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ON THE ACYCLIC POINT-CONNECTIVITY OF THE n-CUBE

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ABSTRACT. The acyclic point-connectivity of a graph G, denoted $\alpha(G)$, is the minimum number of points whose removal from G results in an acyclic graph. In a 1975 paper, Harary stated erroneously that $\alpha(Q_n) = 2^{n-1} - 1$ where Q_n denotes the n-cube. We prove that for $n > 4$, $7 \cdot 2^{n-4} \le \alpha(Q_n) \le 2^{n-1} - 2^{n-y-2}$, where $y = [\log_2(n - 1)]$. We show that the upper bound is obtained for $n \le 8$ and conjecture that it is obtained for all n.

KEY WORDS AND PHRASES. n-cube, acyclic point-connectivity.
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1. INTRODUCTION.

Definitions and examples for this paper can be found in Harary [1], which has a readable survey of the general area to which our results and conjectures apply. For current work, consult especially the Journal of Graph Theory and the Journal of Combinatorics, Series B. More specifically, in [2] Harary introduced the acyclic point-connectivity of a graph G, denoted $\alpha(G)$, as the minimum number of points whose removal from G results in an acyclic graph. He then stated without proof that $\alpha(K_p) = p - 2$, $\alpha(K_{m,n}) = \min\{m,n\} - 1$ and $\alpha(Q_n) = 2^{n-1} - 1$, where Q_n denotes the n-cube. The first two equalities are indeed trivial; the last one is incorrect. In this paper we obtain upper bounds for $\alpha(Q_n)$, show that the upper bound is obtained for $n \le 8$, and conjecture that the upper bound is obtained for all n.

Before beginning the proof we introduce the following notation and definitions. For any real number x, [x] denotes the greatest integer not greater than x. The distance between two points v and w is denoted by d(v,w). The n-cube Q_n is the graph with point set consisting of all n-tuples of 0's and 1's, and two points are adjacent iff they differ in exactly one coordinate. We observe that Q_n is bipartite and all points of Q_n with fixed ith coordinate, $1 \le i < n$, induce the subgraph Q_{n-1}. In fact, the subgraph of Q_n induced by all points with nth coordinate 0(1) is denoted Q'_{n-1} (Q''_{n-1}).

2. MAIN RESULTS.

THEOREM 1. For $n > 4$, $7 \cdot 2^{n-4} \le \alpha(Q_n) \le 2^{n-1} - 2^{n-y-2}$, where $y = [\log_2(n-1)]$.

We begin the proof of Theorem 1 with a number of lemmas. Some of the easy proofs are omitted.

LEMMA 1. Let $v,w \in V(Q_n)$; $d(v,w) = j$ iff v and w differ in exactly j coordinates.

LEMMA 2. For $u,v \in V(Q_n$, $n > 1$, if $d(u,v) = 2$, then u and v are contained in exactly one 4-cycle.

LEMMA 3. If $\alpha(Q_n) = t$, then $\alpha(Q_{n+1}) \ge 2t$.

PROOF. This follows immediately from the observation that Q_{n+1} contains two disjoint copies of Q_n.

LEMMA 4. $\alpha(Q_2) = 1$ and $\alpha(Q_3) = 3$.

PROOF. The first equality is clear and with Lemma 3 implies that $\alpha(Q_3) \ge 2$. Let $u,v \in V(Q_3)$. If $d(u,v) \le 2(=3)$, then $Q_3 - \{v,u\}$ contains a 4-cycle (6-cycle). Thus $\alpha(Q_3) \ge 3$. Equality follows from the fact that the removal of (1, 1, 0), (0, 1, 1), (1, 0, 1) from Q_3 results in a disconnected graph consisting of K_1 and $K_{1,3}$, which we denote by $K_1 \cup K_{1,3}$. We also note that the five point path P_5 results from the removal of (0, 0, 0), (1, 1, 0), and (0, 0, 1) from Q_3.

LEMMA 5. The only induced acyclic subgraphs of Q_3 with order 5 are P_5 and $K_1 \cup K_{1,3}$.

PROOF. Exactly one of Q_2' and Q_2'' has three points of such a subgraph H. These three points induce P_3, and since H contains no cycles and is induced, it follows that $H = P_5$ or $K_1 \cup K_{1,3}$.

LEMMA 6. $\alpha(Q_4) = 6$ and the resulting induced acyclic subgraph H of Q_4 consists of two disjoint copies of $K_{1,4}$.

PROOF. From Lemmas 4 and 3 it follows that $\alpha(Q_4) \geq 6$. However, Figure 1 shows that six points may be removed to form $K_{1,4} \cup K_{1,4}$.

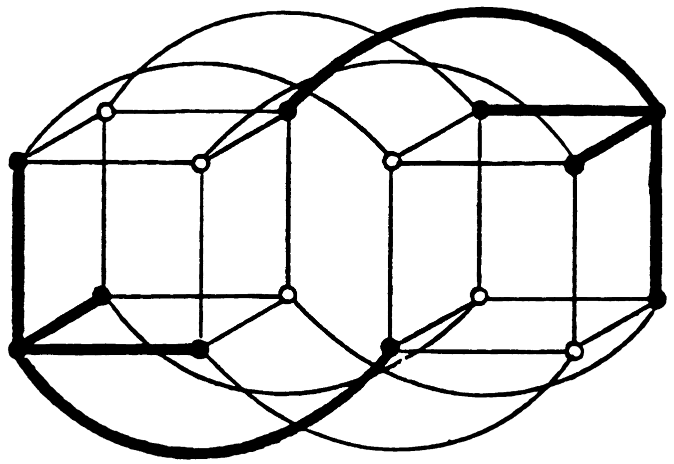

FIGURE 1 G_4

Thus $\alpha(Q_4) = 6$, and according to Lemma 5 when H is formed each of Q_3' and Q_3'' must have three points removed forming respectively H' and H". Also, H' and H" are P_5 or $K_1 \cup K_{1,3}$. Assume $H' = P_5$. Now H' is adjacent with two points of H" and these points must be in different components of H". Thus $H'' = K_1 \cup K_{1,3}$, and points of H' are adjacent to the three points removed from Q_3' and the isolated point of H". However, this implies that H' contains $K_{1,3}$. Thus $H' = H'' = K_{1,3} \cup K_1$.

Let $v_i(w_i)$, $1 \leq i \leq 8$, be the points of $Q_3'(Q_3'')$ and let v_i be adjacent to w_i, $1 \leq i \leq 8$. Also let $\{v_1, v_2, v_3, v_4\}$ induce $K_{1,3}$ and v_8 be the isolated point in H'. Now it can be checked that if w_i, $1 \leq i \leq 7$, is the required point of degree three in H", then H contains a cycle. Thus w_8 must be the degree three point and the Lemma is proved.

LEMMA 7. $\alpha(Q_5) = 14$.

PROOF. According to Lemmas 6 and 3, $\alpha(Q_5) \geq 12$. Figure 2 shows that $\alpha(Q_5) \leq 14$.

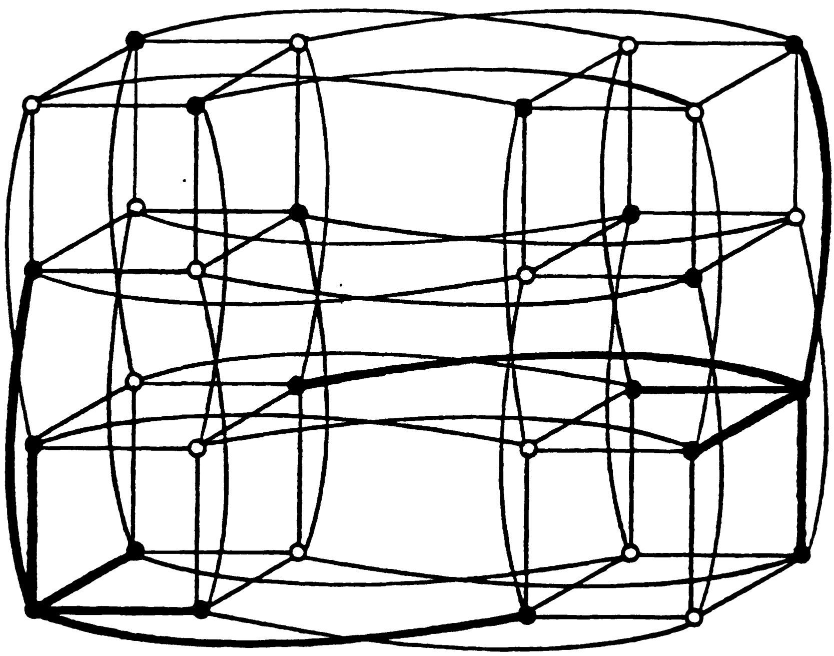

FIGURE 2 G_5

Assume that $(Q_5) = 12$ or 13, and let H be an acyclic subgraph of Q_5 formed by the removal of at most 13 points. Now Q_5 may be viewed as 4 disjoint copies, say M_1, M_2, M_3, and M_4 of Q_3. It follows from Lemma 4 that exactly three points have been removed from M_1, M_2, and M_3, and 3 or 4 points have been removed from M_4. Furthermore, Lemma 6 implies that the darkly shaded lines of Figure 3 are contained in H.

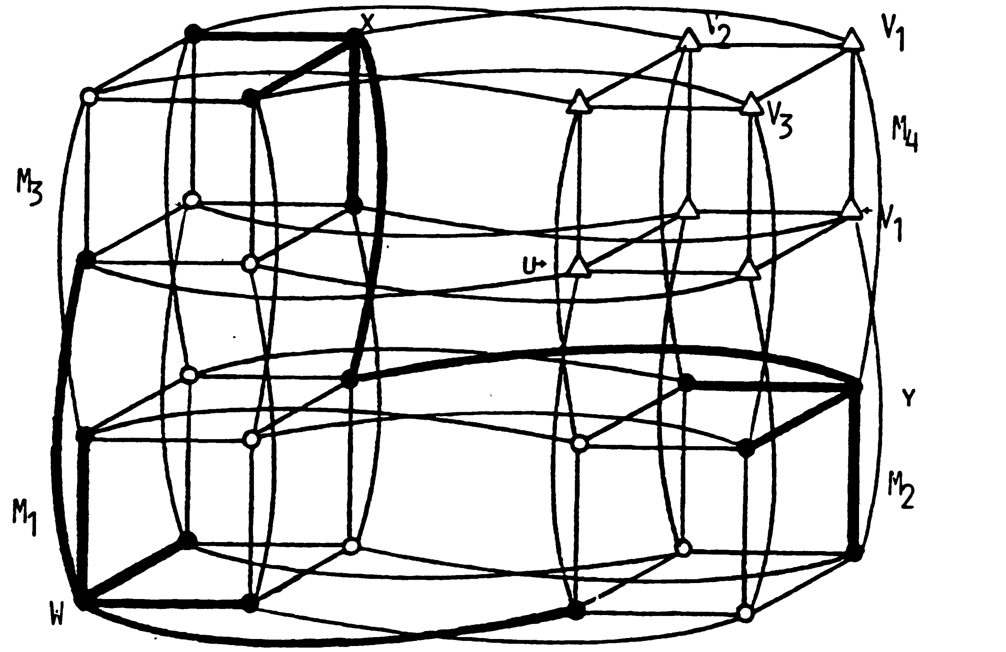

FIGURE 3

However, as shown in Figure 3, points v_2, v_3, and v_4 lie in 6-cycles with x and y, point v_1 is in a 4-cycle containing x and y, and point u is in a 4-cycle containing w. Hence $\alpha(Q_5) > 14$.

Before we prove the main result two additional definitions are required. Let $1 < k + 1 < n_0 < n$. Then $\theta_{1,k+1,n_0,n}$ denotes the mapping which assigns to each v in $V(Q_n)$ the point which differs from v in exactly the coordinates 1, k + 1, n_0, and n. It is clear that $\theta_{1,k+1,n_0,n}$ is an automorphism of Q_n. Also if $\alpha_i \in \{0,1\}$, then $\bar{\alpha}_i$ denotes the element in $\{0,1\} - \{\alpha_i\}$.

THEOREM 2. Let $n > 2$ and $y = [\log_2(n-1)]$. Then there exists a set of $2^{n-1} - 2^{n-y-2}$ points of Q_n whose removal results in an acyclic graph G_n which consists of exactly 2^{n-y-2} mutually disjoint copies $K_{1,n}$ and $2^{n-1} - n2^{n-y-2}$ isolated points. The set of points of degree n is denoted by $C(G_n)$, and these points are called <u>cluster points</u>. $C(G_n)$ is a subset of one bipartite set of Q_n and all points from the other bipartite set of Q_n are contained in G_n. Furthermore, let $n_0 = 2^y + 1$ and $n = n_0 + k$. Then $0 \le k \le 2^y - 1$.

If $k \ne 2^y - 1$, let $v = (\alpha_1, \ldots, \alpha_n) \in C(G_n)$. Then the elements of $C_{n,v} = \{v_i \in V(G_n)\colon\ v_i = (\bar{\alpha}_1, \alpha_2, \ldots, \alpha_{i-1}, \bar{\alpha}_i, \alpha_{i+1}, \ldots, \alpha_{n_0-1}, \bar{\alpha}_{n_0}, \alpha_{n_0+1}, \ldots, \alpha_n),$ $k + 2 \le i \le n_0 - 1$ are isolated points of G_n. Also let $D_{n,v} = C_{n,v} \cup \{v\} \cup \{x \in V(G_n)\colon$ x is adjacent to $v\}$. If $u \ne v$ where $u, v \in C(G_n)$, then $D_{n,v} \cap D_{n,u} = \emptyset$.

PROOF. Lemmas 4, 6, and 7 verify the theorem for $n = 2,3,4,5$. In order to complete the proof we show that the theorem is true for n, assuming it is true for $n - 1$. We consider two cases.

CASE I. $n = 2^y + 1$.

By inductive assumption the theorem is true for Q_{n-1}, $n - 1 = 2^y$. Thus there exist $2^{n-2} - 2^{n-1-(y-1)-2}$ points whose removal from Q_{n-1} results in graph G_{n-1} which consists of $2^{n-1-(y-1)-2} = 2^{n-y-2}$ mutually disjoint copies of $K_{1,n-1}$ and $2^{n-2} - (n - 1)(2^{n-y-2}) = 0$ isolated points. Also $C(G_{n-1})$ is contained in a bipartite set of Q_{n-1}.

Let G'_{n-1} be the subgraph of Q_n induced by all points of the form $(\alpha_1, \ldots, \alpha_{n-1}, 0)$ where $(\alpha_1, \ldots, \alpha_{n-1}) \in V(G_{n-1})$. Let A be the bipartite set of Q_n which contains $C(G'_{n-1})$ and let B be the other bipartite set.

Now define G_n as the subgraph of Q_n induced by $B \cup C(G'_{n-1})$. Note G_n has been formed by removing $2^{n-1} - 2^{n-y-2}$ points from Q_n. Now $C(G'_{n-1}) = C(G_n)$. Furthermore, if $u \ne v$ where $u, v \in C(G_n)$, the points adjacent to v are disjoint from the points adjacent to u. Thus the 2^{n-y-2} copies of $K_{1,n}$ are mutually disjoint. Also G_n contains all points of B and thus $2^{n-1} - n(2^{n-y-2})$ are isolated points.

In this case $k = 0$ and $n = n_0$. Let $v_i \in C_{n,v}$. From Lemma 1, we have that $d(v_i, v) = 3$. Thus $v_i \in B$ and $v_i \in V(G_n)$. Since v_i differs from v in the nth coordinate, $v_i \in V(Q''_{n-1})$. Let w_i be the point of Q'_{n-1} adjacent to v_i. Now $d(w_i, v) = 2$, and by Lemma 2, v and w_i are in a 4-cycle of Q'_{n-1}. However, all points of $B \cap V(Q'_{n-1})$ are in acyclic graph G_n. Thus $w_i \notin V(G_n)$ and v_i is an isolated point.

Let $u \ne v$ be elements of $C(G_n) = C(G'_{n-1})$. Then $d(u,v) \ge 4$. Thus $C_{n,u} \cap C_{n,v} = \emptyset$, for otherwise, $d(u,v) \le 2$. It now follows that $D_{n,u} \cap D_{n,v} = \emptyset$.

CASE II. $n = 2^y + 1 + k$, $1 \le k \le 2^y - 1$.

Again we apply the inductive assumption on $n - 1$ and obtain graph G_{n-1} with the prescribed properties. Let G'_{n-1} be the subgraph of Q_n induced by the points of the form $(x_1, \ldots, x_{n-1}, 0)$ where $(x_1, \ldots, x_{n-1}) \in V(G_{n-1})$. Since $\theta_{1,k+1,n_0,n}$ is an automorphism of Q_n, it follows that $\theta_{1,k+1,n_0,n}(G'_{n-1})$, which we now denote by G''_{n-1}, is isomorphic to G_{n-1}. Thus if $u = (\beta_1, \ldots, \beta_{n-1}, 0) \in C(G'_{n-1})$, then $\bar{u}=\theta_{1,k+1,n_0,n}(u) \in C(G''_{n-1})$. We now define G_n as the subgraph of Q_n induced by $V(G'_{n-1}) \cup V(G''_{n-1})$. It follows that G_n was formed from Q_n by the removal of $2(2^{n-2} - 2^{n-y-3}) = 2^{n-1} - 2^{n-y-2}$ points. From the fact that $d(u,\bar{u}) = 4$ and the inductive assumption, if follows that $C(G_n) = C(G'_{n-1}) \cup C(G''_{n-1})$ is a subset of one bipartite set of Q_n and the other bipartite set of Q_n is contained in $V(G_n)$.

Let $u,v \in C(G_n)$ where $u \neq v$. In order to show that $D_{n,u} \cap D_{n,v} = \emptyset$ we first show that $C_{n,u} \cap C_{n,v} = \emptyset$. The latter is clear if $u \in C(G'_{n-1})$ and $v \in C(G''_{n-1})$. So we suppose that $u,v \in C(G'_{n-1})$. $C_{n,v}$ consists of all but one point, namely v_{k+1}, of $C_{n-1,v}$. Similarly $C_{n,u}$ consists of all but one point of $C_{n-1,u}$. Thus it follows from the induction assumption that $C_{n,u} \cap C_{n,v} = \emptyset$.

Let $v \in C(G_n)$ and $v_i \in C_{n,v}$. Then $k + 2 \leq i \leq n_0 - 1$ and $v \in C(G'_{n-1})$ or $C(G''_{n-1})$, say the former. By induction, v_i was an isolated point of $G'_{n-1} = Q'_{n-1} \cap G_n$. Let $\bar{v} = \theta_{1,k+1,n_0,n}(v)$, and x be the point of Q''_{n-1} which is adjacent to v_i in Q_n. Now $\bar{v} \in C(G''_{n-1})$ and $d(x,\bar{v}) = 2$. Thus $x \notin V(G_n)$, for otherwise, G''_{n-1} would contain a 4-cycle. Hence, v_i is an isolated point of G_n. It now follows easily that $D_{n,u} \cap D_{n,v} = \emptyset$ when $u \neq v$.

Thus G_n consists of $2(2^{n-y-3}) = 2^{n-y-2}$ disjoint copies of $K_{1,n}$ and $2(2^{n-2} - (n - 1)2^{n-y-3}) - 2^{n-y-2} = 2^{n-1} - n2^{n-y-2}$ isolated points. This completes the proof of Theorem 2.

Now the upper bound of Theorem 1 follows immediately from Theorem 2 and Theorem 1's lower bound from Lemmas 4 and 7. We close by stating two conjectures. Lemmas 4, 6, 7, and Theorem 1 show the first conjecture is true for $2 \leq n \leq 8$. Lemmas 4 and 6 show the second conjecture is true for $m = 1$ and 2. Our long proof for $m = 3$ is omitted. It seems possible that Conjecture 2 would be useful in proving Conjecture 1.

CONJECTURE 1. For $n > 1$, $\alpha(Q_n) = 2^{n-1} - 2^{n-y-2}$ where $y = [\log_2(n - 1)]$.

CONJECTURE 2. If $n = 2^m$, $m \geq 1$, any maximum acyclic induced subgraph of Q_n consists of exactly 2^{n-m-1} disjoint copies of $K_{1,n}$.

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