{\sim}{x}) = (1/2)(u_{i,j}(\underset{\sim}{x})+u_{j,i}(\underset{\sim}{x}))$ is the strain tensor, it denotes the pure strain rate in the present case. With these changes in the notation understood, we derive our results in the present case when the guest medium is deformed to the ellipsoid occupying R_2: $x^2/a^2 + y^2/b^2 + z^2/c^2 < 1$ at time t by taking the appropriate limits in the analysis of Section 2:

$$\lambda_1 \to \infty, \quad \lambda_2 \to \infty, \quad \operatorname{div} \underset{\sim}{u}(\underset{\sim}{x}) \to 0, \quad \underset{\sim}{x} \in R_1 \quad \text{and} \quad R_2, \tag{6.2a}$$

such that the hydrostatic pressure $p(x)$:

$$p(\underset{\sim}{x}) = \begin{cases} -\lambda_1 \operatorname{div} \underset{\sim}{u}(\underset{\sim}{x}), & \underset{\sim}{x} \in R_1, \\ -\lambda_2 \operatorname{div} \underset{\sim}{u}(\underset{\sim}{x}), & \underset{\sim}{x} \in R_2, \end{cases} \tag{6.2b}$$

is finite. In view of relations (6.1) we have

$$u^0_{11}(\underset{\sim}{Q}) = U, \quad u^0_{22}(\underset{\sim}{Q}) = -\frac{U}{2}, \quad u^0_{33}(\underset{\sim}{Q}) = -\frac{U}{2}, \tag{6.3a}$$

so that $\operatorname{div} \underset{\sim}{u}^0(\underset{\sim}{x}) = 0$. Also

$$u^0_{ij}(\underset{\sim}{Q}) = 0, \quad i \neq j, \quad a^0_{ij}(\underset{\sim}{Q}) = 0, \quad \text{for all} \quad i,j. \tag{6.3b}$$

Let us note from our elastostatic analysis that, since the inner solution $\underset{\sim}{u}(\underset{\sim}{x})$, $\underset{\sim}{x} \in R_2$ is linear in x,y,z, we have $\operatorname{div} \underset{\sim}{u}(\underset{\sim}{x}) = \sum_{k=1}^{3} u_{kk}(\underset{\sim}{Q})$, $\underset{\sim}{x} \in R_2$.

Now, we take the limits as explained in (6.2) above in the relations (2.13) and (2.14) of elastostatics and get

$$\{1-\frac{2\Delta\mu}{\mu_1}(t_{1122}+t_{1133})\}u_{11}(\underset{\sim}{Q}) + \frac{2\Delta\mu}{\mu_1} t_{1122}u_{22}(\underset{\sim}{Q}) + \frac{2\Delta\mu}{\mu_1} t_{1133}u_{33}(0) = U, \tag{6.4a}$$

$$\frac{2\Delta\mu}{\mu_1} t_{2211}u_{11}(\underset{\sim}{Q}) + \{1 - \frac{2\Delta\mu}{\mu_1}(t_{2233}+t_{2211})\}u_{22}(\underset{\sim}{Q}) + \frac{2\Delta\mu}{\mu_1} t_{2233}u_{33}(\underset{\sim}{Q}) = -\frac{U}{2}, \tag{6.4b}$$

$$\frac{2\Delta\mu}{\mu_1} t_{3311}u_{11}(\underset{\sim}{Q}) + \frac{2\Delta\mu}{\mu_1} t_{3322}u_{22}(\underset{\sim}{Q}) + \{1 - \frac{2\Delta\mu}{\mu_1}(t_{3311}+t_{3322})\}u_{33}(\underset{\sim}{Q}) = -\frac{U}{2}. \tag{6.4c}$$

Also

$$u_{ij}(\underset{\sim}{Q}) = a_{ij}(\underset{\sim}{Q}) = 0, \quad i \neq j, \quad i,j = 1,2,3, \tag{6.4d}$$

where we have used relation (6.3b) and the quantities t_{iijj}, $i \neq j$, are the shape factors of the ellipsoid occupying the region R_2 and their values are given by (3.5b), namely

$$t_{iijj} = -\frac{abc}{4}\int_0^\infty \frac{udu}{(u+a_i^2)(u+a_j^2)R_u}, \quad i \neq j, \quad i,j = 1,2,3, \tag{6.5}$$

where $a_1 = a$, $a_2 = b$, $a_3 = c$ and $R_n = \{(u+a^2)(u+b^2)(u+c^2)\}^{1/2}$. Adding (6.4a), (6.4b) and (6.4c) we find that $u_{11}(\underset{\sim}{Q}) + u_{22}(\underset{\sim}{Q}) + u_{33}(\underset{\sim}{Q}) = 0$, i.e.

$$\operatorname{div} \underset{\sim}{u}(x) = 0, \quad \underset{\sim}{x} \in R_2,$$

so that the equation of continuity is satisfied.

Solving equations (6.4) simultaneously, we obtain

$$u_{11}(\underset{\sim}{Q}) = \frac{U}{\mathcal{D}}\{1-\frac{\Delta\mu}{\mu_1}(t_{1122}+t_{1133}+4t_{2233})\}, \tag{6.6a}$$

$$u_{22}(\underset{\sim}{Q}) = \frac{U}{2\mathcal{D}}\{1-\frac{2\Delta\mu}{\mu_1}(2t_{1133}-t_{1122}+2t_{2233})\}, \tag{6.6b}$$

$$u_{33}(\underset{\sim}{Q}) = -\frac{U}{2\mathcal{D}}\{1-\frac{\Delta\mu}{\mu_1}(3t_{1122}-t_{1133}+4t_{2233})\}, \tag{6.6c}$$

where

$$\mathcal{D} = \{1-\frac{2\Delta\mu}{\mu_1}(t_{2233}+2t_{2211})\}\{1-\frac{2\Delta\mu}{\mu_1}(2t_{3311}+t_{3322})\}$$
$$-\frac{4(\Delta\mu)^2}{\mu_1^2}(t_{2233}-t_{2211})(t_{3322}-t_{3311}), \tag{6.6d}$$

The innner solution at time t is

$$u_1(\underset{\sim}{x}) = u_{11}(\underset{\sim}{Q})x, \quad u_2(\underset{\sim}{x}) = u_{22}(\underset{\sim}{Q})y, \quad u_3(\underset{\sim}{x}) - u_{33}(\underset{\sim}{Q})z, \quad \underset{\sim}{x} \in R_2,$$

where $u_{11}(\underset{\sim}{Q})$ and $u_{22}(\underset{\sim}{Q})$ and $u_{33}(\underset{\sim}{Q})$ are given by (6.6).

Two Important Limiting Cases. Case I. Let $c_0 = b_0 = a_0$, i.e., at time $t = 0$, so that the guest medium consists of a spherical viscous incompressible fluid of viscosity μ_2 occupying the spherical region $\Sigma x^2 < a_0^2$ which is embedded in the infinite host medium of viscous incompressible fluid of viscosity μ_1. This host host medium is subjected to devitorial constant pure strain rate $e_{ij}^0(\underset{\sim}{x})$ whose non-zero components are

$$e_{11}^0(\underset{\sim}{x}) = -2e_{22}^0(\underset{\sim}{x}) = -2e_{33}^0(\underset{\sim}{x}) = U > 0.$$

In this particular case, the spherical inclusion gets deformed to prolate spheroid and at time t occupies the region R_2:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2} < 1, \qquad a > b, \qquad ab^2 = a_j^3 \tag{6.7}$$

Accordingly, we can derive the values of the distinct non-zero shape factors from (6.5) by selling $c = b$, and they are given by (3.6), substituting these values in (6.6) we have, in this case,

$$u_{11}(\underset{\sim}{Q}) = -2u_{22}(\underset{\sim}{Q}) = -2u_{33}(\underset{\sim}{Q}) = \frac{U}{\{1- \frac{6\Delta\mu}{\mu_1} t_{1122}\}}$$

$$= \frac{U}{\{1 - 6\alpha[\frac{1}{12} (1-k^2) - \frac{1}{4} (3-k^2)L_1\}}, \tag{6.8}$$

where $\alpha = (\Delta\mu)/\mu_1 = (\mu_2-\mu_1)/\mu_1$.

Finally, to obtain the values of a and b, which are functions of time t; we appeal to the partial differential equation

$$\frac{DE}{Dt} = 0, \tag{6.9}$$

satisfied by the moving surface

$$F(x,y,z,t) \equiv \frac{x^2}{a^2} + \frac{b^2+z^2}{b^2} - 1 = 0,$$

at time t, where $ab^2 = a_0^3$. Thus

$$\frac{1}{a}\frac{da}{dt} = u_{11}^0(\underset{\sim}{Q}), \tag{6.10}$$

where the constant $u_{11}^0(\underset{\sim}{Q})$ is given by equation (6.8) which when substituted in (6.10) yields the following differential equation for w, defined as $w^2 = a^3/a_0^3$,

$$\frac{2}{3w}\frac{dw}{dt} = \frac{U}{\{1-a\alpha[\frac{1}{12w^2} - \frac{1}{4} (2+1/w^2) \frac{1/w^2}{(1-1/w^2)^2} (\frac{1}{\sqrt{1-1/w^2}} \ell n(w+\sqrt{w^2-1}) - \frac{4w^2+1}{3w^2})]}, \tag{6.11}$$

where we have used the values of L_1 and k^2 as given by (3.7). This differential equation is readily solved by the method of separation of variables and we have

$$\alpha[\frac{1}{w^2-1} - \frac{w \cosh^{-1}w}{(w^2-1)^{3/2}}] + \frac{2}{3} \log w = U++A, \tag{6.12}$$

where A is the constant of integration. To find this constant, we use the initial condition that as $t \to 0$, $w \to 1$. Thus

$$A = - \frac{1}{3} \alpha , \tag{6.13}$$

and (6.12) becomes

$$\alpha[\frac{1}{3} + \frac{1}{w^2-1} - \frac{w \cosh^{-1}w}{(w^2-1)^{3/2}}] = S_H - S, \tag{6.14}$$

where $S = \log(a/a_0)$, is the natural strain of the inhomogeneity and $S_H = e_{11}^0 \times t = Ut$, is the natural strain applied at infinity. Relation (6.14) agrees with the known result [5] and gives $w = (a/a_0)^{3/r}$ in term of time t and expresses the required value of a in terms of t. Substituting this value of a in the relation $ab^2 = a_0^3$, we obtain the value of b in terms of t.

Case II. Let us now consider a two-dimensional limit. Letting $a_0 \to \infty$, $c_0 = b_0$, i.e. at time $t = 0$, the guest medium consists of an infinite circular cylinder of an viscous incompressible fluid of viscosity μ_2 occupying the region $y^2+z^2 < b_0^2$, $|x| < \infty$ embedded in the infinite host medium of viscous incompressible fluid of viscosity μ_1 which is subjected to devitorial uniform pure strain rate $e^0_{ij}(\underset{\sim}{x})$. Its non-zero components

$$e^0_{22}(\underset{\sim}{x}) = -e^0_{33}(\underset{\sim}{x}) = U,$$

where U is a positive constant. In this case, the right circular cylindrical inclusion gets deformed to an infinite elliptic cylinder and at time t occupies the region R_2,

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} < 1, \quad |x| < \infty, \quad b > c,$$

where $\pi bc = \pi b_0^2$ or $bc = b_0^2$. The non-zero distinct shape factors in this case are derived from (6.5) by letting $a \to \infty$ and the values are

$$t_{2222} = -\frac{c(b+2c)}{2(b+c)^2}, \quad t_{3333} = -\frac{b(c+2b)}{2(b+c)^2}, \quad t_{2233} = -\frac{bc}{2(b+c)^2}. \tag{6.15}$$

In this case, the exact inner solution at time t is

$$u_1(\underset{\sim}{x}) = 0, \quad u_2(\underset{\sim}{x}) = u_{22}(\underset{\sim}{Q})y, \quad u_3(\underset{\sim}{x}) = u_{33}(\underset{\sim}{Q})z, \quad \underset{\sim}{x} \in R_2, \tag{6.16}$$

where $u_{22}(\underset{\sim}{Q})$ and $u_{33}(\underset{\sim}{Q})$ satisfy equations (6.4b) and (6.4c) which, in view of (6.15), become

$$\{1 + \frac{\Delta\mu bc}{\mu_1(b+c)^2}\} u_{22}(\underset{\sim}{Q}) - \frac{\Delta\mu}{\mu_1} \frac{bc}{(b+c)^2} u_{33}(\underset{\sim}{Q}) = U, \tag{6.17a}$$

$$-\frac{\Delta\mu bc}{\mu_1(b+c)^2} u_{22}(\underset{\sim}{Q}) + \{1 + \frac{\Delta\mu bc}{\mu_1(b+c)^2}\} u_{33}(\underset{\sim}{Q}) = -U. \tag{6.17b}$$

These equations yield

$$u_{22}(\underset{\sim}{Q}) = -u_{33}(\underset{\sim}{Q}) = \frac{U}{\{1 + \frac{2\Delta\mu bc}{\mu_1(b+c)^2}\}}. \tag{6.18}$$

Substituting these values in (6.16), we obtain the required inner solution at time t.

To find the values of b and c in terms of t, we appeal to the partial differential equation $DF/Dt = 0$, where

$$F(x,y,z) = \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0,$$

and get

$$\frac{1}{b}\frac{db}{dt} = u_{22}(\underset{\sim}{Q}) = \frac{U}{\{1 + \frac{2\alpha bc}{(b+c)^2}\}}, \tag{6.19}$$

where $\alpha = (\Delta\mu)/\mu_1 = (\mu_2-\mu_1)/\mu_1$. Since $bc = b_0^2$, the above relation becomes

$$\frac{1}{b}\frac{db}{dt} = \frac{U}{\{1 + \frac{2\alpha b^2 b_0^2}{(b^2+b_0^2)^2}\}} .$$

Its solution is

$$\ell n\, b - \frac{\alpha b_0^2}{b^2+b_0^2} = Ut + B, \tag{6.20}$$

where B is the constant of integration. Since, when $t \to 0$, $b \to b_0$, we find (6.20) that

$$B = \log b_0 - \frac{\alpha}{2} ,$$

so that

$$\log \frac{b}{b_0} + \frac{\alpha}{2} [\frac{(b/b_0)^2-1}{(b/b_0)^2+1}] = Ut,$$

or

$$S + \frac{\alpha}{2} \tanh S = S_H, \tag{6.21}$$

where $S = \log b/b_0$ is the natural strain of elliptical inhomogeneity and $S_H = e_{22}^0 t = Ut$ is the natural strain applied at infinity. Relation (6.21) agrees with the known result [5]. It gives b in terms of t and using the relation $bc = b_0^2$ we can determine c in terms of t.

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