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SOME FIXED POINT THEOREMS FOR HARDY-ROGERS TYPE MAPPINGS

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ABSTRACT. The first result establishes a fixed point theorem for three maps of a complete metric space. The contractive definition is a generalization of that of Hardy and Rogers, and the commuting condition of Jungck is replaced by the concept of weakly commuting. The other results are extensions of some theorems of Kannan.

KEY WORDS AND PHRASES. *Asymptotically regular sequences, common fixed points, metric space, strictly convex Banach space, weakly commuting mappings.*
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1. INTRODUCTION

There is a multitude of metrical fixed point theorems for mappings satisfying certain contractive type conditions. In each of these results one considers sequences of iterates, which, due to the contractive condition, becomes a Cauchy sequence whose limit is a fixed point of the mapping. In the case of a common fixed point theorem, a joint sequence of iterates is usually suitable for the purpose.

In this paper we prove some common fixed point theorems using sequences which are not necessarily obtained as a sequence of iterates of certain mappings. In doing so we are motivated by the following result of Jungck [1], who replaced the identity mapping with a continuous function in order to generalize the celebrated Banach contraction principle. Jungck [1] showed that a continuous selfmap T of a complete metric space (X,d) has fixed point if and only if there exsits a

$q \in (0,1)$ and a map $A : X \to X$ which commutes with T and satisfies:
(a) $A(X) \subset T(X)$ (b) $d(Ax,Ay) \le qd(Tx,Ty)$, for all x , $y \in X$. Indeed, A and T have a unique common fixed point. It should be noted that in all extensions and generalizations of Jungck's theorem, a family of commuting mappings has been considered. Our first result deals with mappings satisfying a condition weaker than commutativity. The technique of our proof can also be used to prove other results in the literature and thus one need not consider the sequence of successive approximations in order to prove the existence of fixed points. Other results in compact spaces are also given. The structure of the set of common fixed points is also studied.

2. PRELIMINARIES.

Throughout this section (X,d) stands for a metric space.

Definition 2.1. Let A and B be two selfmaps of X and $\{x_n\}$ a sequence in X . Then $\{x_n\}$ is said to be asymptotically A-regular with respect to B if $\lim_{n\to\infty} d(Bx_n,Ax_n) = 0$.

When B is the identity map, the above definition reduces to that of Engl [2]. The following notion was introduced by Sessa [3].

Definition 2.2. Let f and g be two selfmpas of X . Then $\{f,g\}$ is said to be a weakly commuting pair if

$$d(fgx,gfx) \le d(gx,fx) \ , \text{ for all } x \in X \ .$$

Clearly, a commuting pair is weakly commuting, but the converse is not necessarily true as is shown by the following simple example in Sessa [3].

EXAMPLE 2.3. Let $X = [0,1]$ with the usual metric. Define $f(x) = x/2$ and $g(x) = x/(2+x)$. Then, for all x in X , one obtains

$$d(fgx,gfx) = \frac{x}{4+x} - \frac{x}{4+2x} = \frac{x^2}{(4+x)(4+2x)} \le \frac{x^2}{4+2x} = \frac{x}{2} - \frac{x}{2+x} = d(fx,gx) \ ,$$

and f and g commute weakly.

But, for any non-zero $x \in X$, we have

$$gfx = \frac{x}{4+x} > \frac{x}{4+2x} = fgx$$

and f and g do not commute.

Definition 2.4 (Hardy and Rogers [4]). A mapping $T : X \to X$ is said to be generalized nonexpansive if for all x , y in X the inequality

$$d(Tx,Ty) \le a_1 d(x,Tx) + a_2 d(y,Ty) + a_3 d(x,Ty) + a_4 d(y,Tx) + a_5 d(x,y)$$

holds, where $a_i \geq 0$, $i = 1,2,3,4,5$ and $\sum_{i=1}^{5} a_i \leq 1$. Due to symmetry, one can choose $a_1 = a_2$ and $a_3 = a_4$.

Mappings of the above type were first introduced by Hardy and Rogers [4]. Since then these contractive conditions have been extensively studied by many mathematicians for single valued as well as multivalued mappings.

Definition 2.5 (Kannan [5]). A mapping T is said to have property (K) on a subset G of X if for every closed subset F of G which contains more than one point is mapped into itself by T , there exists an $x \in F$ such that $d(x,Tx) < \sup_{y\in F} d(y,Ty)$.

3. RESULTS IN COMPLETE SPACES.

Now we present our main theorem which is motivated by the contractive conditions studied by Hardy and Rogers [4], Mukherjee [6] and Fisher [7], and is a natural extension of the result of Jungck to these maps.

THEOREM 3.1. Let A , S , and T be selfmaps of a complete metric space X satisfying

(i) $$d(Ax,Ay) \leq a_1 d(Sx,Ax) + a_2 d(Tx,Ax) + a_3 d(Sy,Ay) + a_4 d(Ty,Ay) + a_5 d(Sx,Ay) + a_6 d(Tx,Ay) + a_7 d(Sy,Ax) + a_8 d(Ty,Ax) + a_9 d(Sx,Ty) + a_{10} d(Sy,Tx) ,$$

where the $a_i = a_i(x,y)$ are nonnegative functions of x and y satisfying

(ii) $$\max \{\sup_{x,y\in X}(b_3 + b_3' + b_4 + b_4' + b_5 + b_5') , \sup_{x,y\in X}(b_1' + b_2 + b_3 + b_4' + b_5 + b_5') , \sup_{x,y\in X}(b_1' + b_2' + b_3 + b_4)\} < 1 ,$$

and b_1 , b_2 are bounded, the b_i , b_i' as defined in (vi),

(iii) S and T are continuous,

(iv) $\{A,S\}$ and $\{A,T\}$ are weakly commuting pairs,

and

(v) there exists an asymptotically A-regular sequence with respect to both S and T .

Then A , S , and T have a common unique fixed point. Further A is continuous at the fixed point if $\sup_{x,y\in X}(b_1 + b_2 + b_3' + b_4') < 1$.

PROOF. Interchanging the roles of x and y and then adding the two resulting inequalities yields

$$\text{(i')} \quad d(Ax,Ay) \le b_1(x,y)d(Sx,Ax) + b_1(y,x)d(Sy,Ay) + b_2(x,y)d(Tx,Ax) + b_2(y,x)d(Ty,Ay) + b_3(x,y)d(Sx,Ay) + b_3(y,x)d(Sy,Ax) + b_4(x,y)d(Tx,Ay) + b_4(y,x)d(Ty,Ax) + b_5(x,y)d(Sx,Ty) + b_5(y,x)d(Sy,Tx),$$

where

$$\text{(vi)} \quad 2b_1(x,y) = a_1(x,y) + a_3(y,x), \quad 2b_2(x,y) = a_2(x,y) + a_4(y,x), \quad 2b_3(x,y) = a_5(x,y) + a_7(y,x),$$

$$2b_4(x,y) = a_6(x,y) + a_8(y,x), \quad 2b_5(x,y) = a_9(x,y) + a_{10}(y,x), \text{ and}$$

$$b_i(y,x) = b_i'(x,y).$$

Let $\{x_n\}$ satisfy (v). Then, from (i')

$$(1 - b_3 - b_3' - b_4 - b_4' - b_5 - b_5')d(Ax_n, Ax_m) \le (b_1 + b_3 + b_5)d(Sx_n, Ax_n) + (b_1' + b_3' + b_5')d(Sx_m, Ax_m) + (b_2 + b_4 + b_5')d(Tx_n, Ax_n) + (b_2' + b_4' + b_5)d(Tx_m, Ax_m).$$

From (ii) and (v), taking the limit as $m, n \to \infty$ shows that $\{Ax_n\}$ is Cauchy, hence convergent. Call the limit z. $d(Sx_n,z) \le d(Sx_n,Ax_n) + d(Ax_n,z) \to 0$ as $n \to \infty$ so $Sx_n \to z$. Similarly, $Tx_n \to z$. The continuity of S and T imply $SAx_n \to Sz$, $S^2x_n \to Sz$, $STx_n \to Sz$, $TAx_n \to Tz$, $T^2x_n \to Tz$ and $TSx_n \to Tz$.

$$d(ATx_n,Tz) \le d(ATx_n,TAx_n) + d(TAx_n,Tz) \le d(Tx_n,Ax_n) + d(TAx_n,Tz) \to 0,$$

so $ATx_n \to Tz$. Similarly, $ASx_n \to Sz$.

$$d(STx_n,TSx_n) \le d(STx_n,ASx_n) + d(ASx_n,ATx_n) + d(ATx_n,TSx_n).$$

Using (i'),

$$d(ASx_n, ATx_n) \le b_1d(S^2x_n, ASx_n) + b_1'd(STx_n, ATx_n) + b_2d(TSx_n, ASx_n) + b_2'd(T^2x_n, ATx_n) + b_3d(S^2x_n, ATx_n) + b_3'd(STx_n, ASx_n) +$$

$$b_4 d(TSx_n , ATx_n) + b_4' d(T^2x_n , ASx_n) + b_5 d(S^2x_n , T^2x_n) + b_5' d(STx_n , TSx_n)$$

$$\le b_1 d(S^2x_n , ASx_n) + b_2' d(T^2x_n , ATx_n) + b_3' d(STx_n , ASx_n) + b_4 d(TSx_n , ATx_n)$$

$$+ (b_1' + b_2 + b_3 + b_4' + b_5 + b_5')\max\{d(STx_n , ATx_n) , d(TSx_n , ASx_n) , d(S^2x_n , ATx_n) , d(T^2x_n , ASx_n) , d(S^2x_n , T^2x_n) , d(STx_n , TSx_n)\} .$$

Hence

$$\lim \sup_n d(STx_n , TSx_n) \le 0 + \lim \sup_n (b_1' + b_2 + b_3 + b_4' + b_5 + b_5') d(Sz , Tz) ,$$

i.e.

$$d(Sz , Tz) \le \sup_{x,y \in X} (b_1' + b_2 + b_3 + b_4' + b_5 + b_5') d(Sz , Tz) ,$$

which, from (ii) implies that $Sz = Tz$.

From (i'),

$$d(ATx_n , Az) \le b_1 d(STx_n , ATx_n) + b_1' d(Sz , Az) + b_2 d(T^2x_n , ATx_n) + b_2' d(Tz , Az) + b_3 d(STx_n , Az) + b_3' d(Sz , ATx_n)$$

$$+ b_4 d(T^2x_n , Az) + b_4' d(Tz , ATx_n) + b_5 d(STx_n , Tz) + b_5' d(Sz , T^2x_n)$$

$$\le b_1 d(STx_n , ATx_n) + b_2 d(T^2x_n , ATx_n) + b_3' d(Sz , ATx_n) + b_4' d(Tz , ATx_n) + b_5 d(STx_n , Tz) + b_5' d(Sz , T^2x_n)$$

$$+ (b_1' + b_2' + b_3 + b_4)\max\{d(Sz , Az) , d(Tz , Az) , d(STx_n , Az) , d(T^2x_n , Az)\} .$$

Taking the lim sup of both sides yields

$$d(Tz , Az) \le \lim \sup_n (b_1' + b_2' + b_3 + b_4)\max\{d(Tz , Az) , d(Sz , Az)\}$$

$$\le \sup_{x,y \in X} (b_1' + b_2' + b_3 + b_4) d(Tz , Az) ,$$

which, from (ii), implies that $Tz = Az$, since $d(Sz,Az) \le d(Sz,Tz) + d(Tz,Az)$.

From (1'),

$$d(Az,A^2z) \le b_1d(Sz,Az) + b_1'(SAz,A^2z) + b_2d(Tz,Az) + b_2'd(TAz,A^2z)$$
$$+ b_3d(Sz,A^2z) + b_3'd(SAz,Az) + b_4d(Tz,A^2z)$$
$$+ b_4'd(TAz,Az) + b_5d(Sz,TAz) + b_5'd(SAz,Tz) .$$

$d(SAz,A^2z) \le d(SAz,ASz) + d(ASz,A^2z)$. Since $Sz = Az$, $ASz = A^2z$. From (iv), $d(SAz,ASz) \le d(Az,Sz) = 0$, and $SAz = A^2z$. Similarly, $TAz = A^2z$. Therefore

$$d(Az,A^2z) \le b_3d(Sz,Az) + b_3d(Az,A^2z) + b_3'd(SAz,A^2z) + b_3'd(A^2z,Az)$$
$$+ b_4d(Tz,Az) + b_4d(Az,A^2z) + b_4'd(TAz,A^2z) + b_4'd(A^2z,Az)$$
$$+ b_5d(Sz,Az) + b_5d(Az,A^2z) + b_5'd(A^2z,Az) + b_5'd(Az,Tz) .$$

Thus $(1 - b_3 - b_3' - b_4 - b_4' - b_5 - b_5')d(Az , A^2z) \le 0$ which, from (ii), implies $Az = A^2z$

Set $p = Az$.

$$d(Sp,p) = d(SAz,Az) \le d(SAz,ASz) + d(ASz,Az) \le d(Az,Sz) + d(A^2z,Az) = 0 .$$

Similarly, $Tp = p$.

Suppose p and q are common fixed points of A , S , and T . Then

$$d(p,q) = d(Ap,Aq) \le b_3d(p,q) + b_3'd(q,p) + b_4d(p,q) + b_4'd(q,p) + b_5d(p,q) + b_5'd(q,p)$$

which, from (ii), implies $p = q$ and the fixed point is unique.

Let $\{y_n\}$ be any sequence in X with limit p and assume that $\sup_{x,y\in X}(b_1 + b_2 + b_3' + b_4') < 1$. From (i'),

$$d(Ay_n , Ap) \le b_1d(Sy_n , Ay_n) + b_1'd(Sp , Ap) + b_2d(Ty_n , Ay_n) + b_2'd(Tp , Ap)$$
$$+ b_3d(Sy_n , Ap) + b_3'd(Sp , Ay_n) + b_4d(Ty_n , Ap) + b_4'd(Tp , Ay_n)$$
$$+ b_5d(Sy_n , Tp) + b_5'd(Ap , Ty_n)$$
$$\le b_1d(Sy_n , Ap) + b_1d(Ap , Ay_n) + b_2d(Ty_n , Ap) + b_2d(Ap , Ay_n)$$
$$+ b_3d(Sy_n , Ap) + b_3'd(Sp , Ap) + b_3'd(Ap , Ay_n) + b_4d(Ty_n , Ap)$$
$$+ b_4'd(Tp , Ap) + b_4'd(Ap , Ay_n) + b_5d(Sy_n , Tp) + b_5'd(Ap , Ty_n)$$
$$\le b_1d(Sy_n , Ap) + b_2d(Ty_n , Ap) + b_3d(Sy_n , Ap) + b_4d(Ty_n , Ap)$$
$$+ b_5d(Sy_n , Tp) + b_5'd(Ap , Ty_n) + (b_1 + b_2 + b_3' + b_4')d(Ap , Ay_n) .$$

Taking the lim sup of both sides yields

$$\limsup_n d(Ay_n , Ap) \le \limsup_n (b_1 + b_2 + b'_3 + b'_4) \limsup_n d(Ap , Ay_n)$$
$$\le \sup_{x,y\in X}(b_1 + b_2 + b'_3 + b'_4) \limsup_n d(Ap , Ay_n)$$

which, from, the assumption, yields $\limsup_n d(Ap,Ay_n) = 0$. Therefore $\lim_n d(Ap,Ay_n) = 0$ and A is continuous at p .

COROLLARY 3.2. Let S be a complete metric space, A a self-map of X satisfying

$$\text{(vii)}\quad d(Ax,Ay) \le c_1(x,y)d(x,Ax) + c'_1(x,y)d(y,Ay) + c_2(x,y)d(x,Ay) + c'_2(x,y)d(y,Ax) + c_3(x,y)d(x,y) ,$$

where the c_i are nonnegative and bounded and satisfy

$\max\{\sup_{x,y\in X}(c_2 + c'_2 + c_3) , \sup_{x,y\in X}(c'_1 + c_2)\} < 1$. If there exists an asymptotically regular sequence X , then A has a unique fixed point.

Remarks. 1. Corollary 3.2 is an extension of the fixed point theorem of Hardy and Rogers [4] since the functions c_i , c'_i need not satisfy $c_1 + c'_1 + c_2 + c'_2 + c_3 \le 1$.

2. Putting $S = T$ in Theorem 3.1 yields a generalization of a result of Mukherjee [6].

3. For $a_i = 0$, $i = 1,\dots,8$, $a_{10} = 0$, $a_9 = \alpha$ and $S = T$ in Theorem 3.1 yields the result in Fisher [7]. It then follows that the continuity assumption in A in Fisher [7] is not needed, as noted in Fisher [8].

4. A condition analogous to (a) in Jungck's theorem is not required. Further, the continuity of A is neither assumed nor required in Theorem 3.1.

5. In Goebel et al [9] it has been shown that uniform convexity in a Banach space implies the existence of asymptotically regular sequences. Thus Corollary 3.2 can be used to obtain fixed point theorems in such spaces.

6. A result similar to Corollary 3.2 appears in Mukherjea [10] under different conditions but with the added hypothesis that the map $x \to d(x,Tx)$ be lower semi-continuous.

We now give an example of three maps satisfying (i) of Theorem 3.1.

EXAMPLE 3.3. Let $X = [0,3)$ with the usual metric. Define $A , S , T : [0,3) \to [0,3)$ by putting $Ax = (1+x)/2$, $Sx + (1+3x)/4$, $Tx = (1+7x)/8$. Then for all $x , y \in [0,3)$ we have:

$$d(Ax, Ay) = 1/2\ |x-y|$$

$$= 1/2\ \left|\frac{4}{7}\left(\frac{1+6x-7}{4}\right) + -\frac{2}{7} - \left(\frac{x-1}{2}\right)\right|$$

$$= \frac{4}{7}\left|\frac{1+6x-7y}{8}\right| + \frac{2}{7}\left|\frac{x-1}{4}\right|$$

$$= \frac{4}{7}\, d(Sx, Ty) + \frac{2}{7}\, d(Ax, Sx)\ .$$

So the family $\{A,S,T\}$ satisfies (i) of Theorem 3.1 with $a_1 = 2/7$, $a_i = 0$, $i = 2,\dots,8$, $a_9 = 4/7$, $a_{10} = 0$. The following example contains three maps satisfying condition (iv) of Theorem 3.1.

EXAMPLE 3.4. Let $X = [0,2/3]$ be with usual metric and $A, S, T : X \to X$ given $A(x) = x/18$, $S(x) = s/6 + x$, $T(x) = x/2$ for any $x \in X$. Of course, A and T commute whereas the pair (A,S) is weakly commuting since, for any $x \in X$;

$$d(SAx, ASx) = \frac{x}{108+x} - \frac{x}{108+18x} = \frac{17x^2}{18(108+x)(6+x)}$$

$$\le \frac{x(12-x)}{18(6+x)} = \frac{x}{6+x} - \frac{x}{18} = d(Ax, Sx)\ .$$

Since $6 + x < 12 - x$ for any $x \in X$, we have for $x \ge y$:

$$d(Ax, Ay) = \frac{x-y}{18} \le \frac{x}{18} \cdot \frac{12-x}{6+x} = d(Ax, Sx)\ .$$

If $x < y$, we achieve:

$$d(Ax, Ay) = \frac{y-x}{18} \le \frac{1}{2} \cdot \frac{16y}{18} = \frac{1}{2}\left(\frac{y}{2} - \frac{y}{18}\right) = \frac{1}{2}\, d(Ty, Ay)\ .$$

Therefore condition (i) holds if we assume $a_1 = 1$, $a_2 = a_3 = 0$, $a_4 = 1/2$, $a_i = 0$, $i = 5,\dots,10$.

Now we quote an example from Pal and Maiti [11] of a map $T : X \to X$ which satisfies Corollary 3.2 but not Definition 2.4.

EXAMPLE 3.5. Let $X = \{1,2,3,4,5\}$ be a finite set with metric d defined as

$$d(1,2) = 4.3\ ,\ d(1,3) = 0.6\ ,\ d(1,4) = 3.65\ ,\ d(1,5) = 2.8\ ,$$

$$d(2,3) = 3.95\ ,\ d(2.4) = 3.7\ ,\ d(2.5) = 5\ ,\ d(3,4) = 3.9\ ,$$

$$d(3,5) = 3.35\ ,\ d(4,5) = 1.9$$

and $T : X \to X$ given by $T1 = 2$, $T2 = 3$, $T3 = 4$, $T4 = T5 = 5$.

One can verify all the assumptions of Corollary 3.2 with $c_1 = 1.17$, $c_1' = 0.3$, $c_2 = 0.2$, $c_2' = 0.3$, $c_3 = 0.2$. T is not a generalized nonexpansive mapping, since for $x = 1$ and $y = 4$ we obtain:

$$d(T1, T4) = 5 \le a_1 \cdot 4.3 + a_2 \cdot 1.9 + a_3 \cdot 2.8 + a_4 \cdot 3.7 + a_5 \cdot 3.65$$
$$< (a_1 + a_2 + a_3 + a_4 + a_5) \cdot 4.3 \le 4.3 ,$$

a contradiction.

4. RESULTS IN COMPACT SPACES.

In this section a few theorems of Kannan [5], [12] are extended to generalized nonexpansive mappings defined over compact metric spaces. Our first result is an extension of Theorem 1 of Kannan [5].

THEOREM 4.1. Let X be a compact metric space, T a self-map of X satisfying (vii) with $\sup_{x,y}(c_1 + c_1' + c_2 + c_2' + c_3)(x,y) \le 1$, and having property (K) over X. Suppose also that, if $Y \subset X$ such that $Ty \subset Y$, $x_n \to x$ implies $Tx_n \to x$, $x_n \in Y$. Then T has a unique fixed point in X, provided $c_2 = c_2'$ and $\sup_{x,y\in X}(c_1' + c_2)(x,y) < 1$.

PROOF. Let $CL(X)$ denote the collection of subsets $K \subset X$ which are non-empty, closed and invariant under T. Introduce the following partial ordering $\le$ on the space $CL(X)$: Let $K_1, K_2 \subset X$. Then $K_1 \le K_2$ if $K_1 \subset K_2$, $K_1 \ne K_2$. Using Zorn's Lemma, we can obtain a set K which is minimal with respect to $\le$, and which is non-empty, closed and invariant under T.

If K is singleton, then it is a fixed point of T. If K contains more than one point then, by property (K), there is an $x \in K$ such that

$$(*) \quad d(x,Tx) = r < \sup_{y\in K} d(y,Ty) .$$

Let $K_1 = \{x \in K : d(x,Tx) \le r\}$. Then by (*), K_1 is a non-empty proper subset of K. Also, for $x \in K_1$, since T satisfies (vii) we have:

$$d(Tx,T^2x) \le c_1 d(x,Tx) + c_1' d(Tx,T^2x) + c_2 d(x,T^2x) + c_2' d(Tx,Tx) + c_3 d(x,Tx) .$$

Then

$$d(Tx, T^2x) \le \frac{c_1 + c_2 + c_3}{1 - c_1' - c_2} d(x, Tx) \le d(x, Tx) ,$$

and K_1 is invariant under T.

Let x be a limit point of K_1. Then there exists a sequence $\{x_n\} \subset K_1$ with $x_n \to x$. By hypothesis $Tx_n \to x$.

$$d(x,Tx) \le d(x,Tx_n) + d(Tx_n,Tx)$$
$$\le d(x,Tx_n) + c_1 d(x_n,Tx_n) + c_1' d(x,Tx) + c_2 d(x_n,Tx)$$
$$+ c_2' d(x,Tx_n) + c_3 d(x_n,x)$$

$$\leq d(x,Tx_n) + c_1 r + c_1' d(x,Tx) + c_2 d(x_n,x) + c_2 d(x,Tx) + c_2' d(x,Tx_n) + c_3 d(x_n,x) .$$

Therefore

$$d(x,Tx) \leq \frac{1}{1-c_1'-c_2}[d(x,Tx_n) + c_2 d(x_n,x) + c_2' d(x,Tx_n) + c_3 d(x_n,x)] + \frac{c_1 r}{1-c_1'-c_2}$$

$$\leq \frac{(1+c_2+c_2'+c_3)}{1-c_1'-c_2}\max\{d(x,Tx_n), d(x_n,x)\} + r$$

$$\leq \frac{2}{1-\sup_{x,y\in X}(c_1'+c_2)}\max\{d(x,Tx_n), d(x_n,x)\} + r .$$

Taking the limit as $n \to \infty$ yields $d(x,Tx) \leq r$ and K_1 is closed. Thus the minimality of K_1 is contradicted, so K_1 contains only one point, which is the unique fixed point of T.

If, in Theorem 4.1, T is assumed continuous, then the condition $x_n \to x$ implies $Tx_n \to x$ for $x_n \in Y$ can be dropped.

EXAMPLE 4.2. Let $X = [0,1]$ with usual metric and $T : X \to X$ defined by $Tx = 3x/8$ if $x \neq 1$ and $T1 = 1/2$.

Clearly X is a compact metric space and T satisfies Theorem 4.1 with $c_1 = c_1' = 1/4$, $c_2 = c_2' = 0$, $c_3 = 1/2$. With $x = 0$ and $y = 1/3$, $d(T0,T1/3) = 1/8 > 1/2 \cdot 5/24 = 1/2[d(0,T0)+d(1/3,T1/3)]$ and T is not a Kannan map. Further, T is not a contraction either, because it is discontinuous at 1.

Further results about generalizations of Kannan maps can be found in Tasković [13].

The next theorem generalizes Theorem 1 of Kannan [12].

THEOREM 4.2. Let X be a compact metric space, T a continuous selfmap of x satisfying (vii) with $\sup_{x,y\in X}(2c_1'+c_2+c_2'+c_3)(x,y) \leq 1$, $c_2 = c_2'$ and $\sup_{x,y\in X}(c_1'+c_2)(x,y) < 1$. Suppose that T satisfies property (K) over X and that $d(Tx,p) < d(x,p)$ for each $x \neq p$, p the unique fixed point of T. Then, for each x in X, $T^n x \to p$.

PROOF. From the remark following Theorem 4.1, T has a unique fixed point p Let $x \in X$. Then, from the compactness of X, $\{T^n x\}$ contains a convergent subsequence $\{T^{n_i}x\}$. Let $z = \lim_i T^{n_i}x$. Using (vii),

$$d(p,T^n x) = d(Tp,T^n x) \le c_1 d(p,Tp) + c_1' d(T^{n-1}x,T^n x) + c_2 d(p,T^n x) + c_2' d(T^{n-1}x,Tp) + c_3 d(p,T^{n-1}x) ,$$

or

$$d(p,T^n x) \le \frac{(c_1' + c_2' + c_3)}{1 - c_1' - c_2} d(p,T^{n-1}x) \le d(p,T^{n-1}x) .$$

Therefore $\{d(p,T^n x)\}$ is a nonnegative decreasing sequence and hence converges. This, along with the convergence of $\{T^{n_i}x\}$ implies $d(T^n x,p) \to d(z,p)$.

If $z \ne p$, then the hypothesis of the theorem is contradicted.

In Theorem 4.2, the continuity of T can be replaced by orbital continuity at the point x , by adding the hypothesis that the unique fixed point of T exists.

5. RESULTS IN BANACH SPACES.

This section deals with the structure of the set of common fixed points of three mappings. Our theorem extends Theorem 29 of Rhoades [14] and 7 of Rhoades [15].

THEOREM 5.1. Let X be a strictly convex Banach space, K a closed convex subset of X . Let A , S , and T be selfmaps of K satisfying:

(a) S and T are continuous

(b) (i') is satisfied with $\sum_{i=1}^{5} (b_i + b_i')(x , y) \le 1$, and

(viii) $\sup_{x,y\in X}(b_1' + b_2' + b_3 + b_4)(x , y) < 1$.

Then the set F of common fixed points of S , T , and A is closed.

PROOF. Let $\{x_n\}$ be a Cauchy sequence in F with limit x . From (ii),

$$\begin{aligned} d(x , Ax) &\le d(x_n , x) + d(Ax_n , Ax) \\ &\le d(x_n , x) + b_1 d(Sx_n , Ax_n) + b_1' d(Sx , Ax) + b_2 d(Tx_n , Ax_n) \\ &\quad + b_2' d(Tx , Ax) + b_3 d(Sx_n , Ax) + b_3' d(Sx , Ax_n) + b_4 d(Tx_n , Ax) \\ &\quad + b_4' d(Tx , Ax_n) + b_5 d(Sx_n , Tx) + b_5' d(Sx , Tx_n) \\ &\le [(1 + b_1' + b_2' + b_3 + b_4) d(x_n , x) + (b_1' + b_3' + b_5') d(Sx , x_n) \end{aligned}$$

$$+ (b_2' + b_4' + b_5) d(Tx , x_n)] \div (1 - b_1' - b_2' - b_3 - b_4)$$

$$\leq [2d(x_n , x) + d(Sx , Sx_n) + d(Tx , Tx_n)] \div M ,$$

where $M = 1 - \sup_{x,y \in X}(b_1' + b_2' + b_3 + b_4)(x , y)$.

Taking the limit as $n \to \infty$ yields $d(x,Ax) = 0$. Hence $x = Ax$ and F is closed.

COROLLARY 5.2. Let X , K , A , S , and T be as in Theorem 5.1 with $b_1' = b_2 = b_3' = b_4 = b_5' = 0$, $(b_1 + b_2' + b_3 + b_4' + b_5)(x , y) \leq 1$, and $\sup_{x,y \in K}(b_2' + b_3 + b_4')(x , y) < 1$. If T is linear and S commutes with A and T , then F is closed and convex.

PROOF. That F is closed follows from Theorem 5.1. Let $x_1 , x_2 \in F$ and $x = (x_1 + x_2)/2$. Without loss of generality we may assume that $\|x_2 - Ax\| \leq \|x_1 - Ax\|$.

$$\|x - Ax\| \leq 1/2[\|x_1 - Ax\| + \|x_2 - Ax\|] \leq \|x_1 - Ax\| = \|Ax_1 - Ax\|$$

$$\leq b_1\|Sx_1 - Ax_1\| + b_2'\|Tx - Ax\| + b_3\|Sx_1 - Ax\| + b_4'\|Tx - Ax_1\|$$

$$+ b_5\|Sx_1 - Tx\|$$

$$= b_2'\|Tx - Ax\| + b_3\|Ax_1 - Ax\| + b_4'\|Tx - x_1\| + b_5\|x_1 - Tx\| .$$

$$\|Tx - Ax\| = 1/2\|Tx_1 + Tx_2 - 2Ax\| \leq \|Ax_1 - Ax\| .$$

Therefore

$$\|x_1 - Ax\| \leq \frac{(b_4' + b_5)}{1 - b_2' - b_3}\|x_1 - Tx\| \leq \|x_1 - Tx\| \leq 1/2\|x_1 - x_2\| .$$

Also,

$$\|x_1 - x_2\| \leq \|x_1 - Ax\| + \|x_2 - Ax\| \leq 2\|x_1 - Ax\| \leq \|x_1 - x_2\| .$$

Since X is strictly convex, $x_1 - Ax$, and hence Ax , must lie in the line segment joining x_1 and x_2 . The inequalities imply that Ax is the midpoint. Therefore $x = Ax$.

Since $x = Tx$, and S and T commute, $Sx = STx = TSx$. Using the contractive condition, and the commutativity of S and A ,

$$\|x - ASx\| = \|Ax - ASx\|$$

$$\leq b_1\|Sx - ASx\| + b_2'\|Tx - ASx\| + b_3\|Sx - ASx\|$$

$$+ b_4'\|TSx - Ax\| + b_5\|Sx - TSx\|$$

$$= b_2'\|x - Sx\| + b_4'\|Sx - x\| ,$$

which implies $x = Sx$. Therefore $x \in F$ and F is convex.

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