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REPRESENTATION OF FUNCTIONS AS THE POST-WIDDER INVERSION OPERATOR OF GENERALIZED FUNCTIONS

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ABSTRACT. A study is made of the Post-Widder inversion operator to a class of generalized functions in the sense of distributional convergence. Necessary and sufficient conditions are proved for a given function to have the representation as the rth operate of the Post-Widder inversion operator of generalized functions. Some representation theorems are also proved. Certain results concerning the testing function space and its dual are established. A fundamental theorem regarding the existence of the real inversion operator (1.6) with r = 0 is proved in section 4. A classical inversion theory for the Post-Widder inversion operator with a few other theorems which are fundamental to the representation theory is also developed in this paper.

KEY WORDS AND PHRASES. *Post-Widder inversion operator, Representation theorems, Testing function space and its dual, generalized functions.*
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1. INTRODUCTION.

Schwartz ([1], pp 202-203), Zemanian ([2], pp 235-237; [3], pp 70-71), Cooper [4], and Benedetto [5] have investigated the necessary and sufficient conditions for a function to be the Laplace transform of a generalized function. They have used a complex inversion formula of the Laplace transform. However, these authors except Schwartz have studied the same or similar problems under different conditions.

Another type of representation theorem known as structural formula has been treated by Gelfand and Shilov ([6], pp 110-113), and Treves ([7], pp 273-274). Pandey [8] has obtained one such structural formula for the Hirschman - Widder convolution transform which was extended to generalized functions by Zemanian [9].

The Post-Widder inversion operator $L_{k,t}[f]$ ([10], pp. 288) is defined by

$$L_{k,t}[f] = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-\frac{k}{t}u} u^k F(u)\, du \qquad t > 0, \tag{1.1}$$

$$f(x) = \int_0^\infty e^{-xt} F(t)\, dt \tag{1.2}$$

In the right hand side of (1.1) f does not appear and hence we denote by $L_{k,t}[F]$ the Post-Widder inversion operator corresponding to the determining function f of the Laplace transform F.

Iterates of the Laplace transform are well-known and have been studied by Widder [10], Boas and Widder [11], Pollard [12] and Akutowicz [13]. Unlike the iterate of Laplace transform, we define an operate of the Post-Widder inversion operator as an operation defined by

$$L_{k,v}\, L_{k,t}[F] = \int_0^\infty h_k(t,v) \int_0^\infty h_k(u,t)\, F(u)\, du\, dt \tag{1.3}$$

so that, for any non-negative integer r, the rth operate o .he Post-Widder inversion operator may be written as

$$^{r+1}L_{k,t}^{(.)}[F] = \int_0^\infty h_k(u,t)\, {}^{r}L_{k,u}[F]\, du, \tag{1.4}$$

where

$$\begin{aligned} {}^{r}L_{k,u}[F] &= L_{k,u_r}\, L_{k,\cdot_{r-1}}\, L_{k,\cdot_{r-2}} \cdots L_{k,\,1}[F] && r \neq o \\ &= F(u) && r = o \end{aligned} \tag{1.5,a,b}$$

and

$$h_k(u,t) = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{-\frac{k}{t}u}\, u^k. \tag{1.6}$$

We next define, as in [14]-[15], a real inversion operator in the sense of Rooney [16] by

$$P_{n,t}^{(\beta,\nu)}\left[{}^{r+1}L_{k,t}^{(.)}[F]\right] = A(k) \int_0^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, {}^{r+1}L_{k,t}^{(x)}[F]dx, \tag{1.7}$$

where

$$^{r+1}L_{k,t}^{(x)}[F] = \int_0^\infty h_k^{(x)}(u,t)\, {}^{r}L_{k,u}[F]\, du, \tag{1.8}$$

$$h_k^{(x)}(u,t) = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} e^{\frac{-ku}{t}}\, u^k, \tag{1.9}$$

and

$$A(k) = \frac{2k!\, k^{-k+\nu}}{\Gamma(\nu+1)} \tag{1.10}$$

and $Q_\nu^{(\alpha,\beta)}[x]$ is a Jacobi function of the second kind ([17], p 170), defined by

$$Q_\nu^{(\alpha,\beta)}[x] = \frac{2^{\alpha+p+\nu}\Gamma(\alpha+\nu+1)\Gamma(\beta+\nu+1)}{\Gamma(\alpha+\beta+2\nu+2)(x-1)^{\alpha+\nu+1}}(1+x)^{-\beta}\, {}_2F_1\left[\begin{matrix}\nu+1,\alpha+\nu+1; \\ \alpha+\beta+?\nu+2;\end{matrix}\; 2(1-x)^{-1}\right] \tag{1.11}$$

When $r = 0$, (1.7) reduces to the real inversion operator studied in [14-15] apart from a factor $A(k)$.

In spite of the above works, almost no attention has been given to the Post-Widder inversion operator of generalized functions. The main purpose of this paper is to study necessary and sufficient conditions for a given function to have the representation as the rth operate of the Post-Widder inversion operator of certain class of generalized functions. Section 2 deals with the testing function space $\mathfrak{F}_{a,b,n}$ and its dual space $\mathfrak{F}'_{a,b,n}$, and the Post-Widder inversion operator of generalized functions. Using a method similar to that of Zemanian [2], certain representation theorems are proved. These results are related to these of Pandey [8] in the sense that every element in $\mathfrak{F}'_{a,b,n}$ is the linear combination of the finite order distributional derivative of continuous functions. In Section 3, certain results concerning the testing function space and its dual space are established. A fundamental theorem regarding the existence of the real inversion operator (1.7) with $r = 0$ is proved in Section 4. Then a classical inversion theory for the Post-Widder inversion operator with a few other theorems which are fundamental to the representation theory is also developed in this section. The final section is devoted to a few representation theorems for the Post-Widder inversion operator of generalized functions.

2. TESTING FUNCTION SPACE

Let I denote an open internal on the real line. Let $C^m(I)$ denote the vector space of all complex valued functions defined on I having continuous derivatives of all orders $\leq m$ where m is a non-negative integer. Clearly

$$C^\infty(I) = \bigcap_{m \geq o} C^m(I).$$

The elements of $C^\infty(I)$ are called infinitely smooth functions on I. A testing function on I is a C^∞ function on I having a compact support on I. The collection of all testing functions which is denoted by $D(I)$, forms a vector space. We assign to $D(I)$ the customary topologies that make the dual space $D'(I)$ the Schwartz space of distributions.

For any two real numbers a and b with $o \leq a < b < \infty$, n and k, non-negative integers such that $n \leq k$, and u, a real variable, we define

$$k_{a,b,n}(u) = \begin{cases} (\sqrt{2}e)^{-2n} \exp\left(-\frac{2bn^2}{k} + a\right)u, & o < u < \infty \\ (\sqrt{2}e)^{-2n} \exp\left(+\frac{2n^2}{k} + 1\right)ub, & -\infty < u < 0 \end{cases} \tag{2.1}$$

and

$$k_{a,b,o}(u) = \begin{cases} e^{au}, & o < u < \infty \\ e^{bu}, & -\infty < u < o \end{cases} \tag{2.2}$$

Clearly, $k_{a,b,n}(u)$, $n = 0,1,2,...$ are C^∞ functions in $-\infty < u < \infty$, and have the property

$$o < k_{a,b,n}(u) / k_{c,d,m}(u) \le 1 \tag{2.3}$$

provided $a \le c < d \le b$ and $m \le n$.

We define $\mathfrak{J}_{a,b,n}$ to be the space of C^{∞} functions in $-\infty < u < \infty$ for which

$$\gamma_n(\phi) = \sup_{-\infty<u<\infty} | k_{a,b,n}(u)u^n D_u^n \phi(u)| \tag{2.4}$$

is bounded for all u in $-\infty < u < \infty$ and tends to zero as n tends to infinity. Then it is clear that $\mathfrak{J}_{a,b,n}$ is a linear space under the usual definition of addition and multiplication by a complex number, the zero element being the identically zero function. The topology in $\mathfrak{J}_{a,b,n}$ is generated by the collection of semi norms $\{\gamma_n\}_{n=o}^{\infty}$. Since γ_o is a norm, the collection is separating, and thereby making it countably multi normed space. We denote by $\mathfrak{J}'_{a,b,n}$ the dual space of $\mathfrak{J}_{a,b,n}$.

The sequence $\{\phi_\nu\} \subset \mathfrak{J}_{a,b,n}$ converges to ϕ in $\mathfrak{J}_{a,b,n}$ if for every n, $\gamma_n(\phi_\nu - \phi) \to o$ as $\nu \to \infty$.

In view of (2.4), the sequence

$$\{k_{a,b,n}(u)u^n D_u^n \phi_\nu(u)\} \tag{2.5}$$

represents a Cauchy sequence on $-\infty < u < \infty$. Then, by lemma 3.2.1 in [3], $\mathfrak{J}_{a,b,n}$ is complete and therefore a Fréchet space.

For $a \le c < d \le b$, the inequality

$$|k_{a,b,n}(u)u^n D_u^n \phi(u)| \le |k_{c,d,n}(u)u^n D_u^n \phi(u)| \tag{2.6}$$

implies $\gamma_{a,b,n}(\phi) \le \gamma_{c,d,n}(\phi)$ from which by lemma 1.6.3 in [3], it follows that $\mathfrak{J}_{c,d,n} \subseteq \mathfrak{J}_{a,b,n}$.

We denote $\mathfrak{J}_{a,b,n,u}$, $\mathfrak{J}_{a,b,n,t}$ and $\mathfrak{J}_{a,b,n,u,t}$ the spaces of testing functions defined over the Euclidean spaces of the variables u,t, and (u,t) respectively. Similarly, $\mathfrak{J}'_{a,b,n,u}$, $\mathfrak{J}'_{a,b,n,t}$ and $\mathfrak{J}'_{a,b,n,u,t}$ will denote the spaces of generalized functions defined over these Euclidean spaces respectively.

Let $\Omega_t = \{t : a < \frac{k}{t} < b$ with $o \le a < b < \infty\}$ and let $F \in \mathfrak{J}'_{a,b,n}$. Then the Post-Widder inversion operator of generalized functions is defined as an application of F to the kerrel function

$h_k(u,t) = \frac{1}{k!}(\frac{k}{t})^{k+1} e^{-\frac{ku}{t}} u^k \in \mathfrak{J}_{a,b,n}$ for every t in Ω_t by the following equation

$$L_{k,t}[F] = \langle F(u), h_k(u,t) \rangle . \tag{2.7}$$

If $F(u)$ is a locally integrable function such that $\int_o^{\infty} \frac{|F(u)|}{k_{a,b,n}(u)} du$ is finite, then $F(u)$ generates a regular generalized function, and we write

$$L_{k,t}[F] = \int_o^{\infty} \frac{1}{k!}(\frac{k}{t})^{k+1} e^{-\frac{k}{t}u} u^k F(u)\, du . \tag{2.8}$$

3. CERTAIN RESULTS IN TESTING FUNCTION SPACES AND THEIR DUAL SPACES.

We shall prove certain results concerning the testing function spaces $\mathfrak{J}_{a,b,n}$ and their dual spaces $\mathfrak{J}'_{a,b,n}$.

LEMMA 3.1. Let $\Omega_t = \{t;\ a < \frac{k}{t} < b \text{ with } o \le a < b < \infty\}$ and let

$$h_k(u,t) = \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \exp\left(-\frac{k}{t}u\right) u^k.$$

Then, for every $n \ge k$ and for every t in Ω_t, $h_k(u,t) \in \mathfrak{J}_{a,b,n}$.

PROOF. The Leibnitz rule of differentiation yields

$$u^n D_u^n h_k(u,t) = u^n D_u^n \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \exp\left(-\frac{k}{t}u\right) u^k$$

$$= \sum_{p=o}^{k} C_{n,k}(p) \left(\frac{k}{t}u\right)^{n+k-p+1} e^{-\frac{k}{t}u} u^{-1},$$

where

$$C_{n,k}(p) = {}^nC_p \frac{k(k-1)(k-2)\ldots(k-p+1)}{k!}(-1)^{n-p} \qquad p \le k.$$

Then, if $u > o$, $b > \frac{k}{t}$ and $n \ge k$, we see that

$$|k_{a,b,n}(u)\, u^n D_u^n h_k(u,t)|$$

$$\le 2^{-n} e^{-2n\left(\frac{nbu}{k}+1\right)} \sum_{p=o}^{k} |C_{n,k}(p)| \left(\frac{k}{t}u\right)^{n+k-p+1} e^{-\left(\frac{k}{t}-a\right)u}$$

$$\le e^{-2n\left(\frac{nbu}{k}+1\right)} \left(\frac{n}{t}u+1\right)^{2n+1} e^{-\left(\frac{k}{t}-a\right)u}$$

$$< e^{-2n\left[\left(\frac{n}{t}u+1\right) - \log\left(\frac{n}{t}u+1\right)\right]} \left(\frac{nu}{t}+1\right) e^{-\left(\frac{k}{t}-a\right)u},$$

where we have used the following inequality

$$2^{-n} \sum_{p=o}^{n} |C_{n,k}(p)| \le 1, \quad n \le k.$$

Clearly, the right hand side is bounded for all $u \ge o$, every t in Ω_t and all $n \ge k$, and tends to zero as $n \to \infty$. Proceeding similarly, it can be shown that the lemma holds for $u < o$. This completes the proof of the lemma.

COROLLARY 3.1. $h_k(u,t) \in \mathfrak{J}_{a,b,n,t}$ for every t in Ω_t.

PROOF. It can be easily verified by induction that

$$t^n D_t^n h_k(u,t) = (-1)^n (1+uD_u)(2+uD_u)(3+uD_u)\ldots(n+uD_u) h_k(u,t).$$

Then

$$|k_{a,b,n}(u)\, t^n D_t^n h_k(u,t)| = |k_{a,b,n}(u)(n+uD_u)^n h_k(u,t)|$$

$$\le \sum_{p=o}^{n} |a_p(n)|\, |k_{a,b,n}(u)(uD_u)^p h_k(u,t)|$$

$$= \sum_{p=o}^{n} |a_p(n) \frac{k_{a,b,n}(u)}{k_{a,b,p}(u)}|\, |k_{a,b,p}(u)(uD_u)^p h_k(u,t)|.$$

where $a_p(n)$, $p = 0,1,2,3, \ldots n$ are the coefficients of the polynomial $(n+uD_u)^n$, and are given by

$$a_p(n) = {}^nC_p\, n^{n-p}, \qquad p \le n.$$

Since $n^{n-p} < 2^n(n-p)^n$ for $p < n$ and $2^N(n-p)^N e^{-2(n-p)}$ is bounded by unity for positive n and N, We see that

$$\left| a_p(n) \frac{k_{a,b,n}(u)}{k_{a,b,p}(u)} \right| \begin{cases} < {}^nC_p\, 2^{-(N-n)}(n-p)^{-(N-n)} & p < n \\ = 1 & p = n. \end{cases}$$

Therefore, choosing $N \ge 2n$ we find that

$$|k_{a,b,n}(u)\ t^n D_t^n h_k(u,t)| \le |k_{a,b,n}(u)\ (uD_u)^n h_k(u,t)| + + 2^{-n} \sum_{p=o}^{n-1} {}^nC_p |k_{a,b,p}(u)\ (uD_u)^p h_k(u,t)|.$$

Now in view of lemma 1 in [21], and lemma 3.1, it is clear that each term on the right hand side is bounded for all u in $-\infty < u < \infty$, all $n \ge k$ and every t in Ω_t, and tends to zero as $n \to \infty$. This proves the corollary.

COROLLARY 3.2. For any fixed non-negative integer m, $(tD_t)^m h_k(u,t) \in {}'\mathfrak{J}_{a,b,n}$ for each t in Ω_t.

PROOF. Since $h_k(u,t)$ is a C^∞ function with respect to t and u,

$$k_{a,b,n}(u)u^n D_u^n (tD_t)^m h_k(u,t) = k_{a,b,n}(u)(tD_t)^m u^n D_u^n h_k(u,t)$$

Then,

$$k_{a,b,n}(u)\ (tD_t)^m u^n D_u^n h_k(u,t) = (tD_t)^m \sum_{p=o}^{k} C_{n,k}(p)\ (\tfrac{k}{t}u)^{n+k-p+1} e^{-\frac{k}{t}u} u^{-1}$$

$$= \sum_{q=o}^{m} \sum_{p=o}^{k} C_{n,k}(p) d_{n,k,p}(q)\ (\tfrac{k}{t}u)^{n+q+k-p+1} e^{-\frac{k}{t}u} u^{-1}.$$

Since $\left|\frac{d_{n,k,p}(q)}{n^m}\right|$ is bounded, we see, as in lemma 3.1, that

$$| k_{a,b,n}(u)u^n D_u^n\ (tD_t)^m h_k(u,t)| < c \sum_{q=o}^{m} n^m e^{-2n[(\frac{n}{t}u+1) - \log(\frac{n}{t}u+1)]} (\tfrac{n}{t}u+1)^{q+1} e^{-(\frac{k}{t}u - a)u}$$

from which the corollary follows.

LEMMA 3.2. Let $\phi(u)$ $\mathfrak{J}_{a,b,n}$ and let $k_{a,b,n}(u)$ be defined as in (2.1) and (2.2). Then for $a \le c < d \le b$ and $m \le n$, $k_{c,d,m}(u)\ \phi(u)$ belongs to $\mathfrak{J}_{a,b,n}$. If $\{\phi_\nu\}$ converges to zero in $\mathfrak{J}_{a,b,n}$, the sequence $\{k_{c,d,m}(u)\ \phi_\nu(u)\}$ converges to zero function in $\mathfrak{J}_{a,b,n}$.

PROOF. We write

$$k_{a,b,n}(u)\ u^n D_u^n k_{c,d,m}(u)\ \phi_\nu(u) = \sum_{p=o}^{n} k_{a,b,p}(u)\ u^p D_u^p\ \phi_\nu(u) \frac{k_{a,b,n}(u)u^{n-p}D_u^{n-p}k_{c,d,m}(u)}{k_{a,b,p}(u)}$$

For every p, the quantity inside the second brace is bounded, and the quantity inside the first brace always converges to zero function.

LEMMA 3.3. If $\phi \in \mathfrak{J}_{a,b,n}$ and $F \in \mathfrak{J}'_{a,b,n}$, then $F(u)/k_{a,b,n}(u) \in \mathfrak{J}'_{a,b,n}$.

It is a direct consequence of lemma 3.2.

THEOREM 3.1. Let $F \in \mathfrak{J}_{a,b,n}$ and let $\Omega_t = \{t;\ a < \frac{k}{t} < b \text{ with } o \le a < b < \infty\}$. Then the Post-Widder inversion operator

$$L_{k,t}[F] = \langle F(u), h_k(u,t) \rangle \tag{3.1}$$

is a C^∞ function, and belongs to $\mathfrak{J}_{a,b,n}$ for every t in Ω_t, and for every non-negative integer m

$$D_t^m L_{k,t}[F] = \langle F(u), \frac{\partial^m}{\partial t^m} h_k(u,t) \rangle . \tag{3.2}$$

PROOF. By the hypothesis, (3.1) has sense. For some fixed t in Ω_t, consider

$$\frac{L_{k,t+\Delta t}[F] - L_{k,t}[F]}{\Delta t} - \langle F(u), \frac{\partial}{\partial t} h_k(u,t) \rangle = \langle F(u)\ \psi_t(u) \rangle, \tag{3.3}$$

where

$$\psi_t(u) = \begin{cases} \dfrac{h_k(u,t+\Delta t) - h_k(u,t)}{\Delta t} - \dfrac{\partial}{\partial t} h_k(u,t) & \Delta t \ne o \\ = o & \Delta t = o. \end{cases} \tag{3.4}$$

We shall first show that, as $\Delta t \to o$, $\psi_t(u)$ converges to zero in $\mathfrak{J}_{a,b,n}$. By arguments similar to those given in ([2], pp. 112), it readily follows that $\psi_t(u)$ converges uniformly to zero over every finite u interval as $\Delta t \to o$. A similar argument shows that $\psi^{(m)}_t(u)$ converges uniformly to zero over every finite u interval as $\Delta t \to o$.

To prove the above assertion we show that $k_{a,b,n}(u)\ u^n\ D_u^n\ \psi_t(u)$ is bounded for all u in $-\infty < u < \infty$ and for Δt in any finite interval in Ω_t and tends to zero as $n \to \infty$. Indeed, rewriting (3.4) as

$$\psi_t(u) = \frac{1}{\Delta t} \int_t^{t+\Delta t} D_y\ h_k(u,y)\ dy - D_t\ h_k(u,t)$$

$$= \frac{1}{\Delta t} \int_t^{t+\Delta t} dy \int_t^{y} D_z^2\ h_k(u,z)\ dz$$

we see that

$$|k_{a,b,n}(u)\ u^n\ D_u^n\ \psi_t(u)| \le |\Delta t| \sup_{z \in [t,t+\Delta t]} |k_{a,b,n}(u)\ u^n\ D_u^n\ D_z^2\ h_k(u,z)|$$

which, in view of the fact that for any non-negative integer N

$$|\ k_{a,b,n}(u)\ u^{n+N}\ D_u^n\ h_k(u,t)|$$

is bounded for each t in Ω_t and for all u in $-\infty < u < \infty$, establishes the assertion as $\Delta t \to o$.

Then since $F \in \mathcal{J}'_{a,b,n}$, the right hand side of (3.3) is zero and we get

$$D_t\, L_{k,t}[F] = \left\langle F(u),\ \frac{\partial}{\partial t}\, h_k(u,t) \right\rangle .$$

Upon repeated application of this arguments, it is clear that $L_{k,t}[F]$ is C^∞ function and that (3.2) holds.

Now we shall show that $L_{k,t}[F] \in \mathcal{J}_{a,b,n,t}$. For this, set

$$\lambda_t(u) = k_{a,b,n}(u)\ t^m\, D_t^m\, h_k(u,t) \quad .$$

For each fixed t in Ω_t, $t^m D_t^m h_k(u,t)$ belongs to $\mathcal{J}_{a,b,n,u}$ and hence for such fixed t, $\lambda_t(u)$ belongs to $\mathcal{J}_{a,b,n,u}$ by lemma 3.2. Since $h_k(u,t)$ $\mathcal{J}_{a,b,n,u,t}$, for each non-negative integer r, $\lambda_t(u)$ satisfies the following inequality

$$\left| (1+u^2)^r\ \lambda_t^{(r)}(u) \right| \le N_{mr},$$

where the constant N_{mr} is independent of t and u. It now follows from lemma 3.3 and from the bounded property of generalised function that there exist a constant C and a fixed r such that

$$\left| k_{a,b,m}(1)\ t^m\, D_t^m\, L_{k,t}[F] \right| = k_{a,b,m}(1)\ \left| \left\langle F(u),\ t^m D_t^m\, h_k(u,t) \right\rangle \right|$$

$$\le \left| \left\langle F(u)/k_{a,b,m}(u),\ k_{a,b,m}(u)\ t^m\, D_t^m\, h_k(u,t) \right\rangle \right|$$

$$\le C \sup_{-\infty<u<\infty} \left| (1+u^2)^r\ \lambda_t^{(r)}(u) \right| \le C\, N_{mr}.$$

This completes the proof.

COROLLARY 3.3. For any non-negative integer r, the rth operate of the Post-Widder inversion operator defined by (1.3) and (1.4) is a regular generalised function.

PROOF. By theorem 3.1, the Post-Widder inversion operator

$$L_{k,t}[F] = \langle F(u),\ h_k(u,t) \rangle$$

is in $\mathcal{J}_{a,b,n,t}$. Since by ([2], pp. 104-105), there is one-to-one correspondence between the testing functions in $\mathcal{J}_{a,b,n}$ and the regular generalised functions that are generated by them and can therefore be identified. $\mathcal{J}_{a,b,n} \subseteq \mathcal{J}'_{a,b,n}$. Hence $L_{k,t}[F] \in \mathcal{J}'_{a,b,n}$ as a regular generalised function. Then the first operate of the Post-Widder inversion operator

$$L_{k,t}\, L_{k,.}[F] = \langle L_{k,v}[F],\ h_k(u,t) \rangle$$

has sense and belongs to $\mathcal{J}_{a,b,n,t}$, and hence belongs to $\mathcal{J}'_{a,b,n}$ as a regular generalised function. Repeated application of the above arguments suggests that $^r\, L_{k,t}[F]$ belongs to $\mathcal{J}'_{a,b,n}$ as a regular generalised function.

4 CLASSICAL INVERSION THEORY. In this section we give a classical inversion theorem for the Post-Widder inversion operator. Few other theorems which are fundamental to the representation theory are also proved. First of all these, we establish a fundamental theorem regarding the existence of the real inversion operator (1.7) with $r = o$.

We begin this section by defining a class of functions $M(o,\infty)$ which is wider than those in $L(o,\infty)$ for which Widder ([10], pp. 280-283, 317-320) has given inversion and representation theories for the Laplace transform through the Post-Widder inversion operator (1.1).

DEFINITION 4.1. A function F is said to belong to the class of functions $M(o,\infty)$ if the following conditions are satisfied

(i) $t^{k_1} F(t) \in L(o,r)$ for every finite $r > o$ and every k_1 o;

(ii) $t^{-k_2}F(t) \in L(R,\infty)$ for every finite $R > r$ and every k_2 in $o<k_2< 1$;

(iii) $F(t)$ is bounded for all t in $r \le t \le R$.

Clearly, the class of functions $L(o,\infty)$ is a subclass of $M(o,\infty)$.

THEOREM 4.1. If $F(t) \in M(o,\infty)$ then for every $k_1,k_2 < k$, a positive integer, where $k_1 > o$ and $o < k_2 \le 1$, the Post-Widder inversion operator (1.1) exists for each $t > o$. If, in addition, (1) $\beta > o$, $\nu > o$ (2) $k-2\nu-k_1-3 > o$ hold, then for all $n \ge k$, the real inversion operator defined by (1.7) with $r = 0$ exists for each $t > 0$ and

$$P_{n,t}^{(\beta,\nu)} \left[L_{k,t}^{(\cdot)}[F]\right] = \int_0^\infty H_n(x)\ F(xt)\ dx, \tag{4.1}$$

where

$$H_n(x) = \sqrt{2n} \int_0^\infty \frac{x^{k-\nu-1} e^{-kxy} y^{\beta+\nu+2n}}{(1+y)^{\beta+\nu+2n+1}} dy , \tag{4.2}$$

and

$$\int_0^\infty H_n(x)\ dx \to o \quad \text{as} \quad n \to \infty. \tag{4.3}$$

PROOF. For some fixed $t > o$, assume that $r \ge t(1-k_1/k)$ and $R \ge t(1+k_2/k)$. Then clearly,

$$\left| \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_0^\infty e^{-\frac{k}{t}u}\ u^k\ F(u)\ du \right|$$

$$\le t^{-(1+k_1)} \frac{k^{k+1}}{k!} e^{-k(1-\frac{k_1}{k})} \left(1-\frac{k_1}{k}\right)^{k-k_1} \int_0^r u^{k_1} \left|F(u)\right| du$$

$$+ t^{-(1-k_2)} \frac{k^{k+1}}{k!} e^{-k(1-\frac{k_1}{k})} \left(1+\frac{k_2}{k}\right)^{k+k_2} \int_R^\infty u^{-k_2} \left|F(u)\right| du$$

$$+ (R-r) \underset{r \le u \le R}{\text{u.b.}} \left|F(u)\right| \tag{4.4}$$

But for $k_1,k_2 < k$, the expressions

$$D(k,k_1) = \frac{k^{k+1}}{k!} e^{-k(1-\frac{k_1}{k})}\left(1-\frac{k_1}{k}\right)^{k-k_1} \quad \text{and} \quad D(k,k_2) = \frac{k^{k+1}}{k!} e^{-k(1+\frac{k_2}{k})}\left(1+\frac{k_2}{k}\right)^{k+k_2} \tag{4.5}$$

are bounded and tend to zero as $k \to \infty$.

Thus, $L_{k,t}[F]$ exists for each $t > o$. Then we say that $F(t)$ has Post-Widder inversion operator or that it is the Post-Widder inversion operator transformable function

A simple change of variable and the use of (4.4) yield

$$\left| P_{n,t}^{(\beta,\nu)}\ L_{k,t}^{(.)}[F] \right| = \left| A(k) \int_0^\infty y^{-1}(t/y)^{\nu-k}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}\ L_{k,y}[F]\ dy \right|$$

$$\le M_1\ D(k,k_1)t^{-(1+k_1)} \int_0^\infty A(k)\ s^{\nu-k+k_1} \left| Q_\nu^{(o,\beta+2n)}[1+2s^{-1}\right|\ ds$$

$$+ M_2\, D(k,k_2)t^{-(1-k_2)} \int_o^\infty A(k)\, s^{\nu-k-k_2} |Q_\nu^{(o,\beta+2n)}[1+2s^{-1}]|\, ds$$

$$+ M_3 \int_o^\infty A(k)\, s^{\nu-k-1} |\, Q_\nu^{(o,\beta+2n)}[1+2s^{-1}]|\, ds,$$

where M_1 and M_2 are the values of the integrals in (4.4) and M_3, the value of the last term in (4.4) and $D(k,k_1)$, $D(k,k_2)$ are given by (4.5).

In view of (4.5), it is sufficient to show that

$$\int_o^\infty A(k)\, s^{\nu-k+A_i} \,|\, Q_\nu^{(o,\beta+2n)}[1+2s^{-1}] \,|\, ds, \; i = 1,2,3 \tag{4.6}$$

exists, where $A_i = k_1, -k_2, -1$ according as $i = 1,2,3$ respectively.

If (1)-(2) together with $n \geq k$ are satisfied, (4.6) exists provided $k_1 > o$ and $o < k_2 \leq 1$. Indeed, we observe that if (1) holds, the Euler's representation of hypergeometric function ([17], p 114) yields

$$o < {}_2F_1 \begin{bmatrix} \nu+1, \nu+1; \\ \beta+2\nu+2n+2; \end{bmatrix} -\nu \leq 1 \quad \text{for all } \nu \geq o \tag{4.7}$$

which with (1.11) implies that

$$Q_\nu^{(o,\beta+2n)}[1+2\nu^{-1}] \geq o \quad \text{for all } \nu \geq o\,. \tag{4.8}$$

Hence, using this with (4.7) we see that if (1)-(2) hold, the integral (4.6) exists for each $t > o$ and for all $n \geq k$.

Now to complete the theorem it remains to obtain (4.2) and (4.3). Using (1.9) we get

$$P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(\cdot)}[F]\right] = A(k) \int_o^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, dx\, \frac{1}{k!}\left(\frac{k}{t}\right)^{k+1} \int_o^\infty e^{-\frac{k}{t}u} u^k F(u)\, du$$

$$= \int_o^\infty F(yt)\, dy\, \frac{k^{k+1}}{k!} A(k) y^k \int_o^\infty x^\nu Q_\nu^{(o,\beta,+2n)}[1+2x^{-1}]\, e^{-kxy}\, dx. \tag{4.9}$$

Evaluating by a known result ([18], p. 212) the inner integral

$$\frac{k^{k+1}}{k!} A(k) \int_o^\infty y^k x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, e^{-kxy}\, dx$$

$$= \frac{\Gamma(\nu+1)A(k)}{2\Gamma(k-1)\Gamma(\beta+\nu+2n+1)} \; \frac{E\begin{bmatrix} \beta+\nu+2n+1, \\ \beta+\nu+2n+1 \end{bmatrix}::ky}{(ky)^{\beta+2\nu+2n-k+2}} \tag{4.10}$$

provided $\beta + 2\nu+2n + 2 > o$ and $\beta + \nu$ is not an integer, where E[ ] is a MacRobert function [19]. Then (4.2) follows when the right hand side of (4.10) with the integral representation of E function ([19], p. 348), is placed on (4.10) and (4.3) follows easily from (4.2).

To complete the proof, we must justify the change in the order of integration performed in (4.9). For this, we show that the following iterated integrals exist.

$$\frac{k^{k+1}}{k!} \int_o^r y^k |F(yt)|\, dy\, A(k) \int_o^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, e^{-kky}\, dx, \tag{4.11}$$

$$\frac{k^{k+1}}{k!} \int_R^\infty y^k |F(yt)|\, dy\, A(k) \int_o^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, e^{-kxy}\, dx, \tag{4.12}$$

$$\frac{k^{k+1}}{k!} \int_r^R y^k |F(yt)|\, dy\, A(k) \int_o^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, e^{-kxy}\, dx. \tag{4.13}$$

Using the relation $e^{-z} < z^{-M}$ for every positive M, we see that(as in the proof of (4.6)) if (1)-(2) together with $n \geq k$ are satisfied and if $k-M > o$, (4.11) exists. Similarly, (4.12) exists if we choose M such that $M > k$ and $o < M-k \leq 1$ and (4.13) exists if $M = k$.

LEMMA 4.1. If $F(t) \in M(o,\infty)$ and if $k_1-k_2 > o$ with $k_1 > o$ and $o < k_2 \leq 1$, then the function

$$G(x) = \int_{c_1}^{c_2} |F(xt) - F(t)| \; dt, \quad c_1, c_2 \text{ real and } o < c_1 < c_2, \tag{4.14}$$

satisfies the following order properties

$$G(x) = \begin{cases} O(x^{-k_1-1}) & x \to o \\ O(1) & r \leq x \leq R \\ O(x^{k_1}) & x \to \infty \end{cases} .$$

PROOF: For every finite r and R such that $o < r < R$, we set

$$\zeta(x,r) = \int_x^r |F(t)| \; dt \text{ and } \zeta(R,y) = \int_R^y |F(t)| \; dt .$$

Then, for all positive k_1 and k_2 such that $k_1 > o$ and $o < k_2 \leq 1$

$$\zeta(x,r) \leq x^{-k_1} \int_x^r t^{k_1} |F(t)| \; dt$$

and

$$\zeta(R,y) \leq y^{k_2} \int_R^y t^{-k_2} |F(t)| \; dt.$$

Since

$$G(x) \leq \int_o^\infty |F(xt) - F(t)| \; dt$$
$$= (1+x^{-1}) \int_o^\infty |F(t)| \; dt,$$

it is sufficient to replace the last integral by the following limits

$$\lim_{x\to o} [\zeta(x,r) + \zeta(r,r) + \zeta(R,x^{-1})]$$

and

$$\lim_{x\to\infty} [\zeta(x^{-1},r) + \zeta(r,R) + \zeta(R,x)]$$

from which the first and third properties follow provided $k_1-k_2 > o$.

The second property is obvious from (4.14).

LEMMA 4.2. If $G(x) = O(x^{k_1})$ as $x \to \infty$ for every $k_1 > o$ and if the condition (1) of theorem 4.1 with $\nu - k_1 > o$ holds, then for all $n \geq k$ and for some $\eta > o$

$$\lim_{n\to\infty} \int_{1+\eta}^\infty H_n(x) \, G(x) \, dx = o,$$

where $H_n(x)$ and $G(x)$ are given by (4.2) and (4.14) respectively.

PROOF. Let $\psi(y) = \frac{1}{1+y}$ and $g(y) = \log \frac{y}{1+y} - \frac{kxy}{n}$.

If $h_x = \frac{1}{2x}\left[- x + (x^2 + \frac{4nx}{k})^{\frac{1}{2}}\right]$, it can readily verified that

$$g'(h_x) = o \text{ and } g''(h_x) < o .$$

If the condition (1) of Theorem 4.1 is satisfied, we may apply theorem 2b ([10], pp. 278) to the function $H_n(x)$ to get

$$|H_n(x)| \le \left[\frac{h_x e^{-\frac{kx}{n}h_x}}{1+h_x}\right]\left[\frac{-\pi}{2ng''(h_x)}\right]^{\frac{1}{2}} |\psi(h_x)| \tag{4.15}$$

Therefore,

$$\int_{1+\eta}^{\infty} H_n(x)\, G(x)\, dx \le \left(\frac{\pi}{2n}\right)^{\frac{1}{2}} \int_{1+\eta}^{\infty} x^{k-\nu-1}\left[\frac{h_x e^{-\frac{kx}{n}h_x}}{1+h_x}\right]^n \frac{|\psi h_x)|}{(-g''(h_x))^{\frac{1}{2}}}\, G(x)\, dx.$$

If $n \ge k$, we observe that the function

$$\frac{x^{k/n} h_x}{1+h_x}\, e^{-\frac{kx}{n}h_x}$$

is strictly decreasing for $x > 1$. Indeed,

$$\frac{d}{dx}\left[\frac{x^{k/n} h_x}{1+h_x}\, e^{-\frac{hx}{n}h_x}\right] = \frac{x^{k/n}h_x}{1+h_x}\, e^{-\frac{hx}{n}h_x}\left\{h_x' \quad \frac{1}{h_x(1+h_x)} - \frac{kx}{n} + \frac{k}{nx}(1 - rh_x)\right\}$$

where the expression inside the first brace is always zero and that in the second brace is negative for all $x > 1$.

Since

$$\frac{|\psi(h_x)|}{(-g''(h_x))^{\frac{1}{2}}} \to o \quad \text{as} \quad x \to \infty ,$$

given as $\varepsilon > o$ there exists a $\bar{x}$ such that for all $x \ge \bar{x} > 1$

$$\frac{|\psi(h_x)|}{(-g''(h_x))^{\frac{1}{2}}} < \varepsilon.$$

Therefore, writing x_o for $1+\eta$, we have, if $\nu - k_1 > o$,

$$\int_{1+\eta}^{\infty} |H_n(x)|\, G(x)\, dx \le \varepsilon\left(\frac{\pi}{2n}\right)^{\frac{1}{2}} \frac{x_o^{k/n} h_{x_o}}{1+h_{x_o}}\left[e^{-\frac{kx_o}{n}h_{x_o}}\right]^n \int_{1+\eta}^{\infty} G(x)\, x^{-\nu-1}\, dx$$

which tends to zero as $n \to \infty$. This completes the proof.

LEMMA 4.3. If $G(x) = o(x^{-k_1-1})$ as $x \to o+$ for every $k_1 > o$ and if (1) and (2) of theorem 4.1 are satisfied, then

$$\lim_{n\to o} \int_{o}^{1-\eta} H_n(x)\, dx = o.$$

PROOF. Since $o < \frac{h_x}{1+h_x} e^{-\frac{kx}{n}h_x} < 1$ for all $x \ge o$, we have by the ineguqlity (4.15)

$$\int_{o}^{1-\eta} |H_n(x)|\, G(x)\, dx < \left(\frac{\eta}{2n}\right)^{\frac{1}{2}} \sup_{o\le x\le 1}\left[\frac{h_x}{1+h_x} e^{-\frac{kx}{n}h_x}\right]^n \int_{o}^{1-\eta} \frac{|\psi(h_x)|}{(-g''(h_x))^{\frac{1}{2}}}\, x^{k-\nu-1} G(x)\, dx.$$

By the estimation $\frac{|\psi(h_x)|}{(-g''(h_x))^{\frac{1}{2}}} = o(x^{-\frac{1}{4}})$ as $x \to o+$, the right hand side of the above inequality tends to zeιo as $n \to \infty$ provided the condition (2) holds.

THEOREM 4.2. Let $F(t) \in M(o,\infty)$ with r, R, k_1 and k_2 as defined in definition 4.1. If the conditions (1) and (2) of theorem 4.1 together with $\nu - k > o$, $k_1 - k_2 > o$ and $\beta + \nu$ is not an integer hold, then for all $n \ge k$.

$$\lim_{n\to\infty} P^{(\beta,\nu)}_{n,t}\left[L^{(.)}_{k,t}[F]\right] = F(t)$$

for almost all $t > o$.

PROOF. If conditions (1) and (2) of theorem 4.1 together with $k_1 > o$ and $o < k_2 \le 1$ are satisfied, by the same theorem the real inversion operator (1.7) with $r = o$ exists for each $t > o$. By (4.3), there exists a positive integer n_o such that

$$\int_o^\infty H_n(x)\, dx < 1 \quad \text{for all} \quad n \ge n_o.$$

Then, assuming $P^{(\beta,\nu)}_{n,t}\; L^{(.)}_{k,t}[F]$ and $F(t)$ have the same sign at least in the interval $o < c_1 < t < c_2$

$$\left| |P^{(\beta.\nu)}_{n,t}\; L^{(.)}_{k,t}[F]| - |F(t)| \right| < \int_o^\infty H_n(x)\, |F(xt) - F(t)|\, dx.$$

Therefore,

$$\int_{c_1}^{c_2} |P^{(\beta,\nu)}_{(n,t)}\; L^{(.)}_{k,t}[F] - F(t)|\, dt$$

$$< \int_o^\infty H_n(x)\, dx \int_{c_1}^{c_2} |F(xt) - F(t)|\, dt$$

$$= \int_o^\infty H_n(x)\, G(x)\, dx$$

$$= \left(\int_o^r + \int_r^R + \int_R^\infty\right) H_n(x)\, G(x)\, dx\,.$$

If $r < 1$ and $R > 1$, from lemma 4.2 and lemma 4.3 it follows that the first and third integrals tend to zero as $n \to \infty$. By the second conclusion of lemma 4.1 and by (4.3), the second integral tends to zero as $n \to \infty$. This completes the theorem.

Moreover, the uniqueness of the real inversion operator (1.7) with $r = o$ is implicit in theorem 4.2. In other words, we have:

THEOREM 4.3. If the conditions stated in theorem 4.2 hold, if both $F(t)$ and $G(t)$ belong to $M(o,\infty)$ and if their respective real inversion operators $P^{(\beta,\nu)}_{n,t}\left[L^{(.)}_{k,t}[F]\right]$ and $P^{(\beta,\nu)}_{n,t}\left[L^{(.)}_{k,t}[G]\right]$ are equal, then $F(t) = G(t)$ for almost all $t > o$.

PROOF. For all those values of t for which the theorem 4.2 hold

$$F(t) = \lim_{n\to\infty} P^{(\beta,\nu)}_{n,t}\left[L^{(.)}_{k,t}[F]\right] = \lim_{n\to\infty} P^{(\beta,\nu)}_{n,t}\left[L^{(.)}_{k,t}[G]\right] = G(t)\,.$$

Other results that we shall need subsequently are given by the following theorems which are fundamental to the representation theory that will be established in the next section.

THEOREM 4.4. Let $F(t) \in M(o,\infty)$ and let $k_1 > o$ and $o < k_2 \le 1$. Then for all $k_1, k_2 < k$, the first operate of the Post-Widder inversion operator

$$L_{k,t}\, L_{k,.}\, [F] = \int_o^\infty h_k(u,t)\, L_{k,u}\, [F]\, du \tag{4.16}$$

exists for each $t > o$, where $h_k(u,t)$ is given by (1.5.). If. in addition, conditions (1) and (2) of theorem 4.1 are satisfied, then for all $n \ge k$ the real inversion operator

$$P_{n,t}^{(\beta,\nu)} L_{k,t}^{(.)} L_{k,.}[F] = A(k) \int_0^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}] L_{k,t}^{(x)} L_{k,.}[F]dx \quad (4.17)$$

exists for each $t > o$, where $L_{k,t}^{(x)} L_{k,.}[F]$ is defined by (1.9) with $r = 1$, and

$$\int_0^\infty h_k(u,t) P_{n,u}^{(\beta,\nu)} \left[L_{k,u}^{(.)}[F]\right] du = P_{n,t}^{(\beta,\nu)} \left[L_{k,t}^{(.)} L_{k,.}[F]\right]. \quad (4.18)$$

PROOF. It is a simple calculation to show that $L_{k,t}[F]$ belongs to $M(o,\infty)$. Then by theorem 4.1, the first operate of the Post-Widder inversion operator defined by (4.16) exists for each $t > o$ and by the same theorem, the real inversion operator defined by (4.17) exists for each $t > o$.

By formal change of order of integration twice we have

$$\int_0^\infty h_k(u,t) P_{n,u}^{(\beta,\nu)}\left[L_{k,u}^{(.)}[F]\right]du$$

$$= \int_0^\infty h_k(u,t)\, du\, A(k) \int_0^\infty v^\nu Q_\nu^{(o,\beta+2n)}[1+2v^{-1}] L_{k,u}^{(v)}[F]\, dv$$

$$= \int_0^\infty h_k(u,t)\, du\, A(k) \int_0^\infty v^{\nu-k-1} Q_\nu^{(o,\beta+2n)}[1+2v^{-1}] L_{k,uv^{-1}}[F]\, dv,$$

$$\left[v^{k+1} L_{k,u}^{(v)}[F] = L_{k,uv^{-1}}[F]\right]$$

$$= \int_0^\infty h_k(u,t)\, du\, A(k) \int_0^\infty (us)^{\nu-k} Q_\nu^{(o,\beta+2n)}[1+2(us)^{-1}]\, s^{-1} L_{k,s^{-1}}[F]\, ds$$

$$(v = us)$$

$$= \int_0^\infty s^{-1} L_{k,s^{-1}}[F]\, ds\, A(k) \int_0^\infty (us)^{\nu-k} Q_\nu^{(o,\beta+2n)}[1+2(us)^{-1}] h_k^{(us)}(s^{-1},t)\, du,$$

$$[(us)^{-k} h_k(u,t) = h_k^{(us)}(s^{-1},t)]$$

$$= \int_0^\infty s^{-2} L_{k,s^{-1}}[F]\, ds\, A(k) \int_0^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, h_k^{(x)}(s^{-1},t)\, dx,$$

$$(us = x)$$

$$= A(k) \int_0^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]dx \int_0^\infty h_k^{(x)}(s^{-1},t)\, s^{-2} L_{k,s^{-1}}[F]\, ds$$

$$= A(k) \int_0^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\, dx \int_0^\infty h_k^{(x)}(y,t) L_{k,y}[F]\, dy,$$

$$(s^{-1} = y)$$

$$= A(k) \int_0^\infty x^\nu Q_\nu^{(o,\beta+2n)}[1+2x^{-1}] L_{k,t}^{(x)} L_{k,.}[F]\, dx.$$

Thus we have established (4.18). Now it remains to justify the change in the order of integration performed above. The second change of order of integration follows since (4.17) exists. For the first change of order of integration, we note that

$$\int_{r^{-1}}^{\infty} s^{-k_1-2} L_{k,s^{-1}}[F]\,ds = \int_{o}^{r} s^{k_1} L_{k,s}[F]\,ds$$

$$\int_{R^{-1}}^{r^{-1}} s^{-2} L_{k,s^{-1}}[F]\,ds = \int_{r}^{R} L_{k,s}[F]\,ds$$

$$\int_{o}^{R^{-1}} s^{k_2-2} L_{k,s^{-1}}[F]\,ds = \int_{R}^{\infty} s^{-k_2} L_{k,s}[F]\,ds .$$

By the fact that $L_{k,s}[F] \in M(o,\infty)$, the function $s^{-2} L_{k,s^{-1}}[F]$ $M(o,\infty)$. Hence the proof of the theorem will be complete if we show that the following iterated integrals exist

$$\int_{o}^{R^{-1}} s^{k_2-2} L_{k,s^{-1}}[F]\,ds\, A(k)\int_{o}^{\infty} x^{\nu} Q_{\nu}^{(o,\beta+2n)}[1+2x^{-1}] s^{-k_2} h_k^{(x)}(s^{-1},t)\,dx$$

$$\int_{R^{-1}}^{r^{-1}} s^{-2} L_{k,s^{-1}}[F]\,ds\, A(k)\int_{o}^{\infty} x^{\nu} Q_{\nu}^{(o,\beta+2n)}[1+2x^{-1}] h_k^{(x)}(s^{-1},t)\,dx$$

$$\int_{o}^{\infty} s^{-k_1-2} L_{k,s^{-1}}[F]\,ds \int_{o}^{\infty} A(k)\, x^{\nu} Q_{\nu}^{(o,\beta+2n)}[1+2x^{-1}]\, s^{k_1} h_k^{(x)}(s^{-1},t)\,dx. \tag{4.21}$$

If conditions (1) and (2) of theorem 4.1 together with $n \geq k$ are satisfied, (4.19), (4.20) and (4.21) exists for each $t > o$ in the same way as the integrals (4.11), (4.12) and (4.13) exist for each $t > o$ if $N = k+k_2$, k and $k-k_1$ respectively.

THEOREM 4.5. If the hypothesis of theorem 4.2 are satisfied, then

$$\lim_{n\to\infty} \int_{o}^{\infty} h_k(u,t)\, P_{n,u}^{(\beta,\nu)}\left[L_{k,u}^{(.)}[F]\right] du = L_{k,t}[F] .$$

PROOF. By Theorem 4.4, $L_{k,t}[F] \in M(o,\infty)$ and $L_{k,t}\, L_{k,.}[F]$ exists for each $t > o$. Then by the same theorem

$$\int_{o}^{\infty} h_k(u,t)\, P_{n,u}^{(\beta,\nu)}\left[L_{k,u}^{(.)}[F]\right] du = P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(.)}\, L_{k,.}[F]\right] .$$

By Theorem 4.2

$$\lim_{n\to\infty} \int_{o}^{\infty} h_k(u,t)\, P_{n,u}^{(\beta,\nu)}\left[L_{k,u}^{(.)}[F]\right] du$$

$$= \lim_{n\to\infty} P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(.)}\, L_{k,.}[F]\right] = L_{k,t}[F]$$

for almost all $t > o$.

5. REPRESENTATION THEOREMS. In this section we obtain the necessary and sufficient conditions for the representation of a given function as the rth operate of the Post-Widder inversion operator of certain class of generalised functions. For this purpose, we first obtain the necessary and sufficient conditions for a given function to be the

Post-Widder inversion operator of a class of generalised functions from which the general case easily follows.

THEOREM 5.1. Let $\Omega_t = \{t\ ,\ a < \frac{k}{t} < b$ with $o \le a < b < \infty\}$ and let the conditions (1) and (2) of theorem 4.1 together with $k_1 - k_2 > o, \nu - k_1 > o$, $n \ge k$ and $\beta + \nu$ is not an integer be satisfied. Then the necessary and sufficient conditions for the Post-Widder inversion operator $L_{k,t}[F]$ with $F \in M(o,\infty)$ to have the representation as

$$L_{k,t}[F] = \langle F(u),\ h_k(u,t) \rangle$$

of some regular generalized functions is that it should be bounded for every t in Ω_t.

PROOF. By the boundedness property of generalised functions, there exist a constant C and a positive integer n_o such that

$$L_{k,t}[F] = \langle F(u),\ h_k(u,t) \rangle$$

$$\le C \sup_{-\infty < u < \infty}\ \max_{o \le n \le n_o} \gamma_{a,b,n}(h_k(u,t)),$$

where $\gamma_{a,b,n}(\cdot)$ is defined by (2.4).

By lemma 3.1

$$L_{k,t}[F] < C \sup_{-\infty < u < \infty}\ \max_{o \le n \le n_o}\ e^{-2n(\frac{nbu}{k}+1)} (\tfrac{n}{t}u+1)\ e^{-(\frac{k}{t}-a)u}$$

$$< C \sup_{-\infty<u<\infty}\ \max_{o \le n \le n_o}\ e^{-2n[(\frac{n}{t}u+1)-\log(\frac{n}{t}u+1)]} (\tfrac{n}{t}u+1) e^{-(\frac{k}{t}-a)u}$$

from which the necessary part follows.

Sufficiency. To establish the sufficient part we assume that $L_{k,t}[F]$ is bounded for each t in Ω_t. Then we show that there exists a generalized function F such that

$$L_{k,t}[F] = \langle F(u),\ h_k(u,t) \rangle .$$

We further assume that F is a regular generalised function so that

$$L_{k,t}[F] = \int_o^\infty h_k(u,t)\ F(u)\ du.$$

By Theorem 4.2.

$$F(t) = \lim_{n\to\infty} P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(\cdot)}[F]\right].$$

Now to complete the theorem we must show that the real inversion operator $P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(\cdot)}[F]\right]$ is bounded and continuous for all $n \ge k$ and every $t > o$. Then it follows that $F(t)$ is bounded and continuous for every $t > o$.

Let ${}_o\Omega_y$ and ${}_\infty\Omega_y$ be the sets of $y > o$ satisfying $y \le \frac{k}{b}$ and $y \ge \frac{k}{a}$. Then

$$P_{n,t}^{(\beta,\nu)}\ L_{k,t}^{(\cdot)}[F] = \sup_{y\in\Omega_y} |L_{k,y}[F]\ \int_o^\infty A(k) y^{-1} (t/y)^{\nu-k}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}]\ dy$$

$$+ \int_{{}_o\Omega_y} A(k)\ y^{-1}\ (t/y)^{\nu-k}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}]\ L_{k,y}[F]\ dy \qquad (5.1)$$

$$+ \int_{{}_\infty\Omega_y} A(k)\ y^{-1}\ (t/y)^{\nu-k}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}]\ L_{k,y}[F]\ dy.$$

We observe that if conditions (1) and (2) of Theorem 4.1 together with $k_1 > o$ and $o < k_2 \le 1$ are satisfied,

$$\left.\begin{aligned} A(k)\ (t/y)^{\nu-k+k_1+1}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}] &\to o \quad \text{as} \quad y \to \infty \\ A(k)\ (t/y)^{\nu-k-k_2+1}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}] &\to o \quad \text{as} \quad y \to o \end{aligned}\right\} \tag{5.2}$$

for all $n \ge k$.

Then by the fact that $L_{k,t}[F] \in M(o,\infty)$, the second and third integrals exist for each $t > o$ and all $n \ge k$. Furthermore, if the conditions (1) and (2) of Theorem 4.1 hold the first integral exists for each $t > o$ and all $n \ge k$.

To prove the continuity, we assume that $a < \frac{k\,x}{t+h} < \frac{kx}{t} < b$ for some $h > o$ so that $L_{k,(t+h)x^{-1}}[F]$ and $L_{k,tx^{-1}}[F]$ are bounded. We denote by $\bar{\Omega}_x$ the set of x satisfying the above inequality and by ${}_o\bar{\Omega}_x$, and ${}_{-\infty}\bar{\Omega}_x$, the sets of $x > o$ such that $x \le \frac{a(t+h)}{k}$ and $x \ge \frac{tb}{k}$ respectively. Then

$$\left| P_{n,t+h}^{(\beta,\nu)}\left[L_{k,t+h}^{(\cdot)}[F]\right] - P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(\cdot)}[F]\right]\right|$$

$$\le A(k)\int_o^\infty x^\nu\, Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\,\left|L_{k,t+h}^{(x)}[F] - L_{k,t}^{(x)}[F]\right|\,dx$$

$$= A(k)\int_o^\infty x^{\nu-k-1}\, Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\,\left|L_{k,(t+h)x^{-1}}[F] - L_{k,tx^{-1}}[F]\right|\,dx$$

$$\le \sup_{x\in\bar{\Omega}_x}\left|L_{k,(t+h)x^{-1}}[F] - L_{k,tx^{-1}}[F]\right|\int_o^\infty A(k)\ x^{\nu-k-1} Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\ dx$$

$$+\left(\int_{{}_o\bar{\Omega}_x} + \int_{{}_\infty\bar{\Omega}_x}\right)x^\nu\, Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\,\left|L_{k,t+h}[F] - L_{k,t}^{(x)}[F]\right|\,dx$$

$$= I_1 + (I_2+I_3).$$

Obviously, I_1 tends to zero as $h \to o$ since the integral exists by virtue of (4.6) with $A_i = -1$ for all i. We only show that I_2 tends to zero as $h \to o$ since I_3 similarly follows. For this, we consider

$$\left|L_{k,t+h}^{(x)}[F] - L_{k,t}^{(x)}[F]\right| \le \sup_{o\le u<\infty}\left[1 - \frac{e^{-\frac{kxuh}{t(t+h)}}}{(t/(t+h))^{k+1}}\right]\int_o^\infty h_k^{(x)}(u,t+h)\,|F(u)|\ du$$

$$= \sup_{o\le u<\infty}\left[1 - \frac{e^{-\frac{kxuh}{t(t+h)}}}{(t/(t+h))^{k+1}}\right] L_{k,t+h}^{(x)}|[F]|.$$

Then clearly, using (4.8), we have

$$I_2 \le \sup_{x\in{}_o\bar{\Omega}x}\ \sup_{o\le u<\infty}\left[1 - \frac{e^{\frac{kxuh}{t(t+h)}}}{(t/(t+h))^{k+1}}\right] A(k)\int_o^\infty x^\nu\, Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\ L_{k,t+h}^{(x)}|[F]|\ dx$$

$$= \sup_{x\in o\bar{\Omega}x} \sup_{o\le u<\infty} \left[1 - \frac{e^{-\frac{kxuh}{t(t+h)}}}{(t/(t+h))^{k+1}}\right] P^{(\beta,\nu)}_{n,t+h}\left[L^{(\cdot)}_{k,t+h}\,|[F]|\right]$$

which tends to zero as $h \to o$ since $P^{(\beta,\nu)}_{n,t+h}\left[L^{(\cdot)}_{k,t+h}|[F]|\right]$ exists for each $t > o$ by Theorem 4.1. This completes the proof of the theorem.

DEFINITION 5.1. Denote by $\bar{M}(o,\infty)$ the class of functions F satisfying the first two conditions of definition 4.1.

DEFINITION 5.2. A function $F(t)$ belongs to the class of functions $M^*(o,\infty)$ if it belongs to $\bar{M}(o,\infty)$ and if for some negative real number A such that $-k_1-1 < A < k_2-1$ with $k_1 > o$ and $o < k_2 < 1$ and for some constant C it satisfies

$$|F(t)| \le Ct^A \text{ for every } t \text{ in } \Omega_t, \qquad (5.3)$$

where Ω_t is defined in Theorem 5.1.

Clearly, the class $M(o,\infty)$ is a subclass of $M^*(o,\infty)$.

A more general result is contained in the following theorem. The proof is based on the fact that every generalized function is of finite order distributional derivative of continuous function.

THEOREM 5.2. Let Ω_t be defined as in Theorem 5.1 and let the conditions (1)-(2) of Theorem 4.1 together with $k_1-k_2 > o$, $\nu - k_1 > o$, $n > k(\frac{k}{b})^{\frac{1}{4}}$ and $\beta + \nu$ is not an integer be satisfied. Then a necessary and sufficient condition for a function $L_{k,t}[F]$ to have the representation as the Post-Widder inversion operator

$$L_{k,t}[F] = \langle F(u), h_k(u,t) \rangle$$

of certain class of generalized functions is that it has derivatives of all orders and satisfies

$$|L_{k,t}[F]| \le P(t^{-1}) \qquad \text{for every } t \text{ in } \Omega_t, \qquad (5.4)$$

where $P(t^{-1})$ is a polynomial in t^{-1}.

PROOF. Let $F \in \mathcal{Y}'_{a,b,n}$. Then the first part of the necessity follows from Theorem 3.1 and the second part follows as in Theorem 5.1 except that we have to use $n > k(\frac{k}{b})^{\frac{1}{4}}$.

For the sufficiency we need the following lemmas.

LEMMA 5.1. Let $L_{k,t}[F] \in M^*(o,\infty)$. Then the first operate of the Post-Widder inversion operator defined by

$$L_{k,t}L_{k,\cdot}[F] = \int_o^\infty h_k(u,t)L_{k,u}[F] \quad du \qquad (5.5)$$

exists for each $t > o$ and belongs to $M^*(o,\infty)$ for some fixed $k > k_1, k_2 > o$.

PROOF. By Theorem 4.1, $L_{k,t}[F]$ exists for each $t > o$ and has Post-Widder inversion operator. That is, the first operate of the Post-Widder inversion operator exists for each $t > o$ and for all $k > k_1, k_2$ with $k_1 > o$ and $o < k_2 < 1$. Now with the same a of definition of Ω_t, we have

$$|L_{k,t}\, L_{k,.}[F]| \le \int_{\Omega_u} h_k(u,t)\, u^A\, du + \int_{\infty\Omega_u} h_k(u,t)\, L_{k,u}[F]\, du + \int_{o\Omega_u} h_k(u,t)\, L_{k,u}[F]\, du$$

$$\le Ct^A + \int_{\infty\Omega_u} h_k(u,t,k_1\, u^{k_1}\, L_{k,u}[F]\, du + \int_{o\Omega_u} h_k(u,t,k_2)\, u^{-k_2}\, L_{k,u}[F]\, du$$

$$\le Ct^A + t^{-k_1-1}\, D(k,k_1) \int_{o\Omega_u} u^{k_1}\, L_{k,u}[F]\, du + t^{-k-1} \frac{e^{-k}\, D(k,k_2)}{(\frac{1}{t} - \frac{a}{k})^{k+k_2}} \int_{\infty\Omega_u} u^{-k_2}\, L_{k,u}[F]\, du, \tag{5.6}$$

where $D(k,k_1)$ and $D(k,k_2)$ are given by given by (4.5) and $h_k(u,t,k_1)$ and $h_k(u,t,k_2)$ are respectively defined by

$$h_k(u,t,k_1) = \frac{k^{k+1}}{k!}\, t^{-k-1}\, e^{\frac{-ku}{t}}\, u^{k-k_1}$$

and

$$h_k(u,t,k_2) = \frac{k^{k+1}}{k!}\, t^{-k-1}\, e^{\frac{-k_u}{t}}\, u^{k+k_2}$$

with their obvious properties

$$\begin{aligned} \int_o^\infty t^{k_1}\, h_k(u,t,k_1)\, dt &= \int_o^\infty t^{k_1}\, h_k(u,t,k_1)\, du \\ \int_o^\infty t^{-k_2}\, h_k(u,t,k_2)\, dt &= \int_o^\infty t^{-k_2}\, h_k(u,t,k_2)\, du \end{aligned} \tag{5.7}$$

where the integrals are bounded for some fixed $k > k_1,k_2 > o$ and tend to unity as $k \to \infty$.

If $t \in \Omega_t$, $-k_1-1 < A \le o$ and $k > k_1,k_2 > o$ fixed, then from the fact that $L_{k,t}[F] \in M^*(o,\infty)$ it readily follows that the first operate of the Post-Widder inversion operator satisfies (5.3). If $a = o$, $\infty\Omega_u$ becomes a null set and (5.5) still satisfies (5.3).

Using (5.6) and (5.7) it is easy to show that $L_{k,t}\, L_{k,.}[F]$ belongs to $\bar{M}(o,\infty)$ if $-k_1-1 < A < k_2-1$ for all $k > k_1,k_2 > o$ fixed.

Then the lemma follows.

LEMMA 5.2. If in addition to the hypothesis of Lemma 5.1, conditions (1)-(2) of Theorem 4.1 together with $n > k$ are satisfied, then the real inversion operator $P^{(\beta,\nu)}_{n,t}\, L^{(\cdot)}_{k,t}[F]$ defined by (1.7) with $r = o$ exists for each $t > o$ and belongs to $M^*(o,\infty)$.

PROOF. If conditions (1)-(2) of Theorem 4.1 together with $k_1 > o$ and $o < k_2 < 1$ are satisfied, then using (5.2) and the fact that $L_{k,t}[F] \in M^*(o,\infty)$, we see that the real inversion operator $P_{n,t}^{(\beta,\nu)}\ L_{k,t}^{(\cdot)}[F]$ exists for each $t > o$ and all $n > k$. To show that it also satisfies (5.3), in view of (5.1) we need only show that the third integral in (5.1) is majorised by a constant multiple of t^{-k-1}.

If $a = o$, the assertion is obvious from (5.1). Let $a \neq o$. We observe that for any $p > o$ and $t \in \Omega_t$, the function $(ty)^{\frac{p}{2}}/(y+t)^p$ is decreasing for all $y > t$. Hence

$$\int_{\infty\Omega_y} A(k)\ y^{-1}(t/y)^{\nu-k}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}]\,|L_{k,y}[F]|\ dy$$

$$\leq \frac{(\frac{k}{a}t)^{p/2}}{(\frac{k}{a}-t)^p}\int_{\infty\Omega_y} A(k)\ y^{-1}(t/y)^{\nu-k}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}](1+\frac{t}{y})^p\ L_{k,y}[F]\quad dy$$

$$\leq \frac{t^{k_2-1}(\frac{k}{a}t)^{p/2}}{(\frac{k}{a}-t)^p}\int_{\infty\Omega_y} A(k)(t/y)^{\nu-k-k_2+1-p/2}\ Q_\nu^{(o,\beta+2n)}[1+2(t/y)^{-1}](1+\frac{t}{y})^p\ y^{-k_2}\ L_{k,x}[F]\ \ dy.$$

The quantity inside the first brace is bounded for all $y \geq \frac{k}{a}$ and t in Ω_t provided $\beta+2n-p > o$, $n > k$ and $n+1-p/2 > o$. Now the assertion follows by taking $p = 2(k_1+k_2)$. Again, if conditions (1)-(2) of Theorem 4.1 with $k_1 > o$ and $o < k_2 < 1$ are satisfied, then using (4.6) it readily follows by straightforward manipulations that the real inversion operator $P_{n,t}^{(\beta,\nu)}\left[L_{k,t}^{(x)}\ [F]\right]$ belongs to $M^*(o,\infty)$ provided $A < k_2-1$. This completes the proof of the lemma.

We next turn to the sufficiency part.

Sufficiency. Consider a function $G_{k,t}[g]$ defined by

$$G_{k,t}[g] = \int_o^\infty h_k(u,t)\ g(u)\ du \qquad \text{with } g \in M^*(o,\infty) \tag{5.8}$$

which has derivatives of all orders and satisfies (5.3). Then, If conditions (1)-(2) of Theorem 4.1 with $k_1 > o$ and $o < k_2 < 1$ are satisfied, the real inversion operator defined by

$$P_{n,t}^{(\beta,\nu)}\left[G_{k,t}^{(\cdot)}[g]\right] = A(k)\int_o^\infty x^\nu\ Q_\nu^{(o,\beta+2n)}[1+2x^{-1}]\ G_{k,t}^{(x)}[g]\quad dx \tag{5.9}$$

exists for each $t > o$ and all $n \geq k$. If in addition $\nu-k_1 > o, k_1-k_2 > o$ and $\beta + \nu$ is not an integer, then as in Theorem 5.1, $g(t)$ is continuous and bounded for each $t > o$.

By Lemma 5.2, the real inversion operator defined by (5.9) satisfies (5.3), and hence belongs to $M^*(o,\infty)$. Then by Theorem 4.1, it has Post-Widder inversion operator. We have yet to verify that this Post-Widder inversion operator is $G_{k,t}[g]$.

By Lemma 5.1, the first operate of $G_{k,t}[g]$ on itself, that is, $G_{k,t}\ G_{k,.}[g]$ exists for each $t > o$ and belongs to $M^*(o,\infty)$. Then by Lemma 5.2, the real inversion operator $P_{n,y}^{(\beta,\nu)}\left[G_{k,t}^{(\cdot)}\ G_{k,.}\ [g]\right]$ exists for each $t > o$ and belongs to $M^*(o,\infty)$. Moreover, by Theorem 4.4

$$\int_o^\infty h_k(u,t)\ P_{n,u}^{(\beta,\nu)}\left[G_{k,u}^{(\cdot)}[g]\right]\ du = P_{n,t}^{(\beta,\nu)}\left[G_{k,t}^{(\cdot)}\ G_{k,.}\ [g]\right].$$

Finally, the use of Theorem 4.5 and the uniqueness Theorem 4.3 show that the Post-Widder inversion operator of $P_{n,t}^{(\beta,\nu)}\left[G_{k,t}^{(\cdot)}[g]\right]$ is $G_{k,t}[g]$ for almost all $t > o$ and all $n \geq n_o$ some $n_o > k > k_1, k_2 > o$. This verifies the assertion made above.

Now, by (5.4), there exist a positive number d such that $\hat{\Omega}_t = \{t;\ o \leq a < \frac{k}{t} < d \leq b\}$, a negative number A satisfying $-k_1-1 < A < k_2-1$ and a constant C such that

$$\left|L_{k,t}[F]\right| \leq Ct^A \quad \text{for every } t \text{ in } \hat{\Omega}_t. \tag{5.10}$$

For any nonnegative integer m and $A \neq -1$. We set

$$L_{k,t}[F] = (tD_t)^m\, G_{k,t}[g],$$

so that $G_{k,t}[g]$ satisfies (5.3) for every t in $\hat{\Omega}_t$. Then Lemma 3.1 and Corollary 3.2 yield for every t in $\hat{\Omega}_t$

$$\begin{aligned} L_{k,t}[F] &= (tD_t)^m < g(u),\ h_k(u,t) > \\ &= < g(u),\ (tD_t)^m\, h_k(u,t) > \\ &= < g(u),\ (-uD_u-1)^m\, h_k^{(u,t)} > \\ &= < (uD_u)^m\, g(u),\ h_k(u,t) > \\ &= < \sum_{N=o}^{m} a_N(m)\, g^{(N)}(u),\ h_k(u,t) >, \end{aligned}$$

so that the generalized function F has the representation

$$F(u) = \sum_{N=o}^{m} a_N(m)\, g^{(N)}(u). \tag{5.11}$$

This form was obtained by Treves ([7], pp. 274) and Gelfand and Shilov ([6], pp. 117). This completes the proof of the theorem.

The following theorems provide further generalisations of Theorems 5.1 and 5.2.

THEOREM 5.3. Let all the hypothesis of Theorem 5.2 be satisfied. Then a necessary and sufficient condition for a function ${}^{r+1}L_{k,t}[F]$ with $L_{k,t}[F] \in \bar{M}(o,\infty)$ to have the representation as the rth operate of the Post-Widder inversion operator

$${}^{r+1}L_{k,t}[F] = \left\langle {}^{r}L_{k,u}[F],\ h_k(u,t)\right\rangle, \quad F \in \mathcal{J}'_{a,b,n}, \tag{5.12}$$

of regular generalized function in $\mathcal{J}'_{a,b,n}$ is that $L_{k,t}[F]$ has derivatives of all orders and satisfies

$$\left| L_{k,t}[F] \right| \leq P_q(t^{-1}) \qquad \text{for every } t \text{ in } \Omega_t, \tag{5.13}$$

where $P_q(t^{-1})$ is a polynomial of degree q.

PROOF. Let $F \in \mathcal{J}'_{a,b,n}$. Then by Corollary 3.3, ${}^{r}L_{k,u}[F]$, $r = 1,2,3,\ldots\ldots$ defined by(1.4) exist as regular generalized functions in $\mathcal{J}'_{a,b,n}$. Hence by Lemma 3.1, (5.12) has sense.

Necessity follows as in Theorem 5.2. For the sufficiency, in view of (1.4) we note that by repeated applications of Theorem 3.1 and of (5.6) on $L_{k,u}[F]$, it is easy to see that ${}^{r+1}L_{k,t}[F]$, $r = 0,1,2,3,\ldots\ldots$ has derivatives of all orders and satisfies (5.13) provided $1+k_1-q > o$ in case when the constant term of the polynomial is zero. Then, as in Theorem 5.2, there exist a subset $\hat{\Omega}_t \subseteq \Omega_t$, a negative number A satisfying $-k_1-1 < A < k_2-1$ and a constant C such that

$$|{}^{r+1}L_{k,t}[F]| \le Ct^{A} \quad \text{for every } t \text{ in } \hat{\Omega}_t. \tag{5.14}$$

Next, consider a function $G_{k,t}[g]$ as defined by (5.8) and then as in Theorem 5.2, the Post-Widder inversion operator of $P^{(\beta,\nu)}_{n,t}\ G^{(\cdot)}_{k,t}[g]$ is $G_{k,t}[g]$ for almost all $t > o$ and for all $n \ge$ some $n_o > k > k_1, k_2 > o$. For any non-negative integer m and $A \ne -1$, we set

$${}^{r+1}L_{k,t}[F] = (tD_t)^m\ G_{k,t}[g]$$

then $G_{k,t}[g]$ satisfies (5.3) for every t in $\hat{\Omega}_t$. Proceeding as in Theorem 5.2, we finally get the regular generalized functions, which has the representation

$${}^{r}L_{k,u}[F] = \sum_{N=o}^{m} a_N(m)\ g^{(N)}(u). \tag{5.15}$$

This completes the proof of the theorem.

In the preceding theorem the representation of functions by the rth operate of the Post-Widder inversion operator was restricted to the class of regular generalized functions. In the following theorem we extend it to the class of generalized functions in $\mathcal{J}'_{a,b,n}$. For this, we redefine the rth operate of the Post-Widder inversion operator in the form

$${}^{r+1}L_{k,t}[F] = \int_{o}^{\infty} {}^{r+1}h_k(u,t)\ F(u)\ du, \quad r = o,1,2,\ldots\ , \tag{5.16}$$

where

$$\left.\begin{aligned} {}^{r+1}h_k(u,t) &= \int_{o}^{\infty} {}^{r}h_k(u,v)\ h_k(v,t)\ dv \qquad r \ne o \\ &= h_k(u,t) \qquad\qquad\qquad\qquad\quad r \ne o \end{aligned}\right\} \tag{5.17}$$

and $h_k(u,t)$ is given by (1.5).

The relation (5.16) has much similarity with ([20], pp. 18) and the relation (5.17) is a form obtained by Widder ([10], pp. 263) for the iterated Stieltjes transform.

With the function $h_k(u,t)$ the following interesting properties are easy to prove. For any nonnegative integer r

$$\int_0^\infty {}^{r+1}h_k(u,t)\,dt = \int_0^\infty {}^{r+1}h_k(u,t)\,du = 1 \qquad (5.18)$$

$$\int_0^\infty {}^{r+1}h_k(u,v)\,h_k(v,t)\,dv = \int_0^\infty h_k(u,v)\,{}^{r+1}n_k(v,t)\,dv \qquad (5.19)$$

$$\int_0^\infty {}^{r+1}h_k(u,v)\,h_k^{(\cdot)}(v,t)\,dv = \int_0^\infty h_k^{(\cdot)}(u,v)\,{}^{r+1}h_k(v,t)\,dv \qquad (5.20)$$

$$\int_0^\infty {}^{r+1}h_k^{(\cdot)}(u,v)\,h_k(v,t)\,dv = \int_0^\infty h_k(u,v)\,{}^{r+1}h_k^{(\cdot)}(v,t)\,dv \qquad (5.21)$$

$$\int_0^\infty {}^{r+1}h_k(u,v)\,u^{-\alpha}du \le M(k)\,v^{-\alpha}, \qquad (5.22)$$

where α is any positive number less than k and $M(k)$ is bounded and tends to zero as k tends to infinity.

THEOREM 5.4. Let all the hypothesis of Theorem 5.2 be satisfied. Then a necessary and sufficient condition for a function ${}^{r+1}L_{k,t}[F]$ with $L_{k,t}[F] \in \bar{M}(o,\infty)$ to have the representation as the rth operate of the Post-Widder inversion operator

$${}^{r+1}L_{k,t}[F] = \langle F(u),\ {}^{r+1}h_k(u,t)\rangle \qquad (5.23)$$

of a class of generalized functions is that $L_{k,t}[F]$ has derivatives of all orders and satisfies (5.13) for every t in Ω_t.

PROOF. Let $F \in \mathcal{J}'_{a,b,n}$. Consider the first operate of the Post-Widder inversion operator

$$L_{k,t}\,L_{k,\cdot}[F] = \langle L_{k,v}[F],\ h_k(v,t)\rangle$$

$$= \langle\langle F(u),\ h_k(u,v)\rangle\ ,\ h_k(v,t)\rangle \qquad (5.24)$$

$$= \langle F(u), \langle h_k(u,v),\ h_k(v,t)\rangle\rangle\,. \qquad (5.25)$$

In view of the fact $L_{k,v}[F]$ belongs to $\mathcal{J}_{a,b,n,v} \subseteq \mathcal{J}'_{a,b,n}$ by Corollary 3.3, the expressions (5.24) and (5.25) are meaningful if we show that

$$\langle h_k(u,v),\ h_k(v,t)\rangle = \int_0^\infty h_k(u,v)\,h_k(v,t)\,dv = {}^2h_k(u,t)$$

belongs to $\mathcal{J}_{a,b,n}$.

Indeed

$$|k_{a,b,n}(u)\,u^n\,D_u^n\ {}^2h_k(u,t)|$$

$$\le \int_0^\infty |\,k_{a,b,n}(u)\,u^n\,D_u^n\,h_k(u,v)|\ h_k(v,t)\,dv$$

$$\leq \sup_{-\infty<u<\infty} \sup_{0\leq v<\infty} |k_{a,b,n}(u)\, u^n D_u^n h_k(u,v)| \int_0^\infty h_k(v,t) \, dv \tag{5.26}$$

which, by Lemma 3.1, is bounded and tends to zero as $n \to \infty$.

In order to prove the equality between (5.24) and (5.25), we need only show that

$$\int_0^\infty \langle F(u), h_k(u,v) \rangle h_k(v,t) \, dv = \langle F(u), \int_0^\infty h_k(u,v) h_k(v,t) \, dv \rangle \tag{5.27}$$

By using the technique of Riemman sum, we can easily show that

$$\int_0^N \langle F(u), h_k(u,v) \rangle h_k(v,t) \, dv = \langle F(u), \int_0^N h_k(u,v) h_k(u,t) \, dv \rangle . \tag{5.28}$$

Since by (5.26)

$$\int_N^\infty h_k(u,v) h_k(v,t) \, dv \to 0 \quad \text{in} \quad \mathcal{J}_{a,b,n} \quad \text{as} \quad N \to \infty$$

one can readily justify taking limit as $N \to \infty$ in (5.28) to obtain the equality between (5.24) and (5.25). Then by repeated applications of the above arguments for $r = 2,3,\ldots.$, it follows that (5.23) has sense.

The necessity follows as in Theorem 5.2. For the sufficiency, we consider a function $G_{k,t}[g]$ with $g \in M^*(o,\infty)$ defined by

$$G_{k,t}[g] = \int_0^\infty h_k(u,t) \, g(u) \, du$$

which has derivatives of all orders.

It is a simple calculation to show that $g \in M^*(o,\infty)$ implies $G_{k,t}[g] \in M^*(o,\infty)$. Then in view of (5.12), ${}^{r+1}G_{k,t}[g]$, $r = o,1,2,\ldots.$ belongs to $M^*(o,\infty)$. By Lemma 5.2, $P_{n,t}^{(\beta,\nu)}\left[{}^{r+1}G_{k,t}^{(\cdot)}[g]\right]$ exists for each $t > o$ and belongs to $M^*(o,\infty)$. Then using (5.16), (5.20) and Theorem 4.4, we have

$$\begin{aligned}
&\int_0^\infty {}^{r+1}h_k(u,t)\, P_{n,u}^{(\beta,\nu)}\left[G_{k,u}^{(\cdot)}[g] \right] du \\
&= P_{n,t}^{(\beta,\nu)}\left[\int_0^\infty {}^{r+1}h_k(u,t)\, G_{k,u}^{(\cdot)}[g] \right] du \\
&= P_{n,t}^{(\beta,\nu)}\left[\int_0^\infty g(v)\, dv \int_0^\infty {}^{r+1}h_k(u,t)\, h_k^{(\cdot)}(v,u)\, du \right] \\
&= P_{n,t}^{(\beta,\nu)}\left[\int_0^\infty g(v)\, dv \int_0^\infty {}^{r+1}h_k^{(\cdot)}(v,u)\, h_k(u,t)\, du \right] \\
&+ P_{n,t}^{(\beta,\nu)}\left[\int_0^\infty {}^{r+1}G_{k,u}^{(\cdot)}[g]\, h_k(u,t)\, du \right] \\
&= P_{n,t}^{(\beta,\nu)}\left[{}^{r+2}G_{k,t}^{(\cdot)}[g] \right] .
\end{aligned}$$

Then by Theorems 4.5 and 4.2

$$\lim_{n\to\infty} \int_0^\infty {}^{r+1} h_k(u,t)\, P_{n,u}^{(\beta,\nu)} \left[G_{k,u}^{(\cdot)} [g] \right] du$$
$$= \lim_{n\to\infty} P_{n,t}^{(\beta,\nu)} \left[{}^{r+2}\, G_{k,t}^{(\cdot)}[g] \right] = {}^{r+1}\, G_{k,t}[g] \, .$$

Now it remains to justify the changes in the order of integrations performed above. By the arguments used in Theorem 4.4, we need only show that

$$P_{n,t}^{(\beta,\nu)} \left[\int_0^\infty {}^{r+1}\, h_k(u,t)\, G_{k,u}^{(\cdot)}[g] \right] du$$

exists. For this, it is sufficient to show that the following iterated integrals exist.

$$A(k) \int_0^\infty x^{\nu-k+k_1}\, Q_\nu^{(o,\beta+2n)} [1+2x^{-1}]\, dx\, x^{-1} \int_{o}^{\Omega_{u/x}} u^{-k_1}\, {}^{r+1} h_k(u,t) (\tfrac{u}{x})^{k_1} |G_{k,u/x}[g]| \, du$$

$$A(k) \int_0^\infty x^{\nu-k-A-1}\, Q_\nu^{(o,\beta+2n)} [1+2x^{-1}]\, dx \int_{\Omega_{u/x}} {}^{r+1} h_k(u,t)\, u^A \, du$$

$$A(k) \int_0^\infty x^{\nu-k-k_2}\, Q^{(o,\beta+2n)} [1+2x^{-1}]\, dx\, x^{-1} \int_{c\,\Omega_{u/x}} u^{k_2}\, {}^{r+1} h_k(u,t) (\tfrac{u}{x})^{-k_2} |G_{k,u/x}[g]| \, du$$

Using the fact that $G_{k,t}[g] \in M^*(o,\infty)$ and relations (5.17), (5.19) and (5.20), it is easy to see that all the right hand u integrals exist for each $t > o$, and in view of (4.6) all the left hand x-integrals exist if $A_i = k_1, -A-1, -k_2$ according as $i = 1,2,3$ in (4.6).

As in Theorem 5.3, there exist a subset $\hat{\Omega}_t$, a negative number A satisfying $-k-1 < A < k_2-1$ and a constant C such that

$$\left| {}^{r+1} L_{k,t}[F] \right| \le Ct^A \quad \text{for every } t \text{ in } \hat{\Omega}_t .$$

For any nonnegative integer m and $A \neq -1$, set

$${}^{r+1} L_{k,t}[F] = (tD_t)^m\, {}^{r+1} G_{k,t}[g]$$

so that ${}^{r+1} G_{k,t}[g]$ satisfies (5.3) for every t in $\hat{\Omega}_t$.

Now the theorem follows as in Theorem 5.2.

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