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ON CENTER-LIKE ELEMENTS IN RINGS

JOE W. FISHER and MOHAMED H. FAHMY

University of Cincinnati
Cincinnati, Ohio 45221
U.S.A.

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ABSTRACT. In a paper with a similar title Herstein has considered the structure of prime rings which contain an element a which satisfies either $[a,x]^n = 0$ or is in the center of R for each x in R. We consider the structure of rings which contain an element a which satisfies powers of certain higher commutators. The two types which we consider are (1) $[\ldots[[a,x_1],x_2],\ldots,x_m]^n = 0$ or is in the center of R for all $x_1,x_2,\ldots,x_m$ in R and (2) $[a,[x_1,[x_2,\ldots,[x_{m-1},x_m]\ldots]]]^n = 0$ for all $x_1,x_2,\ldots x_m$ in R. We obtain results similar to those obtained by Herstein; however, in some cases we must strengthen the hypotheses.

Also we consider a third type (3) $(ax^m - x^n a)^k = 0$ for all x in R and extend results of Herstein and Giambruno.

KEY WORDS AND PHRASES. Prime and semiprime rings, primitive and semiprimitive rings, rings with involution, commutativity theorems.
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1. INTRODUCTION.

The definition of the center Z of a ring R has recently been generalized in several papers. Herstein [1, Theorem 2] showed that an element a of a prime ring R is central if and only if $[a,u]^n = 0$ for all $u \in U$ where U is a nonzero two sided ideal in R. We generalize this result in two directions. First, we show that (1) if R is prime and $[\ldots[[a,u_1],u_2],\ldots,u_m]^n = 0$ for all $u_1,u_2,\ldots,u_m \in U$, then $a \in Z$. From (1) it follows easily that a semiprime ring satisfying the Lie nilpotent identity $[\ldots[x_1,x_2],\ldots,x_m] = 0$ for all $x_1,x_2,\ldots,x_m$ in R is commutative [2, p. 230]. We also conclude from (1) two commutivity theorems which generalize two well-known theorems due to Kaplansky [2, p. 219] and Herstein [3].

Second, we prove that if $[a,[u_1,[u_1,[u_2,\ldots,[u_{m-1},u_m]\ldots]]]]^n = 0$ for all $u_1,u_2,\ldots u_m$ in U, then $a \in Z$ if either R is semisimple and U is essential, or R is prime with Z infinite and n fixed.

Herstein [1, Theorem 4] proves that if R is prime, $a \notin Z$, and $[a,x]^n \in Z$ for all $x \in R$, then R is an order in a 4-dimensional simple algebra. We show that the same result holds if $[\ldots[[a,x_1],x_2],\ldots,x_m]^n \in Z$ for all $x_1,x_2,\ldots,x_m \in R$.

In another attempt to generalize the structure of the center Z of a ring R without nonzero nil ideas Herstein [4] proved that the subring $T = \{a \in R\colon\ ax^{n(a,x)} = x^{n(a,x)}a$ for all $x \in R\} = Z$. This theorem was generalized by Giambruno [5], who showed that the set $G = \{a \in R\colon\ ax^{m(a,x)} = x^{n(a,x)}a$ for all $x \in R\} = Z$. In an attempt to generalize these results, we show that $G = \{a \in R\colon\ (ax^{m(a,x)} - x^{n(a,x)}a)^k = 0$ for all x in $R\} = Z$ if R is semiprimitive and $2R \neq 0$.

Throughout this paper R is an associative ring with 1 and Z denotes the center of R. Moreover, $[a,x] = ax - xa$ and if X is a subset of R, then $\ell(X) = \{r \in R\colon$ $rx = 0$ for all $x \in X\}$.

2. MAIN RESULTS. We begin this section with a lemma which will be useful in the sequel.

LEMMA 1. Let R be a ring, U an ideal of R, and $a \in R$. If $[[a,u_1],u_2] = 0$ for all $u_1,u_2 \in U$, then $[a,u]^2 = 0$ for all $u \in U$.

PROOF. Let $u \in U$. Since U is an ideal we obtain

$$0 = [[a,au],u] = [a[a,u],u] = a[[a,u],u] + [a,u]^2.$$

However the first term is zero. Hence $[a,u]^2 = 0$.

THEOREM 1. Let R be a prime ring and $U \neq 0$ an ideal of R. If $a \in R$ is such that for fixed positive integers m and n, $[\ldots[[a,u_1]u_2],\ldots,u_m]^n = 0$ for all $u_1,u_2,\ldots,u_m \in U$, then $a \in Z$.

PROOF. The proof goes by induction on m. The result is true for $m = 1$ by Herstein's theorem [1, Theorem 2].

Assume the result is true for $k < m$ and suppose that $[\ldots[[a,u_1],u_2],\ldots,u_m]^n = 0$ for all $u_1,u_2,\ldots,u_m \in U$. Set $b = [a,u_1]$. Then by assumption

$$[\ldots[[b,u_2],u_3],\ldots,u_m]^n = 0$$

for all $u_2,u_3,\ldots,u_m \in U$. Hence $b \in Z$ by the induction hypothesis. By applying Lemma 1 we obtain that $[a,u]^2 = 0$ for all $u \in U$. Therefore $a \in Z$ by Herstein's aforementioned theorem.

As a consequence of Theorem 1, we get the following two corollaries which generalize for prime rings two well-known theorems due to Kaplansky [2 , p. 219] and Herstein [3].

COROLLARY 1. Let R be a prime ring and $U \neq 0$ an ideal of R. If for every $a \in R$ there exists three natural numbers $k(a)$, $m(a)$, and $n(a)$ such that

$$[\ldots[[a^{k(a)},u_1],u_2],\ldots,u_{m(a)}]^{n(a)} = 0$$

where $u_1,u_2,\ldots,u_{m(a)} \in U$, then R is commutative.

PROOF. Evident.

COROLLARY 2. Let R be a prime ring and $U \neq 0$ and ideal of R. If for every $a \in R$, there exists two natural numbers $m(a)$, $n(a)$ and a polynomial $p_a(\lambda)$ with integer coefficients such that

$$[\ldots[[(a - a^2p_a(a),u_1], u_2],\ldots,u_{m(a)}]^{n(a)} = 0$$

where $u_1,u_2,\ldots,u_{m(a)} \in U$, then R is commutative.

PROOF. Evident.

Also as a corollary we obtain a result from [2 , p. 230].

COROLLARY 3. If R is a semiprime ring satisfying the Lie nilpotent identity $[\dots[x_1,x_2],\dots,x_n] = 0$, then R is commutative.

PROOF. Evident.

The next theorem generalizes a theorem of Herstein [1, Theorem 3].

THEOREM 2. Let R be a prime ring with center Z and let $a \in R$, $a \notin Z$ be such that $[\dots[[a,u_1],u_2],\dots,u_m]^n \in Z$ for all $u_1,u_2,\dots,u_m \in U$ where $U \neq 0$ is an ideal of R. Then R is an order in a 4-dimensional simple algebra.

PROOF. If $[\dots[a,u_1],u_2],\dots,u_{m-1}] \in Z$ for all $u_1,u_2,\dots,u_{m-1} \in U$, then $a \in Z$ by Theorem 2. Hence there exists $v_1,v_2,\dots,v_{m-1} \in U$ such that

$$b = [\dots[[a,v_1],v_2],\dots,v_{m-1}] \notin Z.$$

However by hypothesis $[b,u_m]^n \in Z$ for all $u_m \in U$. Ergo, R is an order in a 4-dimensional simple algebra by Herstein [1, Theorem 3].

We now generalize Herstein's Theorem 2 in [1] in another direction. Let $U \neq 0$ be an ideal of R, $a \in R$, m fixed in $\mathbb{Z}^+$. If

$$[a,[u_1,[u_2,\dots,[u_{m-1},u_m]\dots]]]^n = 0$$

for all $u_1,u_2,\dots,u_m \in U$ (Condition A) then we shall prove that $a \in Z$ in the following two cases:

(i) R is semiprimitive and U is an ideal such that $\ell(U) = 0$ (Theorem 3), or

(ii) R is prime, U is an ideal, Z is infinite, and n is fixed (Theorem 4).

First we prove a lemma:

LEMMA 2. Let R be a primitive ring, $U \neq 0$ an ideal of R, and $a \in R$ satisfying condition (A). Then $a \in Z$.

PROOF. (a) If R is a division ring, then $[a,[x_1,[x_2,\dots,[x_{m-1},x_m]\dots]]] = 0$ for all $x_1,x_2,\dots,x_m \in R$. Hence $a \in Z$ by a result of Smiley [6].

(b) If R is primitive, then it has a faithful irreducible R-module V which is also faithful and irreductible as a U-module. By the Density theorem U acts densely on V as a vector space over a division ring D. If $\dim_D V = 1$, then $R = D$ and the result follows from (a). So let $\dim_D V > 1$.

Suppose that there exists a nonzero victor $v \in V$ such that v and va are linearly independent over D. Since U acts densely on V there exists $u_1,u_2 \in U$ such that $vu_1 = v$, $(va)u_1 = v$, $vu_2 = 0$, and $(va)u_2 = va$. Thus

$$v[a,[u_1,[u_1,[u_1,\dots,[u_1,u_2]\dots]]] = v$$

and so $v[a,[u_1,[u_1,\dots,[u_1,u_2]\dots]]]^n = v$. But, by the hypothesis, the expression on the left is zero, which gives that $v = 0$, contrary to our assumption. Thus for every $v \in V$, $va = \lambda(v)v$, where $\lambda(v) \in D$. It follows easily from this that, in fact, $\lambda(v)$ does not depend on v, hence $va = \lambda v$ for all $v \in V$. So, if $x \in R$, then $(vx)a = \lambda vx$ and $(va)x = (\lambda v)x = \lambda(vx)$. Hence $v(xa - ax) = 0$ for all $v \in V$. Since R acts faithfully on V we have $ax - xa = 0$ for all $x \in R$, and so $a \in Z$.

THEOREM 3. If R is a semiprimitive ring, $U \neq 0$ an ideal of R with $\ell(U) = 0$, and $a \in R$ which satisfies condition A, then $a \in Z$.

PROOF. Since $\ell(U) = 0$, U is an essential ideal of R. Hence it can easily be shown that $\cap\{P: P \text{ primitive ideal such that } P \not\supset U\} = 0$. Hence R is the subdirect product of R/P where $P \not\supset U$. It follows from Lemma 2 that a is in the center of each R/P. Therefore $a \in Z$.

THEOREM 4. Let R be prime with Z infinite, $U \neq 0$ on ideal of R, and $a \in R$ which satisfies condition A, then $a \in Z$.

PROOF. Let C be the extended centroid of R [7]. Then $C \supset Z$ and because Z is infinite condition A carries over to the prime ring $S = RC$ and its ideal $V = UC$. If $a \notin Z$ then R satisfies a nontrivial generalized polynomial identity $[a,[u_1x,[u_2x,\ldots,[u_{m-1}x,u_mx]\ldots]]]^n = 0$ for $u_1,u_2,\ldots,u_m \in R$. Hence $S = RC$ is primitive by Martindale's theorem. Since $V = UC$ is an ideal of S which satisfies condition A, we have that $a \in Z(S)$ by Lemma 2. Hence $a \in Z$.

Question 1: In Theorem 4 is the hypothesis that Z be infinite necessary? Note that in Theorem 1 it was not necessary.

We finish our paper with a partial generalization of the results in [5] and [4]. Let a be an element of the ring R such that for all $u \in U$, a nonzero ideal of R, we have

$$(au^{m(u)} - u^{n(u)}a)^{k(u)} = 0 \quad \text{(Condition B)}$$

and let $\bar{G} = \{a \in R: (ax^{m(x)} - x^{n(x)}a)^{k(x)} = 0 \text{ for all } x \text{ in } R\}$. It is clear that $\bar{G} \supseteq G \supseteq T \supseteq Z$.

THEOREM 5. If R is a ring satisfying condition (B) with $2R \neq 0$, then either

(1) R is semiprimitive with $\ell(U) = 0$ or

(2) R is prime with infinite center with fixed integers m, n, and k.

Then $a \in Z$.

PROOF. By using the same technique of proof as that in Theorems 3 and 4, it is enough to prove the result in the primitive case.

Let V and D be as in the proof of Lemma 2. If $\dim_D V = 1$, i.e., R is a division ring, we get that for all $x \in R$, $ax^{m(x)} - x^{n(x)}a = 0$. Hence by a result of Giambruno [8] $a \in Z$. Thus let $\dim_D V > 1$. If $0 \neq v \in V$, then the vectors $\{v,va,va^2\}$ are linearly dependent. Indeed, if they were linearly independent, then by the density theorem, there is $u \in U$ such that $vu = v$, $(va)u = v$, and $(va^2)u = 0$.

Thus we get $v(au^{m(u)} - u^{n(u)}a)^{k(u)} = v$ if $k(u)$ is even and equals $v - va$ if $K(u)$ is odd. But $(au^{m(u)} - u^{n(u)}a)^{k(u)} = 0$ so we get a contradiction in both cases.

Assume that $\{v,va\}$ are linearly independent, then $va^2 = \lambda v + \mu va$ where $\lambda,\mu \in D$. If $\lambda \neq 0$, then by the density theorem there is $w \in U$ such that $vw = v$ and $(va)w = 0$. So $v(aw^{m(w)} - w^{n(w)}a)^{k(w)} = \pm\lambda^s v$ where $s = s(k)$. Contradiction.

However, if $\lambda = 0$, i.e., $va^2 = \mu va$, then there is $y \in U$ such that $vy = v$ and $(va)y = \alpha v$ where $0 \neq \alpha \in D$, $\alpha \neq \mu$ (because $2R \neq 0$ implies $D \neq \mathbb{Z}_2$). Thus $v(ay^{m(y)} - y^{n(y)}a)^{k(y)} = \beta v - \gamma(va) = 0$ where $0 \neq \beta = \beta(k) \in D$ and $\gamma = \gamma(k) \in D$. Contradiction. Therefore $\{v,va\}$ are linearly dependent. The same argument as used in the proof of Lemma 2 shows that $a \in Z$. This completes the proof.

REMARKS: 1) It is of interest to study all the above theorems for rings with involution "*" by applying the same conditions on the set of symmetric elements. For example, it is natural to ask: If $[a,s_1,\ldots,s_n]^n \in Z$ for all $s_1,s_2,\ldots,s_n \in S$ and $a \notin Z$, then what about R? It was shown by Fahmy [9] and Giambruno [8], that if $[s_1,s_2,\ldots,s_n]^n \in Z$, then $\dim_Z R \leq 16$.

2) A second direction in which one may try to extend the above theorems is to generalize the cohypercenter introduced by Chacron in [10], i.e., to study the set $\{a \in R: [a,x - x^2p(x)]^{n(x)} = 0$ for all x in $R\}$ where $p(x)$ is a polynomial with integral coefficients.

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