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STABLE P AND DISTAL DYNAMICAL SYSTEMS

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ABSTRACT. In this paper we study the behavior of dynamical systems in uniform space which are Poisson stable and distal. The main results are that the trajectories of such motions are closed and every almost periodic motion is stable P and distal. In complete uniform space, this motion is periodic, Lagrange stable and recurrent.

KEY WORDS AND PHRASES. *Dynamical systems, stability and periodic motion.*

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1. INTRODUCTION.

Bhatia and Nishihama [1] have studied Lagrange stable distal semidynamical systems. We study Poisson stable distal dynamical systems. That such a dynamical system exists is guaranteed by Example 1.

DEFINITION 1. Let X be a uniform space with a Hausdorff topology generated by a directed set (A, $\geqslant$) and a correspondence V. Let T be a topological group. A DYNAMICAL SYSTEM on X is defined to be a mapping π: X x T $\to$ X subject to the conditions:

(i) $\pi(x,o) = x$ (identity prop.)

(ii) $\pi(\pi(x,t),s) = \pi(x,t+s)$ (group property)

(iii) π is continuous.

We shall be interested in only two groups T=R(the reals) or I(the integers) with usual topology. The dynamical system π is sometimes referred to as a CONTINUOUS FLOW when T=R and a DESCRETE FLOW when T=I.

$\pi(x,t)$ is sometimes denoted by xt. For fixed $x \varepsilon X$, the map π_x: $T \to X$ is called a MOTION THROUGH x and is denoted by $\pi_x(t)$ or $\pi(x,t)$.

DEFINITION 2. Let π be a flow on X. The TRAJECTORY through x is the set $\gamma(x) = \{xt: t \varepsilon T\}$. We define w-LIMIT SET by $\Omega_x = \{y \varepsilon X: y = \lim xt_n$ for some net t_n in T with $t_n \to +\infty\}$. The HULL of a point x is defined as $H(x) = cl\ \gamma(x)$.

LEMMA. $\Omega_x = \Omega_{xt}$ [1,3.5.2. p.23].

DEFINITION 3. A point $x \varepsilon X$ is said to be PERIODIC POINT and the motion $\pi(x,t)$ is said to be PERIODIC, if there is a $\tau > 0$ such that

$$\pi(x,t) = \pi(x,t+\tau).$$

DEFINITION 4. We shall say that a flow π is DISTAL on an invariant set E if, for any pair of distinct points x and y in E, there is an index $a \varepsilon A$ such that

$$\pi(x,t) \notin V_a(\pi(y,t)) \qquad (t \varepsilon T)$$

or π is distal iff $xt_\alpha \to z \leftarrow yt_\alpha \Rightarrow x = y$ for $x,y,z \varepsilon X$ and t_α, a net in T.

DEFINITION 5. A motion $\pi(x,t)$ is said to be POSITIVELY POISSON STABLE (stable P^+) if it is defined for all $t \geqslant 0$ and for every neighborhood U of x there is a net t_n with $t_n \to +\infty$ and $\pi(x,t_n) \varepsilon U$. NEGATIVELY POISSON STABILITY (stable P^-) is defined by demanding that $t_n \to -\infty$. A motion is POISSON STABLE (stable P) if it is both positively and negatively Poisson stable. A point $x \varepsilon X$ is said to be POISSON STABLE if the corresponding motion $\pi(x,t)$ is Poisson stable.

2. MAIN RESULTS.

THEOREM 2.1. Let $\pi(x,t)$ be distal and stable P for $x \varepsilon X$. Then $\gamma(x)$ is closed. Moreover $\gamma(x)$ is a perfect set.

PROOF. Let $\pi(x,t)$ be distal and stable P. Since $\pi(x,t)$ is stable P, every point $p \varepsilon \gamma(x)$ is stable P [4, p.85]. This implies $p \varepsilon \Omega_p$ [4, p.86]. But $\Omega_p = \Omega_x$ (lemma). Therefore p is a point of Ω_x and hence is a limit point of $\gamma(x)$. Now to complete the proof let $y \notin \gamma(x)$; then we show that y is not a limit point of $\gamma(x)$. Clearly $y \neq x$. By distality of π, there exists an index $a \varepsilon A$ such that $xt \notin V_a(yt)$; $(t \varepsilon T)$ implies $yt \notin V_a(xt)$ implies $y \notin V_a(xt)$ for some $a \varepsilon A$ and $t \varepsilon T$. Again X is Hausdorff uniform space and there exists $a \varepsilon A$ such that $xt \notin V_a(y)$ for $t \varepsilon T$. Therefore y is not a limit point of $\gamma(x)$. //

THEOREM 2.2. Let $x \varepsilon X$ which is a complete uniform space. Let $\pi(x,t)$ be distal and stable P; then the motion $\pi(x,t)$ is periodic.

PROOF. $\pi(x,t)$ is stable P; therefore $\Omega_x = H(x)$ [4, p.86]. Since $\pi(x,t)$ is stable P and distal, therefore $\Omega_x = \gamma(x)$ (Theorem 2.1). Therefore, if x is not periodic, then

$$\text{cl }(H(x) - \gamma(x)) = H(x) = \Omega_x \qquad [4, \text{ p. } 87]$$

$$\text{cl }\phi = \phi = H(x) = \Omega_x = \gamma(x)$$

which is a contradiction as $x \varepsilon \gamma(x)$. Hence $\pi(x,t)$ is periodic. //

DEFINITION 6. A motion $\pi(x,t)$ is called POSITIVELY LAGRANGE STABLE if the closure of the positive trajectory $\gamma^+(x) = \{xt: t \varepsilon T^+\}$ is a compact set where $T^+ = \{t \varepsilon T: t \geqslant 0\}$. NEGATIVE LAGRANGE STABILITY is defined analogously and a motion which is at the same time positively and negatively stable according to Lagrange is called LAGRANGE STABLE.

A motion $\pi(x,t)$ is said to be RECURRENT if it is defined for all t and, for every neighborhood U of x, the set

$$\{\tau \varepsilon T: \pi(x,\tau) \varepsilon U$$

is relatively dense [1, p.39] in T. A point $x \varepsilon X$ is said to be RECURRENT if the corresponding motion $\pi(x,t)$ is recurrent.

THEOREM 2.3. Let $x \varepsilon X$ which is a complete uniform space and let $\pi(x,t)$ be stable P and distal; then $\pi(x,t)$ is Lagrange stable and recurrent. Moreover $\gamma(x)$ is compact.

PROOF. By Theorem 2.2, $\pi(x,t)$ is periodic; hence, $\pi(x,t+\tau) = \pi(x,t)$ for all $t \varepsilon T$. Then $\gamma(x)$ is compact, being a continuous image of compact set $0 \leqslant t \leqslant \tau$. Thus $\pi(x,t)$ is Lagrange stable. By Theorem 2.1, $\gamma(x)$ is closed.

cl $\gamma(x) = \gamma(x)$; hence $\gamma(x)$ is minimal [1, p. 37]. Thus $\gamma(x)$ is compact minimal and so $\pi(x,t)$ is recurrent (Birkhoff recurrence theorem). //

DEFINITION 7. A motion $\pi(x,t)$ is LYAPUNOV STABLE (with regard to set D) if $\pi(x,t)$ is defined for all t and, for every index a, there is an index b such that $yt \varepsilon V_a(xt)$, $(t \varepsilon T)$.. whenever $y \varepsilon D$ and $y \varepsilon V_b(x)$. The motion $\pi(x,t)$ is said to be UNIFORMLY LYAPUNOV STABLE (with respect to D) if $\pi(x,t)$ is defined for all t and, for every index a, there is an index b such that $yt \varepsilon V_a(\bar{x},t)$, $(t \varepsilon T)$.. (2.1) whenever $y \varepsilon D$, $\bar{x} \varepsilon \gamma(x)$ and $y \varepsilon V_b(\bar{x})$. We shall say $\pi(x,t)$ is POSITIVELY UNIFORMLY LYAPUNOV STABLE when (2.1) holds for $t \geqslant 0$ and $\bar{x} \varepsilon \gamma^+(x)$.

DEFINITION 8. A motion $\pi(x,t)$ is said to be ALMOST PERIODIC if the function

$\pi(x,.)$: $T \to X$ is almost periodic i.e., the set $\{\tau \in T$: $\pi(x,t+\tau) \in V_a(\pi(x,t))$ for all t in T$\}$ is relatively dense in T for every index a in A.

We shall say that the flow π is EQUICONTINUOUS if the family of mappings $\{\pi_t : t \in T\}$ is equicontinuous; that is, for every index a and every $x \in X$, there is an index b such that

$$\pi_t(y) \in V_a(\pi_t(x)), \quad (t \in T)$$

whenever $y \in V_b(x)$.

Then the following results can be proved easily and therefore their proofs are omitted.

THEOREM 2.4. If a flow π is equicontinuous on an invariant set D, then it is distal on D.

THEOREM 2.5. The motion $\pi(x,t)$ is uniformly Lyapunov stable with respect to $\gamma(x)$ iff the flow is equicontinuous on $\gamma(x)$.

THEOREM 2.6. A compact motion $\pi(x,t)$ is almost periodic iff it is recurrent and uniformly positively Lyapunov stable with respect to $\gamma(x)$.

We prove the following results.

THEOREM 2.7. Every almost periodic motion is stable P and distal [cf, 1,2].

PROOF. Let π be almost periodic flow. Then π is recurrent and uniformly Lyapunov stable in $\gamma(x)$ (Theorem 2.6). That is, π is recurrent and equicontinuous on $\gamma(x)$ (Theorem 2.5). Hence π is recurrent and distal on $\gamma(x)$ (Theorem 2.4). But every recurrent motion is stable P. Hence every almost periodic motion is stable P and distal.

EXAMPLE 1. Let (ϕ,θ), $0 < \phi < 1$, $0 \leqslant \theta < 1$ denote the co-ordinates of points on the torus T^2 and let

$$\frac{d\phi}{dt} = 1, \quad \frac{d\theta}{dt} = \alpha$$

where α is irrational.

Every trajectory on torus T^2 is dense; so cl $\gamma(x) = T^2$ and every point $x \in T^2$ is stable P; therefore,

$$\text{cl}\,\gamma(x) = T^2 = \Omega_x = A_x$$

for every x in T^2. This means that T^2 is a compact minimal set in this flow. It can be shown that π is distal and the motion $\pi(x,t)$ is almost periodic.

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