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THE DIOPHANTINE EQUATION $r^2 + r(x+y) = kxy$

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ABSTRACT. The Diophantine equation of the title is solved in integers.

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1. INTRODUCTION. In Section 3 of this note we will find an infinite family of solutions of

$$r^2 + r(x + y) = kxy, \quad k = 0, \pm 1, \pm 2, \ldots \tag{1.1}$$

and, by a proper choice of a parameter, all solutions will be secured.

This equation, for $k = 1$, arises from a geometric problem [1]. For this case, the problem was solved by C. G. Paradine [2]. While the solution of this note agrees with that of Paradine in case $k = 1$, the procedure for solving the problem is different.

Our solution depends upon the special form of a quadratic that occurs within the problem. In Section 2 we formalize the method to be used within the solution given in Section 3.

2. A METHOD FOR SOLVING CERTAIN QUADRATICS.

Suppose that a, b are nonzero integers such that $a + b = s^2$ for some integer s (possibly zero). Since a and b cannot both be negative, assume that $a > 0$.

Then to solve

$$ax^2 + by^2 = z^2 \tag{2.1}$$

in integers we write

$$a(x^2-y^2) = z^2 - s^2y^2$$

or

$$a(x-y)(x + y) = (z-sy)(z + sy).$$

If x,y,z are integral solutions, then for integers p,q

$$\frac{ax - ay}{z - sy} = \frac{z + sy}{x + y} = \frac{p}{q} \tag{2.2}$$

where we assume that $(p,q) = 1$.

With due regard for vanishing denominators, (2.2) yields two homogeneous equations in the three variables x,y,z which may be solved for these variables as polynomials in p,q. Any integral multiple, c, of these three functions gives a solution of (2.1) and if c takes on, also, certian rational values (those for which its denominator "cancels"), all solutions of (2.1) are secured.

The solution just described is possible because the determinant on the variables x and z in the two linear equations is $\pm(aq^2 + p^2)$ and so cannot be zero because $a > 0$.

3. THE TITLE EQUATION SOLVED.

We now consider equation (1.1).

If $k = 0$, the equation is trivial.

If $k = -1$, then one sees from (1.1) that $r = -y$ or $r = -x$ and so the solutions for $k = -1$ are given by $(x = a,\ y = b,\ r = -a)$ and $(x = a,\ y = b,\ r = -b)$ for all integers a,b.

We now let k be any integer except for 0 and -1. From (1.1) we have

$$r = \frac{1}{2}[-(x + y) \pm \sqrt{(x+y)^2 + 4kxy}$$

and so we require an integer n for which

$$(x + y)^2 + 4kxy = n^2. \tag{3.1}$$

Following the procedure of Section 2, we write (3.1) as

$$(y + (1+2k)x)^2 - (1 + 2k)^2x^2 = n^2 - x^2$$

and then as

$$y(y + 2(1+2k)x) = (n-x)(n + x)$$

from which we secure

$$\frac{y}{n - x} = \frac{n + x}{y + 2(1+2k)x} = \frac{p}{q}\,. \tag{3.2}$$

We pause to consider the denominators of (3.2). If $n = x$, then (using (3.1)) $y = 2(1 + 2k)x$, also. In this case either $y = 0$ and $r = -x$ (which occurred for $k = -1$) or $r^2 + (4k-1)r + (2 + 4k)x^2 = 0$ follows from (3.1). This equation is not possible in non-zero integers because the discriminant of $W^2 + (4k-1)W + (2 + 4k) = 0$ is $(4k + 3)^2 - 16$ which is never a square.

Going back to (3.2) we have the equations

$$px + qy = pn$$

$$(q-2p(1 + 2k))x - py = -qn. \tag{3.3}$$

The determinant on the variables x,y is

$$-p^2 - q^2 + 2pq(1 + 2k)$$

which cannot be zero for non-zero p,q. This is because the quadratic equation

$$W^2 - 2(1 + 2k)W + 1 = 0$$

has discriminant $4[(1 + 2k)^2 - 1]$ which cannot be a square for $k \neq 0, -1$.

Solving system (3.3) we secure

$$x = c(q^2 - p^2)$$

$$y = 2cp[(1+2k)p-q]$$

$$r = c[-q^2 + 2(1+k)pq - (1+2k)p^2]$$

or

$$r = -2ckp(q + p)$$

where c is any integer. This will be all solutions of (1.1) provided c is also allowed to range over all rationals with denominators that divide the fundamental solution of (3.3).

Thus, we have proved the following theorem.

Theorem. *For* $k \neq 0, -1$ all integral solutions of (1.1) are given by

$$x : y : r = [q^2-p^2] : 2p[(1+2k)p-q] : [-q^2 + 2(1+k)pq - (1+2k)p^2]$$

or

$$x : y : r = [q^2-p^2] : 2p[(1+2k)p-q] : [-2p(q+p)].$$

In [2] the solution of (1.1) for $k = 1$ was given as

$$x : y : r = ab : (a-b)(2a-b) : b(a-b) \quad \text{or} \quad a(b-2a).$$

If one lets $a = q - p$, $b = q + p$ then this agrees with the theorem for $k = 1$.

C. V. Gregg [3] stated, with out proof, that if $k = 1$, then

$$x : y : r = m(m-n) : n(m + n) : n(m-n)$$

is a solution of (1.1). This, also, is valid. Let $m = a$, $m - n = b$ to secure one of the solutions of Paradine.

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