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RINGS DECOMPOSED INTO DIRECT SUMS OF J-RINGS AND NIL RINGS

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ABSTRACT. Let R be a ring (not necessarily with identity) and let E denote the set of idempotents of R. We prove that R is a direct sum of a J-ring (every element is a power of itself) and a nil ring if and only if R is strongly π-regular and E is contained in some J-ideal of R. As a direct consequence of this result, the main theorem of [1] follows.

KEY WORDS AND PHRASES. Periodic, potent, J-ring, nil ring, strongly π-regular ring, direct sum.

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1. INTRODUCTION.

Throughout the present note, R will represent a ring (not necessarily with identity), N the set of nilpotent elements of R, and E the set of idempotents of R. We say that R is periodic if for each $r \in R$, there exist distinct positive integers h, k for which $r^h = r^k$. According to Chacron's theorem (see, e.g., [2, Theorem 1]), R is periodic if and only if for each $r \in R$, there exists a polynomial $f(\lambda)$ with integer coefficients such that $r - r^2 f(r) \in N$. An element r of R is called potent if there is an integer $n > 1$ such that $r^n = r$. We denote by I the set of potent elements of R. If R coincides with I, R is called a J-ring. As is well known, every J-ring is commutative (Jacobson's theorem). An ideal of R is called a J-ideal if it is a J-ring. Also, we denote by I_0 the set $\{r \in R \mid r \text{ generates a subring with identity}\}$. It is clear that $E \subseteq I \subseteq I_0$. Furthermore, if I_0 is a subring of R then I_0 coincides with I. In fact, if r is an arbitrary element of I_0 then there exists a polynomial $f(\lambda)$ with integer coefficients such that $r = r^2 f(r)$. This proves that I_0 is a reduced periodic ring, and therefore a J-ring. Especially, R is a J-ring if and only if $R = I_0$. If R is the direct sum of a J-ideal I' and a nil ideal N', then it is easy to see that $I' = I = I_0$ and $N' = N$.

2. MAIN THEOREM.

Now, the main theorem of this note is stated as follows:

THEOREM 1. The following conditions are equivalent:

1) R is right (or left) π-regular and E is contained in some J-ideal A of R.

2) R is periodic and E is contained in some reduced ideal A of R.

3) R is a direct sum of a J-ring and a nil ring.

More precisely, if 1) or 2) is satisfied, then N is an ideal of R, $R = A \oplus N$, and $A = I = I_0$. In particular, if R is right (or left) s-unital, that is, $r \in rR$ (or $r \in Rr$) for all $r \in R$, then each of 1), 2) is equivalent to

4) R is a J-ring.

PROOF. Obviously, 3) $\Rightarrow$ 2) $\Rightarrow$ 1).

1) $\Rightarrow$ 3). By a result of Dischinger (see, e.g., [3, Proposition 2]), R is strongly π-regular. Let r be an arbitrary element of R. Then there exists a positive integer n and elements s', s'' of R such that $r^{2n}s' = s''r^{2n} = r^n$. We put $s = r^n s'^2$. As is easily seen,

$$s = s''r^n s' = s''^2 r^n$$

and

$$r^n s' r^n = s''r^{2n} = r^n = r^{2n}s' = r^n s'' r^n.$$

Hence,

$$r^n s = r^n s'' r^n s' = r^n s' = s'' r^{2n} s' = s'' r^n = s'' r^n s' r^n = s r^n$$

and

$$r^{2n} s = r^n s r^n = r^n s' r^n = r^n.$$

Since $e = r^n s$ is an idempotent with $re = er$ ($\in A$) and $r^n e = r^n$, we see that

$$(r - re)^n = r^n(1 - e)^n = 0.$$

This together with $r = re + (r - re)$ proves that r is represented as a sum of an element in A and a nilpotent element. Now, let $a, b \in A$, and $x, y \in N$. Noting that $xa \cdot yb = xyba$ and $ax \cdot by = baxy$, we can easily see that $xa \in N \cap A = 0$ and $ax = 0$; $NA = AN = 0$. Set $xy = c + u$ and $x + y = d + v$ ($c, d \in A$ and $u, v \in N$), where we may assume that $u^\ell = v^\ell = 0$. In view of $NA = 0$, we obtain

$$(xy)^2 = xy(c + u) = xyu$$

and

$$(x + y)^2 = (x + y)(d + v) = (x + y)v,$$

and therefore

$$(xy)^{\ell+1} = xyu^\ell = 0$$

and

$$(x + y)^{\ell+1} = (x + y)v^\ell = 0.$$

We have thus seen that N forms an ideal of R and $R = A \oplus N$.

Given an integer $n > 1$, we denote by I_n the set $\{r \in R \mid r^n = r\}$. In [1], Abu-Khuzam and Yaqub proved that if R is a periodic ring with N commutative and for which I_n forms an ideal, then R is a subdirect sum of finite fields of at most n elements and a nil commutative ring. The next corollary includes this result.

COROLLARY 1. If R is periodic and I_n forms an ideal of R for some integer $n > 1$ then $R = I \oplus N$ and I is a subdirect sum of finite fields of at most n elements.

REFERENCES

1. ABU-KHUZAM, H. and YAQUB, A. Periodic rings with commuting nilpotents, Internat. J. Math. Math. Sci. 7 (1984) 403-406.

2. BELL, H.E. On commutativity of periodic rings and near rings, Acta Math. Acad. Sci. Hungar. 36 (1980), 293-302.

3. HIRANO, Y. Some studies on strongly π-regular rings, Math. J. Okayama Univ. 20 (1978), 141-149.